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THE KY-FAN METRIC AND THE CHANGE OF SCALE

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Abstract: In [2] Thompson showed how the Lévy distance changes during change of scale. In [1] it was proved these relations for the Prokhorov distance. Here it is proved similar relations for the Ky-Fan distance.

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1. Introduction

Let $\mathcal{F}$ denote the class of all distribution functions on $\mathbb{R}$. The Lévy distance is the function $L : \mathcal{F} \times \mathcal{F} \to [0, 1]$ defined as follows:

$$L(F, G) = \inf\{h > 0 : F(x - h) - h \le G(x) \le F(x + h) + h, x \in \mathbb{R}\}.$$

In [2] Thompson showed how this distance changes during change of scale. Namely, he proved that

$$\begin{aligned} aL(F, G) \le L(F_a\,, G_a) \le L(F, G) \quad &\text{for} \quad 0 < a < 1, \\ L(F, G) \le L(F_a\,, G_a) \le aL(F, G) \quad &\text{for} \quad a > 1, \end{aligned}$$

where $F_a(x) = F(\frac{x}{a})$. In [1] it was proved similar relations for the Prokhorov distance on the space of probability measures defined on a normed linear space. The Lévy and Prokhorov distances metrize the weak convergence. We prove similar relations for the Ky-Fan distance which metrizes the convergence in probability.

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Let S be a normed linear space with the norm $||\cdot||$ and ϱ be the metric generated by this norm. Moreover let $\mathfrak{X}$ stands for the set of all random elements X defined on a probability space $(\Omega, \mathcal{A}, P)$ and taking values in S. The Ky-Fan metric $K : \mathfrak{X} \times \mathfrak{X} \to [0, 1]$ is defined as follows:

$$K(X, Y) = \inf\{h > 0 : P[\varrho(X, Y) \geq h] \leq h\}.$$

2. Main Result

First we introduce a concept of a generalized Ky-Fan distance.

Definition. The generalized Ky-Fan distance is the function $K^\theta : \mathfrak{X} \times \mathfrak{X} \to [0, (\sqrt{2}\sin\theta)^{-1}]$, $0 < \theta < \frac{\pi}{2}$, such that

$$K^\theta(X, Y) = \inf\{h > 0 : P[\varrho(X, Y) \geq \sqrt{2}\, h\cos\theta] \leq \sqrt{2}\, h\sin\theta\}.$$

Note that $K^{\frac{\pi}{4}}(X, Y)$ is the classical Ky-Fan distance.

Lemma. *The function K^θ given by Definition is a metric.*

Proof. It is obvious that:

(i) $K^\theta(X, Y) = 0$ iff $P[X = Y] = 1$,

(ii) $K^\theta(X, Y) = K^\theta(Y, X)$, $X, Y \in \mathfrak{X}$.

Now we see that for any given $\varepsilon > 0$ and $\delta > 0$ we have

$$\begin{aligned} &[\varrho(X, Y) \leq (\varepsilon + \delta)\sqrt{2}\cos\theta] \\ &\qquad \supseteq \ [\varrho(X, Y) \leq \varrho(X, Z) + \varrho(Z, Y) \leq (\varepsilon + \delta)\sqrt{2}\cos\theta] \\ &\qquad \supseteq \ [\varrho(X, Z) \leq \varepsilon\sqrt{2}\cos\theta] \cap [\varrho(Z, Y) \leq \delta\sqrt{2}\cos\theta]. \end{aligned}$$

Hence we get

$$\begin{aligned} &[\varrho(X, Y) > (\varepsilon + \delta)\sqrt{2}\cos\theta] \\ &\qquad \subseteq \ [\varrho(X, Z) > \varepsilon\sqrt{2}\cos\theta] \cup [\varrho(Z, Y) > \delta\sqrt{2}\cos\theta]. \end{aligned}$$

If $K^\theta(X, Z) < \varepsilon$ and $K^\theta(Z, Y) < \delta$, then we have

$$\begin{aligned} P[\varrho(X, Y) > (\varepsilon + \delta)\sqrt{2}\cos\theta] \ &\leq \ P[\varrho(X, Z) > \varepsilon\sqrt{2}\cos\theta] \\ &\quad + P[\varrho(Z, Y) > \varrho\sqrt{2}\cos\theta] \\ &< \ (\varepsilon + \delta)\sqrt{2}\sin\theta. \end{aligned}$$

Hence

$$K^\theta(X, Y) < \varepsilon + \delta.$$

Letting now $\varepsilon \to K^\theta(X, Z)$ and $\delta \to K^\theta(Z, Y)$ we get:

(iii) $K^\theta(X, Y) \leq K^\theta(X, Z) + K^\theta(Z, Y)$

which completes the proof. □

Relations between $K(X,Y) := K^{\frac{\pi}{4}}(X,Y)$ and $K^{\theta}(X,Y)$ gives the following theorem.

Theorem 1. *We have:*

(i) $(\sqrt{2}\cos\theta)^{-1}K(X,Y) \leq K^{\theta}(X,Y) \leq (\sqrt{2}\sin\theta)^{-1}K(X,Y)$ *for* $0 < \theta < \frac{\pi}{4}$, *and*

(ii) $(\sqrt{2}\sin\theta)^{-1}K(X,Y) \leq K^{\theta}(X,Y) \leq (\sqrt{2}\cos\theta)^{-1}K(X,Y)$ *for* $\frac{\pi}{4} < \theta < \frac{\pi}{2}$.

Proof. Let $h_0 > K(X,Y)$. Then $P[\varrho(X,Y) \geq h_0] \leq h_0$. As $\left(\frac{\cos\theta}{\sin\theta}\right) > 1$ for $0 < \theta < \frac{\pi}{4}$ then

$$P[\varrho(X,Y) \geq \sqrt{2}\,h_0(\sqrt{2}\sin\theta)^{-1}\cos\theta] \leq \sqrt{2}\,h_0(\sqrt{2}\sin\theta)^{-1}\sin\theta.$$

Therefore

$$K^{\theta}(X,Y) \leq h_0(\sqrt{2}\sin\theta)^{-1}$$

for all $h_0 > K(X,Y)$. Thus

$$K^{\theta}(X,Y) \leq (\sqrt{2}\sin\theta)^{-1}K(X,Y),$$

which gets the right side of (i).

Let now $h_1 > K^{\theta}(X,Y)$. Then

$$P[\varrho(X,Y) \geq \sqrt{2}\,h_1\cos\theta] \leq \sqrt{2}\,h_1\sin\theta$$

and

$$P[\varrho(X,Y) \geq \sqrt{2}\,h_1\cos\theta] \leq \sqrt{2}\,h_1\cos\theta$$

as

$$\cos\theta > \sin\theta \quad \text{for} \quad 0 < \theta < \frac{\pi}{4}\,.$$

Hence

$$K(X,Y) \leq \sqrt{2}\,h_1\cos\theta$$

for all $h_1 > K^{\theta}(X,Y)$. Thus

$$K(X,Y) \leq (\sqrt{2}\cos\theta)K^{\theta}(X,Y)$$

or

$$(\sqrt{2}\cos\theta)^{-1}K(X,Y) \leq K^{\theta}(X,Y).$$

This completes the proof of part (i).

The proof of (ii) one can get by the similar way. □

Now we show how changes the Ky-Fan distance under change of scale.

Theorem 2. *Let $X, Y \in \mathfrak{X}$. Then*

$$\begin{aligned} aK(X,Y) \leq K(aX,aY) \leq K(X,Y), \qquad & 0 < a < 1, \\ K(X,Y) \leq K(aX,aY) \leq aK(X,Y), \qquad & a > 1. \end{aligned}$$

Proof. We have

$$\varrho(aX,aY) = ||aX - aY|| = ||a(X-Y)|| = |a|\,||X-Y|| = a\varrho(X,Y).$$

Hence

$$P[\varrho(aX,aY) \geq h] = P[a\varrho(X,Y) \geq h] = P\left[\varrho(X,Y) \geq \frac{h}{a}\right].$$

By definition of K we get

$$\begin{aligned} K(aX,aY) &= \inf\{h > 0 : P[\varrho(aX,aY) \geq h] \leq h\} \\ &= \inf\{h > 0 : P\left[\varrho(X,Y) \geq \frac{h}{a}\right] \leq h\}. \end{aligned}$$

Now assume $h' = \frac{h}{\sqrt{2}\sin\theta}$ and $\operatorname{tg}\theta = a$. Hence we have $a = \frac{\sin\theta}{\cos\theta}$ and

$$\begin{aligned} & K(aX,aY) \\ &= \inf\{h'\sqrt{2}\sin\theta : P[\varrho(X,Y) \geq h'\sqrt{2}\cos\theta] \leq h'\sqrt{2}\sin\theta\} \\ &= \sqrt{2}\sin\theta \inf\{h' > 0 : P[\varrho(X,Y) \geq h'\sqrt{2}\cos\theta] \leq h'\sqrt{2}\sin\theta\} \\ &= \sqrt{2}\sin\theta K^{\theta}(X,Y). \end{aligned}$$

Hence and by part (i) of Theorem 1 we get for $0 < a < 1$

$$\operatorname{tg}\theta K(X,Y) \leq \sqrt{2}\sin\theta K^{\theta}(X,Y) = K(aX,aY)$$

and

$$\sqrt{2}\sin\theta K^{\theta}(X,Y) \leq K(X,Y).$$

Thus

$$aK(X,Y) \leq K(aX,aY) \leq K(X,Y).$$

Similarly we get

$$K(X,Y) \leq K(aX,aY) \leq aK(X,Y)$$

for $a > 1$. □

Remark. The definition of K^{θ} is correct for any metric space (S,ϱ). Moreover Lemma and Theorem 1 also are true for any metric space (S,ϱ). In the Theorem 2 the assumption that S is normed linear space one can weaken to be a Fréchet space.

References

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