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ON A CLASS OF HYBRID METHODS FOR $y'' = f(x, y, y')$

Samuel N. Jator
Department of Mathematics
Austin Peay State University
Clarksville, TN 37044, USA
e-mail: Jators@apsu.edu

Abstract: A family of hybrid linear multistep methods (HLMMs) with two nonstep points is proposed for the direct solution of second order initial value problems (IVPs). The methods are applied in block form as simultaneous numerical integrators over non-overlapping intervals. We give specific cases for step number $k = 2, 3, 4$ to illustrate the process. Numerical results obtained using the proposed methods reveal that they are highly competitive with existing methods in the literature.

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1. Introduction

The general second-order ordinary differential equation of the form

$$y'' = f(x, y, y'), \quad y(a) = y_0, \quad y'(a) = \delta_0, \tag{1}$$

where f satisfies a Lipschitz condition as given in Henrici [13] is encountered in several areas of engineering and science, such as circuit theory, control theory, chemical kinetics, and biology. We also note that Keller [16] has given the theorem and proof of the general conditions which ensure that the solution to (1) will exist and be unique. In practice, problem (1) is solved by first reducing it to a system of first-order differential equations and then applying the various methods available for solving systems of first order IVPs. This process leads to more computational work (see Lambert [17], Brugnano and Trigiante [4], Onumanyi et al [22], [21], Hairer and Wanner [12], and Gear [8]).

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Several methods for the direct solution of the special case of (1) which does not involve the first derivative have been extensively discussed in Lambert and Watson [18], Twizell and Khaliq [25], Yusuph and Onumanyi [27], and Henrici [13]. The general form (1) has also been solved directly without first reducing it to an equivalent first-order system. For instance, Hairer and Wanner [11] and Chawla and Sharma [5] proposed Nystrom type methods, which enjoy the advantage of being self starting, while Awoyemi [3] and Vigo-Aguiar and Ramos [26] considered methods of the linear multistep type which require fewer function evaluations. However, the Runge-Kutta type methods involve several function evaluations per step, hence increasing the computer time for the methods. On the other hand, methods of the linear multistep type are not self-starting and are generally implemented in a predictor-corrector mode. Hence, errors in earlier steps can be transferred into later calculations as the integration proceeds. According to Conte and de Boor [6], computer programs associated with predictor corrector methods are often complicated, especially, when incorporating subroutines to supply starting values for the methods, thus, resulting in longer computer time and more computational work.

Recently Jator and Li [15] and Jator [14] proposed linear multistep methods (LMMs) for the direct solution of (1) which were implemented without the need for either predictors or starting values from other methods. The methods were applied as simultaneous numerical integrators over sub-intervals which do not overlap, hence errors in earlier steps were not transferred into later calculations as in the step-by-step procedure. In this paper, we propose more accurate methods by collocating the second derivative of the approximate solution at k points as well as at two nonstep points. We derive a continuous hybrid method with two off-step points through interpolation and collocation (see Lie and Norsett [20], Atkinson [2] and Onumanyi et al [21]). The continuous representation generates main discrete hybrid methods and additional methods which are combined and used as block methods to simultaneously produce approximations

$$y_{n+\frac{1}{2}}, \quad y_{n+\frac{2k-1}{2}}, \quad y_{n+j}, \qquad j = 1, \ldots, k,$$

to the solution of (1) at block points

$$x_{n+\frac{1}{2}}, \quad x_{n+\frac{2k-1}{2}}, \quad x_{n+j}, \qquad j = 1, \ldots, k, \; n = 0, k, 2k, \ldots, N-k,$$

on a partition $[a, b]$, where $a, b \,\epsilon\, \mathbb{R}$, $h = \frac{b-a}{N}$ is the constant step-size, n is a grid index and N is the number of steps.

The paper is organized as follows. In Section 2, we obtain a continuous representation $Y(x)$ for the exact solution $y(x)$ which is used to generates a main discrete hybrid methods and additional methods for solving (1). The specification of the methods is given in Section 3, while the analysis and implementation of the methods are discussed in Section 4. Numerical examples are given in Section 5 to show the efficiency of the methods. Finally, the conclusion of the paper is discussed in Section 6.

2. Development of Method

In this section, our objective is to derive a HLMM of the form

$$\sum_{j=0}^{k} \alpha_j y_{n+j} = h^2 \sum_{j=0}^{k} \beta_j f_{n+j} + h^2 \sum_{j=1}^{2} \beta_{\nu_j} f_{n+\nu_j}, \tag{2}$$

where α_j, β_j, and β_{ν_j} are unknown constants and ν_j is not an integer. We note that $\alpha_k = 1$, $\beta_k \neq 0$, and α_0 and β_0 do not both vanish. In order to obtain (2), we proceed by seeking to approximate the exact solution $y(x)$ by a continuous method $Y(x)$ of the form

$$Y(x) = \sum_{j=0}^{r+s-1} \ell_j x^j, \tag{3}$$

where $x \in [a, b]$, ℓ_j are unknown coefficients to be determined and $1 \leq r < k$ and $s > 0$ are the number of interpolation and collocation points respectively. We then construct our continuous approximation by imposing the following conditions.

$$Y\left(x_{n+j}\right) = y_{n+j}, \quad j = 0, 1, 2, \ldots, r-1, \tag{4}$$

$$Y''\left(x_{n+\mu}\right) = f_{n+\mu}, \quad \mu = \{j, \nu_1, \nu_2\}, \qquad j = 0, 1, 2, \ldots, k, \tag{5}$$

where ν_1 and ν_2 are not integers. We note that $y_{n+\mu}$ is the numerical approximation to the analytical solution $y(x_{n+\mu})$, $f_{n+\mu} = f(x_{n+\mu}, y_{n+\mu}, y'_{n+\mu})$, $\mu = \{j, \nu_1, \nu_2\}, j = 0, 1, 2, \ldots, k$.

Equations (4) and (5) lead to a system of $(r + s)$ equations which is solved by Cramer's rule to obtain ℓ_j. Our continuous approximation is constructed by substituting the values of ℓ_j into equation (3). After some manipulation, the continuous method is expressed as

$$Y(x) = \sum_{j=0}^{k} \alpha_j(x) y_{n+j} + h^2 \sum_{j=0}^{k} \beta_j(x) f_{n+j} + h^2 \sum_{j=1}^{2} \beta_{\nu_j}(x) f_{n+\nu_j}, \tag{6}$$

where $\alpha_j(x)$, $\beta_j(x)$, and $\beta_{\nu_j}(x)$ are continuous coefficients. We note that since

(1) involves the first derivative, the first derivative formula

$$Y'(x) = \frac{1}{h}\left(\sum_{j=0}^{k}\alpha_j'(x)y_{n+j} + h^2\sum_{j=0}^{k}\beta_j'(x)f_{n+j} + h^2\sum_{j=1}^{2}\beta_{\nu_j}'(x)f_{n+\nu_j}\right) \tag{7}$$

is easily obtained from (6) and is then used to provide the first derivatives for the methods by imposing that

$$Y'(x) = \delta(x), \tag{8}$$

$$Y'(a) = \delta_0. \tag{9}$$

3. Particular of Methods

In this section, we use (6) to obtain particular HLMMs by specifying r, s, and k. We emphasize that the main methods are obtained by evaluating (6) at $x = x_{n+k}$. We discuss details of specific methods next.

Case $k = 2$. Choosing $\nu_1 = \frac{1}{2}$, $\nu_2 = \frac{3}{2}$, $r = 2$, $s = 5$, $k = 2$, and evaluating (6) at $x = \{x_{n+2}, x_{n+\frac{3}{2}}, x_{n+\frac{1}{2}}\}$ we generate the following main method and two additional methods.

$$y_{n+2} - 2y_{n+1} + y_n = \frac{h^2}{60}(f_n + 16f_{n+1/2} + 26f_{n+1} + 16f_{n+3/2} + f_{n+2}), \tag{10}$$

$$y_{n+3/2} + \frac{y_n}{2} - \frac{3}{2}y_{n+1} = \frac{h^2}{1920}(17f_n + 252f_{n+1/2} + 402f_{n+1} + 52f_{n+3/2} - 3f_{n+2}), \tag{11}$$

$$y_{n+1/2} - \frac{1}{2}y_n - \frac{1}{2}y_{n+1} = \frac{h^2}{1920}(-19f_n - 204f_{n+1/2} - 14f_{n+1} - 4f_{n+3/2} + f_{n+2}). \tag{12}$$

Since our method is designed to simultaneously provide the values of $y_{n+\frac{1}{2}}$, y_{n+1}, $y_{n+\frac{3}{2}}$, y_{n+2} for the solution of (1) at a block of points $x_{n+\frac{1}{2}}$, x_{n+1}, $x_{n+\frac{3}{2}}$, x_{n+2}, the three equations (10)-(12) are insufficient to provide the solutions for the four unknowns $y_{n+\frac{1}{2}}$, y_{n+1}, $y_{n+\frac{3}{2}}$, y_{n+2}. Thus, we obtain an additional method from (9), given by

$$h\delta_0 = y_1 - y_0 + \frac{h^2}{360}(-53f_0 - 144f_{1/2} + 30f_1 - 16f_{3/2} + 3f_2). \tag{13}$$

The derivatives are obtained from (8) by imposing that $\delta(x_{n+\mu}) = \delta_{n+\mu},\ \mu = \{j, \nu_1, \nu_2\}, j = 0, \dots, 2$, thus, we have

$$h\delta_{n+1/2} = -y_n + y_{n+1} + \frac{h^2}{1440}(39f_n + 70f_{n+1/2} - 144f_{n+1} + 42f_{n+3/2} - 7f_{n+2}),$$

$$h\delta_{n+1} = -y_n + y_{n+1} + \frac{h^2}{360}(5f_n + 104f_{n+1/2} + 78f_{n+1} - 8f_{n+3/2} + f_{n+2}),$$

$$h\delta_{n+3/2} = -y_n + y_{n+1} + \frac{h^2}{1440}(31f_n + 342f_{n+1/2} + 768f_{n+1} + 314f_{n+3/2} - 15f_{n+2}),$$

$$h\delta_{n+2} = -y_n + y_{n+1} + \frac{h^2}{360}(3f_n + 112f_{n+1/2} + 126f_{n+1} + 240f_{n+3/2} + 59f_{n+2}).$$

Case $k = 3$. Choosing $\nu_1 = \frac{1}{2}$, $\nu_2 = \frac{5}{2}$, $r = 2$, $s = 6$, $k = 3$, and evaluating (6) at $x = \{x_{n+3}, x_{n+\frac{5}{2}}, x_{n+2}, x_{n+\frac{1}{2}}\}$ we generate the following main method and three additional methods. The fifth method is provided by the (9).

$$y_{n+3} - 3y_{n+1} + 2y_n = \frac{h^2}{300}(22f_n + 88f_{n+1/2} + 445f_{n+1} + 320f_{n+2} + 8f_{n+5/2} + 17f_{n+3}), \quad (14)$$

$$y_{n+5/2} - \frac{5}{2}y_{n+1} + \frac{3}{2}y_n = \frac{h^2}{3840}(197f_n + 928f_{n+1/2} + 4055f_{n+1} + 2435f_{n+2} - 512f_{n+5/2} + 97f_{n+3}), \quad (15)$$

$$y_{n+2} - 2y_{n+1} + y_n = \frac{h^2}{300}(9f_n + 56f_{n+1/2} + 190f_{n+1} + 65f_{n+2} - 24f_{n+5/2} + 4f_{n+3}), \quad (16)$$

$$y_{n+1/2} - \frac{1}{2}y_{n+1} - \frac{1}{2}y_n = \frac{h^2}{19200}(-187f_n - 2048f_{n+1/2} - 145f_{n+1} - 45f_{n+2} + 32f_{n+5/2} - 7f_{n+3}), \quad (17)$$

$$h\delta_0 = y_1 - y_0 + \frac{h^2}{12600}(-1843f_0 - 5032f_{1/2} + 830f_1 - 515f_2 + 328f_{5/2} - 68f_3). \quad (18)$$

The derivatives are obtained from (8) by imposing that $\delta(x_{n+\mu}) = \delta_{n+\mu},\ \mu = \{j, \nu_1, \nu_2\}, j = 0, \dots, 3$, thus, we have

$$h\delta_{n+1/2} = -y_n + y_{n+1}$$

$$+\frac{h^2}{201600}(5449f_n+9256f_{n+1/2}-17275f_{n+1}+4955f_{n+2}$$
$$-2984f_{n+5/2}+599f_{n+3}),$$
$$h\delta_{n+1}=-y_n+y_{n+1}$$
$$+\frac{h^2}{12600}(166f_n+3704f_{n+1/2}+2545f_{n+1}-200f_{n+2}+104f_{n+5/2}-19f_{n+3}),$$
$$h\delta_{n+2}=-y_n+y_{n+1}$$
$$+\frac{h^2}{2520}(113f_n+248f_{n+1/2}+2182f_{n+1}+1633f_{n+2}-472f_{n+5/2}+76f_{n+3}),$$
$$h\delta_{n+5/2}=-y_n+y_{n+1}$$
$$+\frac{h^2}{201600}(8137f_n+24488f_{n+1/2}+166405f_{n+1}+188635f_{n+2}$$
$$+12248f_{n+5/2}+3287f_{n+3}),$$
$$h\delta_{n+3}=-y_n+y_{n+1}$$
$$+\frac{h^2}{12600}(614f_n+1016f_{n+1/2}+11225f_{n+1}+9880f_{n+2}$$
$$+6376f_{n+5/2}+2389f_{n+3}).$$

Case $k=4$. Choosing $\nu_1=\frac{1}{2}$, $\nu_2=\frac{7}{2}$, $r=2$, $s=7$, $k=4$, and evaluating (6) at $x=\{x_{n+4},x_{n+\frac{7}{2}},x_{n+3},x_{n+2},x_{n+\frac{1}{2}}\}$ we generate the following main method and five additional methods. The sixth method is provided by (9).

$$y_{n+4}-4y_{n+1}+3y_n=\frac{h^2}{17640}(1539f_n+340160f_{n+1/2}$$
$$+37548f_{n+1}+40054f_{n+2}+12852f_{n+3}+4096f_{n+7/2}+279f_{n+4}), \quad (19)$$

$$y_{n+7/2}-\frac{7}{2}y_{n+1}+\frac{5}{2}y_n=\frac{h^2}{774144}(57795f_n+340160f_{n+1/2}+1372980f_{n+1}$$
$$+1316546f_{n+2}+282324f_{n+3}+22208f_{n+7/2}-5133f_{n+4}), \quad (20)$$

$$y_{n+3}-3y_{n+1}+2y_n=\frac{h^2}{35280}(2169f_n+12160f_{n+1/2}$$
$$+49896f_{n+1}+40054f_{n+2}+504f_{n+3}+1408f_{n+7/2}-351f_{n+4}), \quad (21)$$

$$y_{n+2}-2y_{n+1}+y_n=\frac{h^2}{105840}(3009f_n+20224f_{n+1/2}$$
$$+66948f_{n+1}+19474f_{n+2}-7140f_{n+3}+4096f_{n+7/2}-771f_{n+4}), \quad (22)$$

$$y_{n+1/2} - \frac{1}{2}y_{n+1} - \frac{1}{2}y_n = \frac{h^2}{27095040}(-259143 f_n + 2909120 f_{n+1/2} - 182532 f_{n+1} - 63098 f_{n+2} + 57372 f_{n+3} - 38336 f_{n+7/2} + 7977 f_{n+4}), \quad (23)$$

$$h\delta_0 = y_1 - y_0 + \frac{h^2}{211680}(-30519 f_0 - 86272 f_{1/2} + 15876 f_1 - 7966 f_2 + 6300 f_3 - 4096 f_{7/2} + 837 f_4). \quad (24)$$

The derivatives are obtained from (8) by imposing that $\delta(x_{n+\mu}) = \delta_{n+\mu}$, $\mu = \{j, \nu_1, \nu_2\}$, $j = 0, \ldots, 4$, thus, we have

$$h\delta_{n+1/2} = -y_n + y_{n+1} + \frac{h^2}{564480}(14713 f_n + 28000 f_{n+1/2} - 50582 f_{n+1} + 12236 f_{n+2} - 8890 f_{n+3} + 5664 f_{n+7/2} - 1141 f_{n+4}),$$

$$h\delta_{n+1} = -y_n + y_{n+1} + \frac{h^2}{211680}(2835 f_n + 62080 f_{n+1/2} + 42840 f_{n+1} - 2702 f_{n+2} + 1512 f_{n+3} - 896 f_{n+7/2} + 171 f_{n+4}),$$

$$h\delta_{n+2} = -y_n + y_{n+1} + \frac{h^2}{70560}(2963 f_n + 7424 f_{n+1/2} + 61292 f_{n+1} + 40054 f_{n+2} - 10892 f_{n+3} + 6144 f_{n+7/2} - 1145 f_{n+4}),$$

$$h\delta_{n+3} = -y_n + y_{n+1} + \frac{h^2}{211680}(5283 f_n + 41600 f_{n+1/2} + 149688 f_{n+1} + 243026 f_{n+2} + 108360 f_{n+3} - 21376 f_{n+7/2} + 2619 f_{n+4}),$$

$$h\delta_{n+7/2} = -y_n + y_{n+1} + \frac{h^2}{564480}(15685 f_n + 102880 f_{n+1/2} + 412090 f_{n+1} + 628628 f_{n+2} + 453782 f_{n+3} + 80544 f_{n+7/2} - 169 f_{n+4}),$$

$$h\delta_{n+4} = -y_n + y_{n+1} + \frac{h^2}{211680}(4617 f_n + 44800 f_{n+1/2} + 144900 f_{n+1} + 248290 f_{n+2} + 135324 f_{n+3} + 126976 f_{n+7/2} + 35973 f_{n+4}).$$

4. Error Analysis and Zero-Stability

Following Fatunla [7] and Lambert [17] we define the local truncation error associated with (2) to be the linear difference operator

$$L[y(x);h] = \sum_{j=0}^{k}\{\alpha_j y(x+jh) - h^2\beta_j y''(x+jh)\} - h^2\sum_{j=1}^{2}\beta_{\nu_j} y''(x+\nu_j h). \quad (25)$$

Assuming that $y(x)$ is sufficiently differentiable, we can expand the terms in (25) as a Taylor series about the point x to obtain the expression

$$L[y(x);h] = C_0 y(x) + C_1 hy' + \ldots + C_q h^q y^q(x) + \ldots, \quad (26)$$

where the constant coefficients C_q , $q = 0, 1, \ldots$ are given as follows:

$$C_0 = \sum_{j=0}^{k}\alpha_j,$$

$$C_1 = \sum_{j=1}^{k} j\alpha_j,$$

$$\vdots$$

$$C_q = \frac{1}{q!}[\sum_{j=1}^{k} j^q\alpha_j - q(q-1)(\sum_{j=1}^{k} j^{q-2}\beta_j + \sum_{j=1}^{2}\nu_j^{q-2}\beta_{\nu_j})].$$

According to Henrici [13], we say that the method (3) has order p if

$$C_0 = C_1 = \ldots = C_p = C_{p+1} = 0, \quad C_{p+2} \neq 0,$$

therefore, C_{p+2} is the error constant and $C_{p+2}h^{p+2}y^{(p+2)}(x_n)$ the principal local truncation error at the point x_n. It is established from our calculations that the HLMMs have high order and relatively small error constants as displayed in Table 1.

In order to analyze the methods for zero-stability, we normalize the HLMMs and write them as a block method from which we obtain the first characteristic polynomial $\rho(R)$ given by

$$\rho(R) = \det(RA^{(0)} - A^{(1)}) = R^{k+1}(R-1), \quad (27)$$

where $A^{(0)}$ is an identity matrix of dimension $k+2$, $A^{(1)}$ is a matrix of dimension $k+2$.

Case $k = 2$. It is easily shown that (10)-(13) are normalized to give the first characteristic polynomial $\rho(R)$ given by

$$\rho(R) = \det(RA^{(0)} - A^{(1)}) = R^3(R-1),$$

where $A^{(0)}$ is an identity matrix of dimension 4 and $A^{(1)}$ is a matrix of dimension 4 given by

$$A^{(1)} = \begin{pmatrix} 0 & 0 & 1/2 & -1 \\ 0 & 0 & -1/2 & 1 \\ 0 & 0 & -1 & 2 \\ 0 & 0 & -1 & 2 \end{pmatrix}.$$

Case $k = 3$. The methods (14)-(18) are normalized to give the first characteristic polynomial $\rho(R)$ given by $\rho(R) = \det(RA^{(0)} - A^{(1)}) = R^4(R-1)$, where $A^{(0)}$ is an identity matrix of dimension 5 and $A^{(1)}$ is a matrix of dimension 5 given by

$$A^{(1)} = \begin{pmatrix} 0 & 0 & 0 & 1/2 & -3/2 \\ 0 & 0 & 0 & -1 & 3 \\ 0 & 0 & 0 & -3/2 & 9/2 \\ 0 & 0 & 0 & -2 & 6 \\ 0 & 0 & 0 & -1 & 3 \end{pmatrix}.$$

Case $k = 4$. It is also shown that (19)-(24) are normalized to give the first characteristic polynomial $\rho(R)$ given by

$$\rho(R) = \det(RA^{(0)} - A^{(1)}) = R^5(R-1)$$

from (23), where $A^{(0)}$ is an identity matrix of dimension 4 and $A^{(1)}$ is a matrix of dimension 4 given by

$$A^{(1)} = \begin{pmatrix} 0 & 0 & 0 & 0 & 1/2 & -2 \\ 0 & 0 & 0 & 0 & -1 & 4 \\ 0 & 0 & 0 & 0 & -2 & 8 \\ 0 & 0 & 0 & 0 & -5/2 & 10 \\ 0 & 0 & 0 & 0 & -3 & 12 \\ 0 & 0 & 0 & 0 & -1 & 4 \end{pmatrix}.$$

Following Fatunla [7], the block method obtained by combining $k+2$ HLMMs is zero-stable, since from (27), $\rho(R) = 0$ satisfies $|R_j| \le 1$, $j = 1, \ldots, k+2$, and for those roots with $|R_j| = 1$, the multiplicity does not exceed 2. The block method obtained by combining $k+2$ HLMMs is consistent since the HLMMs have order $p > 1$. According to Henrici [13], we can safely assert the convergence of the HLMMs.

Step number	Method	Order	Error Constant
2	(9)	6	-8.2672×10^{-6}
	(10)	5	1.62760×10^{-5}
	(11)	5	-1.6276×10^{-5}
	(12)	5	-1.9841×10^{-4}
	(13)	6	-6.4980×10^{-4}
3	(14)	6	-4.2037×10^{-4}
	(15)	6	-2.1660×10^{-4}
	(16)	6	1.28340×10^{-5}
	(17)	6	1.62860×10^{-4}
	(18)	7	3.14150×10^{-4}
	(19)	7	3.24570×10^{-4}
4	(20)	7	3.14150×10^{-4}
	(21)	7	1.57080×10^{-4}
	(22)	7	-1.0415×10^{-5}
	(23)	7	-1.2806×10^{-4}

Table 1: Orders and error constants for the HLMMs

5. Numerical Examples

The HLMMs are implemented as simultaneous numerical integrators for IVPs without requiring starting values and predictors. We proceed by explicitly obtaining initial conditions at $x_{n+k}, n = 0, k, \ldots, N-k$ using the computed values $Y(x_{n+k}) = y_{n+k}$ and $\delta(x_{n+k}) = \delta_{n+k}$ over sub-intervals $[x_0, x_k], \ldots, [x_{N-k}, x_N]$ which do not overlap. We note that for linear problems, we solve (1) directly from the start with Gaussian elimination using partial pivoting and for non-linear problems, we use a modified Newton-Raphson method. We give four examples to illustrate the efficiency of the methods. We find absolute errors of the approximate solution in π_N. The computations were carried out in *Mathematica 7.0*.

Example 5.1. We consider the Bessel's ODE given by

$$x^2y'' + xy' + (x^2 - 0.25)y = 0, \quad y(1) = \sqrt{\frac{2}{\pi}}\sin 1 \simeq 0.6713967071418031,$$

Steps	Vigo-Aguiar-Ramos ($p = 8$) Absolute Errors	Our Method ($p = 7$) Absolute Errors
67	7.1122×10^{-7}	3.6287×10^{-10}
82	9.2632×10^{-8}	7.8256×10^{-11}
97	8.7834×10^{-9}	2.1474×10^{-11}
112	1.2108×10^{-10}	7.0280×10^{-12}
125	2.7068×10^{-11}	2.9792×10^{-12}

Table 2: Absolute errors, $|y(x) - y|$ at $x = 8$, for Example 5.1 where $y(x) = J_{1/2}(x) = \sqrt{\frac{2}{\pi x}} \sin x$

h	Sallam-Anwar [23] ($p = 6$) Absolute Maximum Errors	Our Method ($p = 7$) Absolute Maximum Errors
$2\pi/300$	9.40000×10^{-9}	2.83774×10^{-8}
$2\pi/600$	1.40000×10^{-10}	1.12849×10^{-10}
$2\pi/1200$	3.80000×10^{-12}	9.20153×10^{-13}

Table 3: Absolute maximum errors, $|y(x) - y|$, for Example 5.2

$$y'(1) = (2\cos 1 - \sin 1)/\sqrt{2\pi} \simeq 0.0954005144474746.$$

$$\text{Exact:} \qquad y(x) = J_{1/2}(x) = \sqrt{\frac{2}{\pi x}} \sin x.$$

The theoretical solution at $x = 8$ is $y(8) = \sqrt{\frac{2}{8\pi}} \sin(8) \simeq 0.279092789108058969$. The errors in the solution were obtained at $x = 8$ using our method of order seven ($p = 7$) as shown in Table 2. Similar results were obtained for the same problem by Vigo-Aguiar and Ramos [26] using the variable-step Falker method of order eight ($p = 8$) in the predictor-corrector mode as displayed in Table 5. It is seen that although we used fixed step-sizes, our method performs better than the method of Vigo-Aguiar and Ramos [26].

Example 5.2. We consider the inhomogeneous ODE given by

$$y'' = -100y + 99 \sin x, \quad y(0) = 1, \quad y'(0) = 11,$$

$$\text{Exact :} \qquad y(x) = \cos(10x) + \sin(10x) + \sin x.$$

h	$k=2$	$k=3$	$k=4$
1.0	1.11852×10^{-4}	1.00468×10^{-4}	4.28437×10^{-5}
0.5	1.68791×10^{-6}	7.06084×10^{-7}	2.33590×10^{-7}
0.25	1.22041×10^{-8}	1.93436×10^{-9}	7.02538×10^{-10}
0.125	9.78576×10^{-11}	7.50178×10^{-12}	1.33620×10^{-12}
0.0625	1.06321×10^{-12}	3.31957×10^{-14}	5.07927×10^{-15}
0.03125	1.53766×10^{-14}	7.77156×10^{-16}	1.38778×10^{-15}

Table 4: Maximum absolute errors, $|y(x) - y|$ on $[0, 10]$, for Example 5.3, for $k = 2, 3, 4$

Example 5.3. We consider the mildly stiff ODE given by

$$y'' = -1001y' - 1000y, \quad y(0) = 1, \quad y'(0) = -1,$$

$$\text{Exact :} \qquad y(x) = e^{-x}.$$

The numerical results for this problem are displayed in Table 4. It is shown that the 4-step method performs better than the 3-step which performs better than the 2-step method. This is expected from the analysis of their orders given by $p = 7$, $p = 6$, and $p = 6$ respectively (see Table 1).

Example 5.4. We consider the given nonlinear ODE which was also solved by Hairer [10] and Ananthakrishnaiah [1] on $[0, 10\pi]$.

$$y'' + y + y^3 = B\cos(1.01x), \quad y(0) = 0.200426728069, \quad y'(0) = 0,$$

$$y(x) = 0.200179477536\cos(1.01x) + 0.246946143 \times 10^{-3}\cos(3(1.01)x) + 0.304016 \times 10^{-6}\cos(5(1.01)x) + 0.374 \times 10^{-9}\cos(7(1.01)x).$$

In this example, our method ($k = 2$) is compared with those given in Hairer [10] and Ananthakrishnaiah [1]. It is seen that the absolute errors produced by our method is smaller than those in [10] and [1]. Details of the numerical results are given in Table 5.

6. Conclusions

We have proposed three specific methods for $k = 2, 3, 4$ for solving second-order IVPs directly without first adapting the second order IVP to an equivalent first

x	Hairer [10]	Ananthakrishnaiah [1]	Our Method ($k = 2$)
π	5.03×10^{-2}	4.53×10^{-5}	8.18×10^{-6}
2π	4.79×10^{-2}	1.88×10^{-4}	4.98×10^{-7}
4π	4.25×10^{-2}	7.46×10^{-4}	1.98×10^{-6}
6π	3.64×10^{-2}	1.63×10^{-3}	4.41×10^{-6}
8π	2.97×10^{-2}	2.78×10^{-3}	7.72×10^{-6}
10π	2.26×10^{-2}	4.11×10^{-3}	1.18×10^{-5}

Table 5: Absolute errors, $|y(x) - y|$ on $[0, 10]$, $h = \pi/5$, $k = 2$, nonlinear second order differential equation for Example 5.4

order system. The methods are implemented without the need for starting values or predictors and hence complicated subroutines are avoided. We have demonstrated the efficiency of the methods on four problems. Details of the numerical results are given in Tables 2-5. Our future research will be focused on developing global error estimates for the methods.

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