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APPROXIMATION OF FOURTH-ORDER VARIATIONAL INEQUALITY

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Abstract: In this paper we shall study the uniqueness, existence, and approximation of the solution of elliptic variational inequality of order 4, and also determine an error estimate for the difference of the two solutions in appropriate norm.

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1. Introduction

In the recent years, variational inequalities have gained importance in analysis, both the theoretical and the practical points of view. Numerous problems in mechanics, physics and control theory lead to the study of system of partial differential equation, the solution of which leans heavily on the techniques of so-called variational inequalities. The theory of variational inequalities is rich and exciting; within it, one can find a wealth of powerful ideas which not reveal fundamental facts on the qualitative behaviour of solutions to important

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classes of non-linear boundary-value problems, but which also provide a natural framework for a host of relatively new numerical methods. Equally important, the theory also enables one to construct a rather elaborate approximation theory which brings to light useful information on the behaviour of numerical solution, error estimates, convergence criteria, etc. Also, at the heart of variational inequalities is their intrinsic inclusion of free boundary problems in which the solution of a differential equation has to satisfy certain conditions on the boundary of a variable domain. In many important cases, the boundary of the domain is not known in advance but has to be determined as part of the solution. Variational inequality theory introduced by Stampacchia [16]. There are two important types of variational inequalities, called the stationary inequalities (of elliptic type) and the inequalities of evolution (of parabolic type). There are numerous earlier works on stationary inequalities in mechanics. The classical approach (see Germain [8], Mandel [13], and the bibliographies of these works), consists in studying stationary elasticity in relation of minimization of quadratic functionals on vector spaces. The minimization of analogous functionals on convex sets that are not vector spaces made its appearance in perfect plasticity (where the stress tensor remains in a closed bounded convex sets) (see Koiter [10], Mandel [13], and bibliographies of these works), subsequently in unilateral elasticity in the problem of Signorini, solved in Fichera [7], then in Lions-Stampacchia [12]. Similarly, the phenomena of cavitation studied by Moreau [14] and the investigation of minimal surfaces with constraints (Nitsche [15]) also lead to problems in variational inequalities.

We introduce some general definitions that will be used in our coming discussion.

Definition 1.1. (Fixed Point) Let F be a mapping of a set A into itself. $F : A \longrightarrow A$. A point $x \in A$ is called a fixed point of F if $F(x) = x$.

Theorem 1.1. (Brouwer) *Let F be a continuous mapping of a closed ball in $\mathbb{R}^n$ into itself. Then F admits of at least one fixed point.*

Definition 1.2. (A Contraction Mapping) Let $X = (X, d)$ be a metric space. A mapping $T : X \longrightarrow X$ is a contraction mapping if

$$d(T(x), T(y)) \leq \alpha d(x, y)$$

or

$$\|T(x) - T(y)\| \leq \alpha \|x - y\|$$

for some α, $0 \leq \alpha < 1$. When we allow $\alpha = 1$ then mapping T is called nonexpansive.

Theorem 1.2. *Let X be a complete metric space and let $T : X \longrightarrow X$ be a contraction mapping. Then there exists a unique point of T.*

Corollary 1.1. *Let K be a closed convex set. Then the projection operator is nonexpansive, that is,*

$$\|P_K x \ - \ P_K x'\| \leq \|x - x'\| \ , \quad \forall x, x' \in K.$$

Let Ω be a bound open set in $\mathbb{R}^2$ with a smooth boundary Γ. In Ω we consider the operator $A : V = H_0^2(\omega) \longrightarrow V^{'}$ is linear continuous and symmetric, satisfying the coercivity condition, that is,

$$(A\psi, \psi) = a(\psi, \psi) \geq \alpha\|\psi\|^2, \quad \alpha > 0, \tag{1.1}$$

that is, $A = -\Delta \cdot \Delta$; let K be a nonempty, closed, and convex subset of a Hilbert space V, i.e.,

$$K = \{\psi | \psi \in V, \quad \psi \geq 0\}. \tag{1.2}$$

Let V a real Hilbert space, whose inner product and norm are denoted by $((.,.))$ and $\|.\|$. Let $V^{'}$ be the dual of V, with the pairing between V and $V^{'}$ denoted by $(.,.)$. Let us define the Sobolev space:

$$H^m(\Omega) = \{\phi \in L^2(\Omega); D^\alpha \phi \in L^2(\Omega), |\alpha| \leq m\}.$$

For $\phi, \psi \in L^2(\Omega)$,

$$(\phi, \psi)_{L^2(\Omega)} = \int_\Omega \phi(x)\psi(x) \, dx,$$

and for $\phi, \psi \in H^m(\Omega)$,

$$(\phi, \psi)_{H^m(\Omega)} = \sum_{|\alpha| \leq m} (D^\alpha \phi, D^\alpha \psi).$$

Let a_{ij} be a family of functions such that

$$a_{ij}(x) \in L^\infty(\Omega), \quad \sum a_{ij}(x) w_i w_j \geq \alpha_0 \sum w_i^2, \quad \alpha_0 > 0, \quad \forall \ w \in \Re^n,$$

and A is the operator associated with $a(.,.)$, so that

$$A\varphi = -\sum_{i,j} \partial^2/\partial x^2 (a_{ij}(x) \partial/\partial x \varphi),$$

$$(\phi, \psi) = \int_\Omega \phi\psi dx, \quad \forall \ \ \phi, \psi \in L^2(\Omega),$$

and

$$a(\phi, \psi) = \int_\Omega \Delta\phi \cdot \Delta\psi dx, \quad \forall \ \phi, \psi \in V,$$

with the associated norms denoted by $|\phi|^2 = (\phi, \phi)$ and $\|\psi\|^2 = a(\psi, \psi)$.

For given f in $L^p(\Re^n)$ we consider

$$\phi \in V, \quad a(\phi, \varphi - \phi) + \mu(\phi, \varphi - \phi) \geq (f, \varphi - \phi), \quad \forall\, \varphi \in V. \tag{1.3}$$

If we define the closed convex sub set K, then the above inequality is equivalent to

$$\phi \in K, \quad a(\phi, \varphi - \phi) + \mu(\phi, \varphi - \phi) \geq (f, \varphi - \phi), \quad \forall\, \varphi \in K. \tag{1.4}$$

The set of inequalities (1.4) (or (1.3)) is what is called a variational inequality. A general variational inequality associated with the operator A is (1.4), where K is an arbitrary closed (nonempty) convex set in V.

1.1. Uniqueness of the Problem

We shall prove the uniqueness of the solution ϕ of (1.4) in the form of the following theorem.

Theorem 1.3. (Uniqueness of the Problem) *Let A be a linear continuous and symmetric operator satisfying* (1.1)*, and let K be a closed convex set of V. There exists a unique solution of* (1.4)*. The map $f \longrightarrow \phi$ is continuous from V' into V.*

Proof. To prove the uniqueness of the problem, let ϕ and ψ be two possible solution of (1.4); if we take $\psi = \varphi$ in (1.4) (respectively, $\varphi = \phi$ in the equation analogous to (1.4) with respect to ψ), we have

$$\phi \in K \quad (A\phi, \psi - \phi) + \mu(\phi, \psi - \phi) \geq (f, \psi - \phi) \quad \forall\, \psi \in K, \tag{1.5}$$

and

$$\phi \in K \quad (A\psi, \phi - \psi) + \mu(\phi, \phi - \psi) \geq (f, \phi - \psi) \quad \forall\, \psi \in K. \tag{1.6}$$

Now, by adding (1.5) and (1.6), we obtain

$$(A(\psi - \phi), \psi - \phi) + \mu(\psi - \phi, \psi - \phi) \leq 0,$$

and by using (1.1), we have

$$\alpha\|\psi - \phi\|^2 + \mu\|\psi - \phi\|^2 \leq (A(\psi - \phi), \psi - \phi) + \mu(\psi - \phi, \psi - \phi) \leq 0.$$

This implies that

$$C\|\psi - \phi\|^2 \leq 0,$$

where C is constant, and since $C \geq 0$, therefore, we obtain $\varphi = \psi$.

Let us denote $A \in L(V; V')$ the map defined, and by setting $\|A\| = M$ and $\|\mu\| = N$. Let Υ be the canonical isomorphism from V' onto V defined, for $f \in V'$, by

$$(f, \varphi) = ((\Upsilon f, \varphi)) \quad \text{for } \varphi \in V, \tag{1.7}$$

then

$$\begin{aligned}\|\Upsilon\|_{L(V,V')} &= \sup \|\Upsilon f\| \qquad \forall\, f \in D(\Upsilon), \|f\| = 1\\ &= \sup(f, f) \qquad \forall\, f \in D(\Upsilon), \|f\| = 1\\ &= 1.\end{aligned}$$

Before proving the existence part of the problem, we prove the following lemma.

Lemma 1.1. *Let λ be a number such that $0 \le \lambda < \frac{2\gamma}{Q}$. Then there exists θ with $0 < \theta < 1$ such that*

$$|((\phi, \varphi)) - \lambda(A\phi, \varphi) - \mu((\phi, \varphi))| \le \theta\|\phi\|\|\varphi\|, \quad \forall\, \phi, \varphi \in V. \tag{1.8}$$

Proof. In fact

$$\begin{aligned}&|((\phi, \varphi)) - \lambda(A\phi, \varphi) - \mu((\phi, \varphi))|\\ &\quad = |((\phi - \lambda\Upsilon(A\phi + \mu\phi), \varphi))|\\ &\quad \le \|\phi - \lambda\Upsilon(A\phi + \mu\phi)\|\|\varphi\|.\end{aligned}$$

On the other hand,

$$\begin{aligned}\|\phi - \lambda\Upsilon(A\phi + \mu\phi)\|^2 &= \|\phi\|^2 + \lambda^2\|\Upsilon(A\phi + \mu\phi)\|^2 - 2\lambda(\phi, \Upsilon(A\phi + \mu\phi))\\ &= \|\phi\|^2 + \lambda^2[\|\Upsilon A\phi\|^2 + \|\Upsilon\mu\phi\|^2\\ &\quad + 2\mu(A\phi, \phi)] - 2\lambda\Upsilon(A\phi + \mu\phi)\phi\\ &\le \|\phi\|^2 + \lambda^2\|A\|^2\|\phi\|^2 + \lambda^2\|\mu\|^2\|\phi\|^2\\ &\quad + 2\lambda^2\mu\alpha\|\phi\|^2 - 2\lambda(\alpha\|\phi\|^2 + \mu\|\phi\|^2)\\ &= (1 + \lambda^2(M^2 + N^2) + 2\lambda^2\mu\alpha - 2\lambda\alpha - 2\lambda\mu)\|\phi\|^2\\ &= (1 + \lambda^2(M^2 + N^2 + 2\mu\alpha) - 2\lambda(\alpha + \mu))\|\phi\|^2\\ &= (1 + \lambda^2 Q - 2\lambda\gamma)\|\phi\|^2,\end{aligned}$$

where $Q = (M^2 + N^2 + 2\mu\alpha)$, and $\gamma = (\mu + \alpha)$. Since $\gamma > 0$, we can choose λ such that

$$(1 + \lambda^2 Q - 2\lambda\gamma)^{\frac{1}{2}} < 1,$$

and hence, (1.8) follows. □

1.2. Existence of the Problem

(i) Suppose that, $(A\phi, \varphi) = ((\phi, \varphi))$. Then we consider the following problem: Find $\phi \in K$ such that

$$((\phi, \varphi - \phi)) \ge (f, \varphi - \phi) = ((\Upsilon f, \varphi - \phi)), \quad \forall\, \varphi \in K, \tag{1.9}$$

that is,

$$((\phi - \Upsilon f, \varphi - \phi)) \geq 0.$$

Now we prove that the problem (1.9) is equivalence the following problem

$$\|\phi - \Upsilon f\| = \min_{\varphi \in K} \|\varphi - \Upsilon f\|. \tag{1.10}$$

If ϕ satisfies (1.9), then

$$\begin{aligned} 0 &\leq ((\phi - \Upsilon f, \varphi - \phi)) \\ &= ((\phi - \Upsilon f, (\varphi - \Upsilon f) + (\Upsilon f - \phi))) \\ &= ((\phi - \Upsilon f, \Upsilon f - \phi)) + ((\phi - \Upsilon f, \varphi - \Upsilon f)) \\ &\leq -\|\Upsilon f - \phi\|^2 + \|\phi - \Upsilon f\| \|\varphi - \Upsilon f\|. \end{aligned}$$

Therefore

$$\|\phi - \Upsilon f\|^2 \leq \|\phi - \Upsilon f\| \|\varphi - \Upsilon f\|.$$

So, finally,

$$\|\phi - \Upsilon f\| \leq \|\varphi - \Upsilon f\| \quad \text{for all } \varphi \in K.$$

Conversely, let $\|\phi - \Upsilon f\| = \min_{\varphi \in K} \|\varphi - \Upsilon f\|$ and since K is convex set $(1-\theta)\phi + \theta\varphi = \phi + \theta(\varphi - \phi)$ for any $\phi, \varphi \in K$, $0 \leq \theta \leq 1$ and hence, by (1.10) the function

$$\begin{aligned} \Phi(\theta) &= \|\Upsilon f - \phi - \theta(\varphi - \phi)\|^2 \\ &= \|\Upsilon f - \phi\|^2 - 2\theta(\Upsilon f - \phi, \varphi - \phi) + \theta^2 \|\varphi - \phi\|^2 \end{aligned}$$

attains its minimum at $\theta = 0$, so $\Phi'(0) \geq 0$ namely,

$$(\phi - \Upsilon f, \varphi - \phi) \geq 0.$$

Finally, we prove that there exists one and only one element of K which minimizes

$$\|\varphi - \Upsilon f\|, \quad \forall \ \varphi \in K.$$

Let φ_n be a minimizing sequence, namely

$$\lim_{n \to \infty} \|\varphi_n - \Upsilon f\| = l = \inf_{\varphi \in K} \|\varphi - \Upsilon f\|. \tag{1.11}$$

Now, we have

$$\begin{aligned} \|\varphi_n - \varphi_m\|^2 &= 2\|\Upsilon f - \varphi_n\|^2 + 2\|\Upsilon f - \varphi_m\|^2 \\ &\quad - 4\|\Upsilon - \frac{1}{2}(\varphi_n + \varphi_m)\|^2. \end{aligned}$$

Now, K is convex, So $\frac{1}{2}(\varphi_n + \varphi_m) \in K$ and

$$l^2 \leq \|\Upsilon f - \frac{1}{2}(\varphi_n + \varphi_m)\|^2.$$

Therefore

$$\|\varphi_n - \varphi_m\|^2 \leq 2\|\Upsilon f - \varphi_n\|^2 + 2\|\Upsilon f - \varphi_m\|^2 - 4l^2$$

and we conclude from (1.11) that

$$\lim_{n,m\to\infty} \|\varphi_n - \varphi_m\| = 0,$$

so φ_n is a Cauchy sequence and since K is a complete, there is an element $\phi \in K$ such that

$$\lim_{n\to\infty} \varphi_n = \phi.$$

Let $\phi_1, \phi_2 \in K$ be any element. (1.10) may be inserted of φ_n, φ_m. This yields.

The solution of variational inequality (1.9) is thus given by

$$\phi = P_K \Upsilon(f),$$

where P_K is the projection of V in K.

(ii) Fix λ as in above lemma. For $\phi \in V$, define $\Phi(\phi) \in V'$ and setting

$$(\Phi(\phi), \varphi) = ((\phi, \varphi)) - \lambda[(A\phi, \varphi) + \mu(\phi, \varphi)] + \lambda(f, \varphi), \quad \forall\, \varphi \in V.$$

Then, for $\phi_1, \phi_2 \in V$ and by using the above lemma, we have

$$\begin{aligned} |(\Phi(\phi_1) - \Phi(\phi_2), \varphi)| \;&=\; |((\phi_1 - \phi_2, \varphi)) \\ &\quad - \lambda[(A(\phi_1 - \phi_2), \varphi) + \mu(\phi_1 - \phi_2, \varphi)]| \\ &\leq\; \theta\|\phi_1 - \phi_2\|\|\varphi\|. \end{aligned} \tag{1.12}$$

Therefore

$$\begin{aligned} \|\Phi(\phi_1) - \Phi(\phi_2)\|^2 &= (\Phi(\phi_1) - \Phi(\phi_2), \Phi(\phi_1) - \Phi(\phi_2)) \\ &\leq \theta\|\phi_1 - \phi_2\|\|\Phi(\phi_1) - \Phi(\phi_2)\|, \end{aligned}$$

or

$$\|\Phi(\phi_1) - \Phi(\phi_2)\| \leq \theta\|\phi_1 - \phi_2\| \tag{1.13}$$

Then by using (i), there exists a unique $\omega \in K$ such that

$$((\omega, \varphi - \omega)) \geq ((\Phi(\phi), \varphi - \omega)) \quad \forall\, \varphi \in K,$$

and ω is given by

$$\omega = P_K \Upsilon \Phi(\phi) = T\phi. \tag{1.14}$$

The problem reduces to finding the fixed points of T. This defines a map $\phi \longrightarrow T\phi$ from V into K. Moreover,

$$\begin{aligned} \|T\phi_1 - T\phi_2\| \;&=\; \|P_K \Upsilon \Phi(\phi_1) - P_K \Upsilon \Phi(\phi_2)\| \\ &\leq\; \|\Upsilon \Phi(\phi_1) - \Upsilon \Phi(\phi_2)\| \\ &\leq\; \|\Phi(\phi_1) - \Phi(\phi_2)\| leq\, \theta\|\phi_1 - \phi_2\|. \end{aligned}$$

Since $\theta < 1$, $T\phi$ is a contraction and thus there exists one and only one solution ϕ such that

$$T\phi = \phi. \tag{1.15}$$

Then ϕ belongs to K and satisfies

$$((\phi, \varphi - \phi)) \geq (\Phi(\phi), \varphi - \phi) = ((\phi, \varphi - \phi)) - \lambda[(A\phi, \varphi) + \mu(\phi, \varphi - \phi)] - (f, \varphi - \phi), \quad \forall\, \varphi \in K. \tag{1.16}$$

It follows ($\lambda > 0$) that ϕ is a solution of (1.4).

1.3. The Approximation of the Problem

Before we discuss the approximation of (1.4), we prove the following result.

Lemma 1.2. *Let $a(\varphi_\epsilon, \phi_\epsilon)$ is a bilinear, continuous form on $H_0^2(\Omega)\times H_0^2(\Omega)$ such that*

$$a(\varphi_\epsilon, \varphi_\epsilon) \geq 0, \quad \forall\, \varphi_\epsilon \in H_0^2(\Omega).$$

Then the function $\phi_\epsilon \longrightarrow a(\phi_\epsilon, \phi_\epsilon)$ is lower-semicontinuous with respect to the weak topology.

Proof. From the bilinearity, we have for all $\varphi_\epsilon \in H_0^2(\Omega), \phi_\epsilon \in H^2(\Omega)$

$$a(\phi_\epsilon, \phi_\epsilon) = a(\varphi_\epsilon, \varphi_\epsilon) + [a(\varphi_\epsilon, \phi_\epsilon - \varphi_\epsilon) + a(\phi_\epsilon - \varphi_\epsilon, \varphi_\epsilon) + a(\varphi_\epsilon - \phi_\epsilon, \varphi_\epsilon - \phi_\epsilon)].$$

Now by using the condition of elipticity, that is,

$$a(\varphi_\epsilon, \varphi_\epsilon) \geq 0; \quad \forall\, \varphi_\epsilon \in H_0^2(\Omega),$$

this implies that

$$a(\phi_\epsilon, \phi_\epsilon) \geq a(\varphi_\epsilon, \varphi_\epsilon) + [a(\varphi_\epsilon, \phi_\epsilon - \varphi_\epsilon) + a(\phi_\epsilon - \varphi_\epsilon, \varphi_\epsilon)]. \tag{1.17}$$

Now, let $\phi_\epsilon \longrightarrow \varphi$ in $H_0^2(\Omega)$ weakly. Since (1.17) is true for all φ_ϵ in $H_0^2(\Omega)$ and $\varphi \in H_0^2(\Omega)$ we replace φ_ϵ by φ in (1.17), we have

$$a(\phi_\epsilon, \phi_\epsilon) \geq a(\varphi, \varphi) + [a(\varphi, \phi_\epsilon - \varphi) + a(\phi_\epsilon - \varphi, \varphi)].$$

So $\phi_\epsilon \longrightarrow \varphi$ in $H_0^2(\Omega)$ weakly then

$$a(\varphi, \phi_\epsilon - \varphi) \longrightarrow 0 \quad \text{and} \quad a(\phi_\epsilon - \varphi, \varphi) \longrightarrow 0.$$

Finally

$$\liminf_{\phi_\epsilon \to \varphi} a(\phi_\epsilon, \phi_\epsilon) \geq a(\varphi, \varphi).$$

Hence map $\phi_\epsilon \longrightarrow a(\phi_\epsilon, \phi_\epsilon)$ is lower-semicontinuous. □

Let us suppose there is given a parameter h converging to zero and a family $\{V_h\}_{0<h\leq 1}$ of closed subspaces of V (in practice, the V_h are finite dimensional

and the parameter h varies over a sequence). We are also given a family $\{K_h\}_h$ of closed convex nonempty subset of V. and approximation of (1.4) generally involves seeking a function ϕ_h in a set K_h which a subset of a finite dimensional subspace V_h of V, h being an appropriate index. The approximation of (1.4) will then involve seeking $\phi_h \in K_h$ such that

$$(A\phi_h, \varphi_h - \phi_h) + \mu(\phi_h, \varphi_h - \phi_h) \geq (f, \varphi_h - \phi_h), \quad \forall\, \varphi_h \in K_h. \tag{1.18}$$

We are interested in determining sequences of solutions $\{\phi_h\}$ to (1.18), $\phi_h \in K_h \subset V_h$, and in investigating the behavior of the approximations as $h \longrightarrow 0$. In particular, we want to determine conditions under which $\{\phi_h\}$ converges in some sense to a solution to (1.4) and estimating the error $\phi - \phi_h$. For a consistent approximation of the set K_h, we use the following conditions:

Let K be a subset of a space V. A sequence of subsets $\{K_h\}_h$ in V_h is said to converge to a set K if:

(a) for every weakly convergent sequence $\{\phi_h\}_h$, $\phi_h \in K_h$, for all h, its weak limit ϕ belongs to K.

(b) There exists $\zeta \subset V$, $\bar{\zeta} = K$ and $\lambda : \zeta \longrightarrow K_h$ such that

$$\lim_{h \to 0} \lambda_h \phi = \phi \text{ strongly in } V, \quad \forall\, \phi \in \zeta.$$

One can easily prove that the problem (1.18) has a unique solution by using Theorem 1.3, taking V to be V_h and K to be K_h.

Theorem 1.4. *Let K be a closed convex nonempty subset of V, and $\{K_h\}_h$ be a family of closed convex nonempty subset of V. Then, we have*

$$\lim_{h \to 0} \|\phi_h - \phi\| = 0, \tag{1.19}$$

where ϕ_h is the solution of (1.18) *and ϕ the solution of* (1.4).

Proof. To prove this kind of convergence results, first, we shall obtain a priori estimates for $\{\phi_h\}_h$, then the weak convergence of $\{\phi_h\}_h$, and finally, with the help of the weak convergence, we shall prove the strong convergence.

For obtaining the estimates for ϕ_h, we shall show that there exists two constants, called β_1 and β_2 independent of h, such that

$$\|\phi_h\|^2 \leq \beta_1 \|\phi_h\| + \beta_2, \quad \forall\, h. \tag{1.20}$$

Since we know that ϕ_h is the solution of (1.18), we have

$$(A\phi_h, \varphi_h - \phi_h) + \mu(\phi_h, \varphi_h - \phi_h) \geq (f, \varphi_h - \phi_h),$$

or

$$(A\phi_h, \phi_h) + \mu(\phi_h, \phi_h) \leq (A\phi_h, \varphi_h) + \mu(\phi_h, \varphi_h) - (f, \varphi_h - \phi_h),$$

by using (1.1), we have

$$\alpha\|\phi_h\|^2 + \|\mu\|\|\phi_h\|^2 \le \|A\|\|\phi_h\|\|\varphi_h\| + \|\mu\|\|\varphi_h\|\|\phi_h\| + \|f\|(\|\phi_h\| + \|\varphi_h\|), \quad \forall\ \varphi_h \in K_h. \tag{1.21}$$

Let $\varphi_0 \in \zeta$ and $\varphi_h = \lambda_h\varphi_0 \in K_h$, then by the condition on K_h (that is, (b)), we have, $\lambda_h\varphi_0 \longrightarrow \varphi_0$ strongly in V and hence $\|\varphi_h\|$ is uniformly bounded by a constant m. Hence (1.21) can be written as

$$\|\phi_h\|^2 \le \frac{1}{C}\{(m\|A\| + m\|\mu\| + \|f\|)\|\phi_h\| + m\|f\|\},$$

$$\|\phi_h\|^2 \le \beta_1\|\phi_h\| + \beta_2, \quad \forall\ h,$$

where $\beta_1 = \frac{1}{C}(m\|A\| + m\|\mu\| + \|f\|)$, $\beta_2 = \frac{m}{C}\|f\|$ and $C > 0$ is constant. Then (1.20) implies that

$$\|\phi_h\| \le C, \quad \forall\ \ h,$$

where here, and what follows, C denotes the various constant.

Now we shall discuss the weak convergence of $\{\phi_h\}_h$. Since the relation (1.20) implies that ϕ_h is uniformly bounded. Hence there exists a subsequence, say ϕ_{h_i}, such that ϕ_{h_i} converge to Φ^* weakly in V. Then, by the condition of $\{K_h\}_h$ (that is, (a)), we have $\Phi^* \in K$. Now we shall prove that Φ^* is a solution of (1.4). Since we know that

$$(A\phi_{h_i}, \varphi_{h_i} - \phi_{h_i}) + \mu(\phi_{h_i}, \varphi_{h_i} - \phi_{h_i}) \ge (f, \varphi_{h_i} - \phi_{h_i}),$$

or

$$(A\phi_{h_i}, \phi_{h_i}) \le (A\phi_{h_i}, \varphi_{h_i}) + \mu(\phi_{h_i}, \varphi_{h_i} - \phi_{h_i}) - (f, \varphi_{h_i} - \phi_{h_i}), \tag{1.22}$$

by taking $\varphi_{h_i} = \lambda_{h_i}\varphi$ and $\varphi \in \zeta$, then (1.22) becomes

$$(A\phi_{h_i}, \phi_{h_i}) \le (A\phi_{h_i}, \lambda_{h_i}\varphi) + \mu(\phi_{h_i}, \lambda_{h_i}\varphi - \phi_{h_i}) - (f, \lambda_{h_i}\varphi - \phi_{h_i}). \tag{1.23}$$

Since $\lambda_{h_i}\varphi$ converges strongly to φ and ϕ_{h_i} converges to Φ^* weakly as h_i tends to zero, so by taking the limit as h_i tends to zero in (1.22), we have

$$\lim_{h_i\to 0}(A\phi_{h_i}, \phi_{h_i}) \le (A\Phi^*, \varphi) + \mu(\Phi^*, \varphi - \Phi^*) - (f, \varphi - \Phi^*),\ \forall\ \varphi \in \zeta. \tag{1.24}$$

Since $\phi_{h_i} \longrightarrow \Phi^*$ weakly, so by using Lemma 1.2 we have

$$\liminf_{h_i\to 0}(A\phi_{h_i}, \phi_{h_i}) \ge (A\Phi^*, \Phi^*), \tag{1.25}$$

so, from (1.24) and (1.25), we obtain, $\quad \forall\ \varphi \in \zeta$

$$(A\Phi^*, \Phi^*) \le \liminf_{h_i\to 0}(A\phi_{h_i}, \phi_{h_i}) \le (A\Phi^*, \varphi) + \mu(\Phi^*, \varphi - \Phi^*) - (f, \varphi - \Phi^*).$$

Therefore, we have

$$(A\Phi^*, \varphi - \Phi^*) + \mu(\Phi^*, \varphi - \Phi^*) - (f, \varphi - \Phi^*) \ge 0, \quad \forall\ \varphi \in \zeta,\ \Phi^* \in K. \tag{1.26}$$

Since ζ is dense in K and A is continuous, so from (1.26), we get

$$(A\Phi^*, \varphi - \Phi^*) + \mu(\Phi^*, \varphi - \Phi^*) - (f, \varphi - \Phi^*) \geq 0, \quad \forall\, \varphi \in K,\ \Phi^* \in K. \quad (1.27)$$

Hence Φ^* is a solution of (1.4). Since by Theorem 1.3, the solution of (1.4) is unique and so $\Phi^* = \phi$ is the unique solution. Hence ϕ is the only limit point of $\{\phi_h\}_h$ in the weak topology of V.

Hence the whole $\{\phi_h\}_h$ converges to ϕ weakly.

Now we shall discuss the strong convergence. By using (1.1), we have

$$\begin{aligned} 0 \leq \alpha\|\phi_h - \phi\|^2 &\leq (A(\phi_h - \phi), \phi_h - \phi) \\ &= (A\phi_h, \phi_h) - (A\phi_h, \phi) - (A\phi, \phi_h) + (A\phi, \phi), \end{aligned} \quad (1.28)$$

where ϕ_h is solution of (1.18) and ϕ is solution of (1.4) and $\lambda_h\varphi \in K_h$ for any $\varphi \in \zeta$, so we get (1.28) of the form

$$(A\phi_h, \phi_h) \leq (A\phi_h, \lambda_h\varphi) + \mu(\phi_h, \lambda_h\varphi - \phi_h) - (f, \lambda_h\varphi - \phi_h), \quad \forall\, \varphi \in \zeta. \quad (1.29)$$

Since we know that $\lim_{h\to 0}\phi_h = \phi$ weakly in V and $\lim_{h\to 0}\lambda_h\varphi = \varphi$ strongly in V (by condition (b)), we obtain from (1.28) and (1.29):

$$\begin{aligned} 0 \leq \alpha \liminf_{h\to 0} \|\phi_h - \phi\|^2 &\leq \alpha \limsup_{h\to 0} \|\phi_h - \phi\|^2 \leq (A(\phi_h - \phi), \phi_h - \phi)) \\ &\leq (A\phi_h, \lambda_h\varphi) + \mu(\phi_h, \lambda_h\varphi - \phi_h) - (f, \lambda_h\varphi - \phi_h) \\ &\quad - (A\phi_h, \phi) - (A\phi, \phi_h) + (A\phi, \phi). \end{aligned}$$

After taking the limit, we have

$$\begin{aligned} 0 \leq \alpha \liminf_{h\to 0} \|\phi_h - \phi\|^2 &\leq \alpha \limsup_{h\to 0} \|\phi_h - \phi\|^2 \\ &\leq (A\phi, \varphi - \phi) + \mu(\phi, \varphi - \phi) - (f, \varphi - \phi), \quad \forall\, \varphi \in \zeta. \end{aligned} \quad (1.30)$$

By density and continuity, (1.30) also holds for all $\varphi \in K$; then taking $\varphi = \phi$ in (1.30), we get

$$\lim_{h\to 0} \|\phi_h - \phi\| = 0. \quad (1.31)$$

We shall now determine the error estimate of the difference of the solution of related variational inequality for the appropriate norm in the form of the following theorem.

Theorem 1.5. *Let us consider an operator $A : V = H_0^2(\Omega) \longrightarrow V'$, which is linear continuous and symmetric, and satisfies coercivity condition* (1.1), *and let $\phi \in K$ and $\phi_h \in K_h$ be respective solutions of the variational inequalities:*

$$\phi \in K \quad (A\phi, \varphi - \phi) + \mu(\phi, \varphi - \phi) \geq (f, \varphi - \phi), \quad \forall\, \varphi \in K, \quad (1.32)$$

and $\forall\, \varphi_h \in K_h$,

$$\phi_h \in K_h \quad (A\phi_h, \varphi_h - \phi_h) + \mu(\phi_h, \varphi_h - \phi_h) \geq (f, \varphi_h - \phi_h), \quad (1.33)$$

where K and K_h are closed convex subsets of V. If $(f-(A\phi+\mu\phi))\in L^2(\Omega)$, then

$$\|\phi-\phi_h\| \leq \{\frac{2}{\alpha}\|f-A\phi-\mu\phi\| \inf_{\varphi\in K}\|\phi_h-\varphi\| + \inf_{\varphi_h\in K_h}[\frac{2}{\alpha}\|f-A\phi-\mu\phi\|\|\phi-\varphi_h\| + \frac{C^2}{\alpha^2}\|\phi-\varphi_h\|]\}^{\frac{1}{2}}.$$

Proof. Adding the inequalities (1.32) and (1.33), we obtain

$$(A\phi,\phi-\varphi)+(A\phi_h,\phi_h-\varphi_h)+\mu(\phi,\phi-\varphi)+\mu(\phi_h,\phi_h-\varphi_h) \leq (f,\phi-\varphi)+(f,\phi_h-\varphi_h).$$

After, regrouping, we obtain

$$\begin{aligned}(A(\phi-\phi_h),\phi-\phi_h)+\mu(\phi-\phi_h,\phi-\phi_h) &\leq (f,\phi-\varphi_h)+(f,\phi_h-\varphi)\\ &+(A\phi,\varphi-\phi_h)+(A\phi_h,\varphi_h-\phi)\\ &+\mu(\phi,\varphi-\phi_h)+\mu(\phi_h,\varphi_h-\phi)\\ &=(f,\phi-\varphi_h)+(f,\phi_h-\varphi)+(A\phi,\phi-\varphi_h)+(A\phi,\varphi-\phi_h)\\ &+(A\phi,\phi-\varphi_h)-(A\phi,\phi-\varphi_h)+\mu(\phi,\phi-\varphi_h)-\mu(\phi,\phi-\varphi_h)\\ &+\mu(\phi,\varphi-\phi_h)+\mu(\phi_h,\varphi_h-\phi).\end{aligned}$$

Since $\mu(\phi-\phi_h,\phi-\phi_h)\geq 0$, so, by using the hypotheses of the theorem, we have

$$\begin{aligned}\alpha\|\phi-\phi_h\|^2 &\leq \|f-A\phi-\mu\phi\|\|\phi-\varphi_h\|+\|f-A\phi-\mu\phi\|\|\phi_h-\varphi\|\\ &+\|A\|\|\phi_h-\phi\|\|\phi-\varphi_h\|+\|\mu\|\|\phi_h-\phi\|\|\phi-\varphi_h\|,\end{aligned}$$

or

$$\begin{aligned}\alpha\|\phi-\phi_h\|^2 &\leq \|f-A\phi-\mu\phi\|\|\phi-\varphi_h\|+\|f-A\phi-\mu\phi\|\|\phi_h-\varphi\|\\ &+C\|\phi_h-\phi\|\|\phi-\varphi_h\|,\end{aligned}$$

where C denotes various constant. Since we know (Young's inequality) that $ab\leq a^2/2\epsilon+\epsilon b^2/2$ for all $\epsilon>0$, so we make the following estimate

$$C\|\phi-\varphi_h\|\|\phi_h-\phi\| \leq \frac{C^2}{2\alpha}\|\phi-\varphi_h\|^2+\frac{\alpha}{2}\|\phi_h-\phi\|^2.$$

So for all $\varphi\in K$ and $\varphi_h\in K_h$, we obtain

$$\begin{aligned}\alpha\|\phi-\phi_h\|^2 &\leq \|f-A\phi-\mu\phi\|\|\phi-\varphi_h\|+\|f-A\phi-\mu\phi\|\|\phi_h-\varphi\|\\ &+\frac{C^2}{2\alpha}\|\phi-\varphi_h\|^2+\frac{\alpha}{2}\|\phi_h-\phi\|^2,\end{aligned}$$

or

$$\|\phi-\phi_h\| \leq \{\frac{2}{\alpha}(\|f-A\phi-\mu\phi\|\|\phi-\varphi_h\|+\|f-A\phi-\mu\phi\|\|\phi_h-\varphi\|)$$

$$+\frac{C^2}{\alpha^2}\|\phi-\varphi_h\|^2\}^{\frac{1}{2}}.$$

This implies that

$$\|\phi-\phi_h\| \leq \{\frac{2}{\alpha}\|f-A\phi-\mu\phi\| \inf_{\varphi\in K}\|\phi_h-\varphi\| + \inf_{\varphi_h\in K_h}[\frac{2}{\alpha}\|f-A\phi-\mu\phi\|\|\phi-\varphi_h\| + \frac{C^2}{\alpha^2}\|\phi-\varphi_h\|]\}^{\frac{1}{2}}$$

which proves the theorem. □

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