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ON THE EXCHANGE OF CENSORSHIP TYPES AND MAXIMUM LIKELIHOOD ESTIMATION FOR PARAMETERS IN LIFETIME DISTRIBUTIONS

G.C. Jacobs[1], A.J. Watkins[2] §

[1,2]School of Business and Economics
Swansea University
Singleton Park, Swansea, SA2 8PP, U.K.
[2]e-mail: a.watkins@swansea.ac.uk

Abstract: This note examines the impact of exchanging censorship types on the properties of maximum likelihood estimates of parameters in lifetime distributions. We show that considerable differences can exist between estimates obtained from analysing data under various plausible assumptions on censorship types, and it is therefore important for analysts to ensure that the most appropriate analysis of the data.

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1. Introduction and Motivation

This note examines censorship types and the impact of exchanging such types on the properties of maximum likelihood estimates of parameters in lifetime distributions. The statistical basis for including information on non-failed components in the analysis along with that on failed components is uncontroversial, although still occasionally overlooked. However, there may be several plausible interpretations of a single data, and analysts should therefore undertake the most appropriate analysis in order to ensure that the most accurate picture of product performance is obtained. We show that considerable differences can

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§Correspondence author

exist between estimates obtained from analysing data under various plausible censorship assumptions, with such differences often having important practical ramifications. We consider Type-I and Type-II censoring regimes, and a Weibull regression model frequently used in accelerated life-testing studies. We conclude that significant differences may emerge when we analyse data actually derived from Type-I censorings, as observations from an experiment subjected to a Type-II censoring. As one would expect, the differences are greatest for small sample sizes. In contrast, we see that there is a smaller penalty for making the incorrect assumption of a Type-I censoring regime for data actually derived from an experiment subject to a Type-II censoring.

The statistical analysis of reliability data is a well-established, specialised area (see, for instance, Nelson, [6] and Crowder et al, [4]). The collection and analysis of such data facilitates the monitoring of improvements in product design, and thus applies to areas in which engineers and scientists seek to improve on existing products; Nelson [6], for example, cites data on mechanical breakdowns and electrical failures, while Ansell and Phillips [1] include cases on the failure of telecommunication systems, on repairable electronic systems, and on the dielectric breakdown of metal-oxide-semiconductor integrated circuits. More recently, Ciappa and Fichtner [3] summarise the scope of current contributions to the assessment of the reliability in the rapidly evolving area of electronic-device technology.

Two themes emerge from the above examples. The first is the use of reliability distributions to model lifetimes of items in order to make inferences concerning such items. Here, the use of the Weibull distribution, which includes the negative exponential and Rayleigh distributions as special cases, is widespread. The second theme is the importance of including so-called censored data in the statistical analysis; this is a particular feature of reliability analysis, and enables data on items which have failed to be augmented by information on those items which, at the time of data collection, have not failed but are still operational – thus, these eventual times to failure are said to be 'censored' from the analyst.

This paper is concerned with one aspect of the second theme, and continues a discussion initiated by Bugaighis [2] on the exchange of censorship types and its impact on the behaviour and properties of maximum likelihood estimates of parameters in lifetime distributions. It is now comparatively routine within manufacturing to collect operational data, and failure times and modes are often recorded electronically. The statistical basis for analysing the resulting data set is uncontroversial, but several interpretations of a single data set are some-times possible, and it is therefore vital for reliability analysts and managers of manufacturing processes to ensure that the most appropriate analysis of the data is performed: only by carefully considering which analysis is to be used can the most accurate picture of product improvement be obtained. Bugaighis [2] was primarily concerned with the effects on the efficiency of estimators; here, we argue that considerable differences may exist between estimates obtained from analysing data using the correct censorship guidelines, and counterparts obtained from analysing data under incorrect censorship guidelines. Such differences have important theoretical implications, simply because one censoring regime is correct and the other not, and we may wish to gauge the robustness of our analysis to such specification errors. There are also significant practical ramifications, because it will often be necessary accurately to interpret the results of a single analysis. We consider Type-I and Type-II censoring regimes, since these regimes are of considerable practical importance.

To illustrate the above concerns, we return to the data given in the illustrative example in Bugaighis [2]. Under the stated Type-I censoring guidelines (to observe items at 1000 volts for 800 hours, and items at 1600 volts for 600 hours), the data for analysis (in hours) is

1000 volts:	450	550	600	650	800*,
1600 volts:	250	300	400	600*	600*.

Here – as throughout – we use * to denote a censored observation. The data is modelled using the Weibull regression model, for which the cumulative distribution function is, following Bugaighis [2], defined as

$$F\left(t;\alpha,\beta,\sigma\right)=1-\exp\left\{-\left(\frac{t}{\exp\left[\alpha+\beta X\right]}\right)^{\frac{1}{\sigma}}\right\} \tag{1}$$

for $t \geq 0$, where X denotes voltage. The estimates of parameters in (1) are then found (to four significant figures) to be $\sigma = 0.3353, \alpha = 6.835, \beta = -0.000312$; we can also consider the analysis which would follow from applying the Type-II censoring guidelines of stopping the experiment after four failures at 1000 volts, and three failures at 1600 volts. The data for analysis (again in hours) is then

1000 volts:	450	550	600	650	650*,
1600 volts:	250	300	400	400*	400*,

and the estimates of parameters in (1) are now found (again to four significant figures) to be $\sigma = 0.1596, \alpha = 6.835, \beta = -0.000719$. The central point to note is that different assumptions give rise to different estimates.

It is also pertinent to emphasise that σ is the reciprocal of the (common) Weibull shape parameter, so the reported difference between the two estimates of σ is quite large; and we again emphasise that such differences may have considerable practical ramifications. Although the above example is based on small sample sizes, it does illustrate that it is possible to obtain considerable differences between estimates based on the correct censoring regime and counterparts based on an incorrect regime; we further emphasise here that our primary concern is the potential differences between the two sets of estimators.

2. Censoring Regimes

In this section, we briefly outline the basic censoring regimes, and the changes to data which may occur when the guidelines of one censoring regime are (incorrectly) used in place of those of another. For brevity, we consider an experiment in which n identical items are simultaneously entered into test, and observed. We assume that the duration of the test is c (in appropriate units, such as hours), that m items have failed – with times to failure $t_1, \ldots, t_m$, ranked in ascending order as $t_{(1)}, \ldots, t_{(m)}$ – and that the remaining $n - m$ items are still operational, with equal times in service of c units. Type-I and Type-II censoring regimes are based on the following ideas:

— Type-I censoring: the time c is pre-specified; the number of failures m is a random variable, and

— Type-II censoring: the number of failures m is pre-specified; the time $c = t_{(m)}$ is a random variable.

For data actually arising under Type-I censoring, it is possible that the data will be regarded as occurring under Type-II censoring with m failures, with c replaced by $t_{(m)}$; for the data at 1000 volts in the above example, we have $n = 5, m = 4, c = 800, t_{(m)} = 650$, so the change is as follows:

— Type-I: $450, 550, 600, 650, 800^*$,

— Type-II: $450, 550, 600, 650, 650^*$.

Somewhat more generally, the change affects each of the $n-m$ items with a censored time in service, and thus has a potentially sizeable effect on parameter estimates. Here, we are effectively ignoring the fact that the experiment has continued after the m-th failure.

For data actually arising under Type-II censoring, several interpretations are possible: some of these may be summarised as follows:

— Type-II: $450, 550, 600, 650, 650^*$,

— Type-Ia: $450, 550, 600, 650, 650^*$,

— Type-Ib: $450, 550, 600, 650^*, 650^*$,

— Type-Ic: $450, 550, 600, 649^*, 650^*$.

The first possibility, here denoted as Type-Ia, yields precisely the same data set, and hence leads to the same parameter estimates. The other two possibilities given above rely on more subjective arguments concerning possible decisions for truncating the experiment around the last time to failure $t_{(m)}$. However, each of these two possibilities affects only the status (and, possibly, the value) of one item, and thus there is a relatively minor effect on parameter estimates.

3. The Effect on Parameter Estimates

The scenarios outlined above allow us to consider the effect of exchanging censorship types on parameter estimates: to simplify the algebra, we take $\sigma = 1, \beta = 0$, so that (1) reduces to the negative exponential distribution with mean $\theta = \exp(\alpha)$. It is well known that the maximum likelihood estimator of θ is

$$\hat{\theta} = m^{-1} \sum_{i=1}^{n} t_i;$$

that is, the sum of all lifetimes (failed and censored) divided by the number of failures.

3.1. Data under Type-I Censoring

Thus, for data actually arising under Type-I censoring, we *should* estimate θ by

$$\hat{\theta}_I = m^{-1} \left\{ t_{(1)} + \cdots t_{(m)} + (n-m)\, c \right\},$$

but, if the data is regarded as occurring under Type-II censoring with m failures and c is replaced by $t_{(m)}$, then this changes to

$$\begin{aligned} \hat{\theta}_{II} &= m^{-1} \left\{ t_{(1)} + \cdots t_{(m)} + (n-m)\, t_{(m)} \right\} \\ &= \hat{\theta}_I - m^{-1} (n-m) \left\{ c - t_{(m)} \right\}; \end{aligned}$$

for the small data set above, this means the change from:

— Type-I: $450, 550, 600, 650, 800^* : \hat{\theta}_I = 762.5$

to

— Type-II: $450, 550, 600, 650, 650^* : \hat{\theta}_{II} = 725.0$

3.2. Data under Type-II Censoring

For data actually arising under Type-II censoring, we should estimate θ by

$$\hat{\theta}_{II} = m^{-1}\left\{t_{(1)} + \cdots t_{(m)} + (n-m)\,t_{(m)}\right\},$$

and would obtain the same estimator under Type-Ia censoring. For Type-Ib censoring, we obtain

$$\hat{\theta}_{Ib} = (m-1)^{-1}\left\{t_{(1)} + \cdots t_{(m-1)} + (n-m+1)\,t_{(m)}\right\},$$

and, under Type-Ic censoring, we obtain

$$\hat{\theta}_{Ic} = (m-1)^{-1}\left\{t_{(1)} + \cdots t_{(m-1)} + (n-m+1)\,t_{(m)} - 1\right\}.$$

For the small data set above, these changes can be summarised as:

— Type-II: $450, 550, 600, 650, 650^* : \hat{\theta}_{II} = 725.0$

— Type-Ia: $450, 550, 600, 650, 650^* : \hat{\theta}_{Ia} = \hat{\theta}_{II} = 725.0$

— Type-Ib: $450, 550, 600, 650^*, 650^* : \hat{\theta}_{Ib} = 966.67$

— Type-Ic: $450, 550, 600, 649^*, 650^* : \hat{\theta}_{Ic} = 966.33$

3.3. Discussion

On the basis of the above formulae and examples, we see that there is a relatively small penalty for replacing the assumption of Type-II censoring by the incorrect assumption of a Type-I censoring regime; and perhaps the biggest concern is the possible resultant change in status accorded to one observation, which then reduces the denominator from m to $m-1$. This, in turn, can have a considerable effect on the values of estimators of θ when m is small. In contrast, there is a relatively large penalty for replacing the assumption of Type-I censoring by the incorrect assumption of Type-II censoring; this is particularly so when $n-m$ and/or $c - t_{(m)}$ are large.

4. Generalisations and Summary

We expect the broad themes established above for the negative exponential distribution to generalise to the two-parameter Weibull distribution, and, further, to parameters in Weibull regression models. However, it is no longer possible to consider analytical results summarising the effect of the above exchanges on the estimates of all parameters. For instance, in the two parameter Weibull distribution, we must resort to numerical studies to consider the effect on the estimates of the shape parameter; thereafter, we can study the effect on estimates of the scale parameter using the well-known link expressing the estimate of scale in terms of shape (see, for example, Kalbfleisch, [5]). The interested reader will appreciate that further generalisation to Weibull regression models is also heavily dependent on numerical investigations.

In this note, we have argued that our primary concern regarding the exchange of censorship types should be the effect on the estimates of parameters in the chosen lifetime distribution; we have used sample data from Bugaighis [2] to illustrate that this effect can be large. We have also presented some results which further quantify the effect of this exchange on the estimation of the mean parameter in the negative exponential distribution, and have briefly discussed the generalisation of these results to the two parameter Weibull distribution and Weibull regression models.

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