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# ON ($\omega$)TOPOLOGICAL PARACOMPACTNESS

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**Abstract:** Some characterizations of ($\omega$)topological paracompactness involving closure preserving property are obtained.

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## 1. Introduction

The notion of ($\omega$)topological spaces is introduced in Bose and Tiwari [1]. Some important results on ($\omega$)topology including some characterizations of ($\omega$)paracompactness are obtained in [1] and [2]. In this paper, a stronger notion of paracompactness, namely ($\omega^s$)paracompactness, is introduced. We then obtain characterizations of ($\omega^s$)paracompactness which involve closure preserving property, analogous to the characterizations of paracompactness obtained in Michael [5].

## 2. Preliminaries

If $(X, \mathcal{J})$ is a topological space, the closure of a set $A \subset X$ with respect to $\mathcal{J}$ is denoted by $(\mathcal{J})clA$. By '$G$ is $(\mathcal{J})$open', we mean $G$ is open with respect to

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$\mathcal{J}$ and so on. The set of natural numbers is denoted by $N$ and the members of $N$ are denoted by $i, j, k, l, m, n$ etc.

In the sequel, we require the following definitions of Bose and Tiwari [1].

**Definition 1.** If $\{\mathcal{J}_n\}$ is a sequence of topologies on a set $X$ with $\mathcal{J}_n \subset \mathcal{J}_{n+1}$ for all $n \in N$, then the pair $(X, \{\mathcal{J}_n\})$ is called an $(\omega)$*topological space*.

**Definition 2.** Let $(X, \{\mathcal{J}_n\})$ be an $(\omega)$topological space. A set $G \in \mathcal{J}_n$ for some $n$ is called an $(\omega)$*open* set. A set $F$ is said to be $(\omega)$*closed* if $X - F$ is $(\omega)$open.

We also require the following definitions and results.

**Definition 3.** (see Michael [5]) A collection $\mathcal{A}$ of subsets of a topological space $(X, \mathcal{J})$ is said to be $(\mathcal{J})$*closure preserving* if for every subcollection $\mathcal{B}$ of $\mathcal{A}$, $\cup\{(\mathcal{J})clB|B \in \mathcal{B}\} = (\mathcal{J})cl(\cup\{B|B \in \mathcal{B}\})$.

**Definition 4.** (see Burke et al [3]) A collection $\mathcal{A}$ of subsets of a topological space $(X, \mathcal{J})$ is *hereditarily* $(\mathcal{J})$*closure preserving* if, whenever a subset $B_A$ of $A$ is chosen for each $A \in \mathcal{A}$, the resulting collection $\mathcal{B} = \{B_A \mid A \in \mathcal{A}\}$ is $(\mathcal{J})$closure preserving.

**Definition 5.** (see Michael [5]) A collection $\mathcal{A}$ of subsets of a topological space $(X, \mathcal{J})$ is said to be $(\mathcal{J})$*discrete* (resp. $(\mathcal{J})$*locally finite*) if every $x \in X$ has a $(\mathcal{J})$open neighborhood intersecting at most one (resp. finitely many) $A \in \mathcal{A}$.

**Definition 6.** (see Burke et al [3], Michael [5]) A collection $\mathcal{A}$ of subsets of $(X, \mathcal{J})$ is said to be $\sigma-$*hereditarily* $(\mathcal{J})$*closure preserving* (resp. $\sigma-(\mathcal{J})$*discrete*, $\sigma-(\mathcal{J})$*locally finite*) if $\mathcal{A}$ can be written as a countable union of hereditarily $(\mathcal{J})$closure preserving (resp. $(\mathcal{J})$discrete, $(\mathcal{J})$locally finite) subcollections.

**Definition 7.** (see Bose and Tiwari [2]) An $(\omega)$topological space $(X, \{\mathcal{J}_n\})$ is said to be $(\omega)$*paracompact* if every $(\omega)$open cover of $X$ has, for some $n$, a $(\mathcal{J}_n)$locally finite $(\mathcal{J}_n)$open refinement.

**Lemma 8.** (see Michael [5]) *If $\{C_\alpha \mid \alpha \in A\}$ is a $(\mathcal{J})$locally finite collection of subsets of a topological space $(X, \mathcal{J})$ and if, for every $\alpha$, $\mathcal{B}_\alpha$ is a $(\mathcal{J})$closure preserving collection of subsets of $C_\alpha$, then the collection $\cup\{\mathcal{B}_\alpha \mid \alpha \in A\}$ is also $(\mathcal{J})$closure preserving.*

**Lemma 9.** (see Dowker [4]) *If $\{V_\lambda \mid \lambda \in \Lambda\}$ be a disjoint collection of $(\mathcal{J})$open subsets of a $(\mathcal{J})$normal space $(X, \mathcal{J})$, if $D_\lambda \subset V_\lambda$ for each $\lambda$, and if $\cup_\lambda D_\lambda$ is $(\mathcal{J})$closed, then there exist a $(\mathcal{J})$discrete family $\{W_\lambda \mid \lambda \in \Lambda\}$ of $(\mathcal{J})$open subsets of $X$ such that $D_\lambda \subset W_\lambda \subset V_\lambda$, for all $\lambda$.*

## 3. ($\omega^s$)paracompactness

We introduce the following definitions. In the sequel, the ($\omega$)topological space $(X, \{\mathcal{J}_n\})$ is denoted, simply, by $X$.

**Definition 10.** An ($\omega$)topological space $X$ is said to be ($\omega^s$)*paracompact* if for every ($\omega$)open cover $\mathcal{U}$ of $X$, there exists an $m$ such that for any $n$, each ($\mathcal{J}_n$)open refinement of $\mathcal{U}$ has a ($\mathcal{J}_m$)locally finite ($\mathcal{J}_m$)open refinement.

Obviously ($\omega^s$)paracompactness is stronger than ($\omega$)paracompactness.

**Example 11.** Taking $X = [0, 1]$, let us define an ($\omega$)topological space $(X, \{\mathcal{J}_n\})$ as follows:

Let $I_n = [0, 1 - \frac{1}{n+1})$ and $\mathcal{T}_n = \mathcal{U}/I_n$, where $\mathcal{U}$ denotes the usual topology on $R$. Now for any $n \in N$, define $\mathcal{J}_n$ to be the topology generated by the base $\mathcal{T}_n \cup \mathcal{S}$, where $\mathcal{S}$ is the collection of all the left open subintervals of $X$ having 1 as the right end point. Then the space $(X, \{\mathcal{J}_n\})$ is ($\omega$)paracompact but not ($\omega^s$)paracompact.

As in Bose and Tiwari [2], we can prove the following theorem.

**Theorem 12.** *If the* ($\omega$)*topological space* $X$ *is* ($\mathcal{J}_n$)*regular for all* $n$*, then the following statements are equivalent:*

($i$) $X$ *is* ($\omega^s$)*paracompact.*

($ii$) *For every* ($\omega$)*open cover* $\mathcal{U}$ *of* $X$*, there exists an* $m$ *such that for any* $n$*, each* ($\mathcal{J}_n$)*open refinement of* $\mathcal{U}$ *has a* $\sigma-$($\mathcal{J}_m$)*locally finite* ($\mathcal{J}_m$)*open refinement.*

($iii$) *For every* ($\omega$)*open cover* $\mathcal{U}$ *of* $X$*, there exists an* $m$ *such that for any* $n$*, each* ($\mathcal{J}_n$)*open refinement of* $\mathcal{U}$ *has a* ($\mathcal{J}_m$)*open refinement* $\mathcal{U}_0$ *and* $\mathcal{U}_0$ *has a* ($\mathcal{J}_m$)*locally finite refinement.*

($iv$) *For every* ($\omega$)*open cover* $\mathcal{U}$ *of* $X$*, there exists an* $m$ *such that for any* $n$*, each* ($\mathcal{J}_n$)*open refinement of* $\mathcal{U}$ *has a* ($\mathcal{J}_m$)*open refinement* $\mathcal{U}_0$ *and* $\mathcal{U}_0$ *has a* ($\mathcal{J}_m$)*locally finite* ($\mathcal{J}_m$)*closed refinement.*

We now prove the following theorem.

**Theorem 13.** *Suppose the* ($\omega$)*topological space* $X$ *is* ($\mathcal{J}_n$)*regular for all* $n$*. Then the following statements are equivalent:*

($i$) $X$ *is* ($\omega^s$)*paracompact.*

($ii$) *For every* ($\omega$)*open cover* $\mathcal{U}$ *of* $X$*, there exists an* $m$ *such that for any* $n$*, each* ($\mathcal{J}_n$)*open refinement of* $\mathcal{U}$ *has a hereditarily* ($\mathcal{J}_m$)*closure preserving* ($\mathcal{J}_m$)*open refinement.*

*(iii) For every $(\omega)$open cover $\mathcal{U}$ of $X$, there exists an $m$ such that for any $n$, each $(\mathcal{J}_n)$open refinement of $\mathcal{U}$ has a $\sigma-$hereditarily $(\mathcal{J}_m)$closure preserving $(\mathcal{J}_m)$open refinement.*

*(iv) For every $(\omega)$open cover $\mathcal{U}$ of $X$, there exists an $m$ such that for any $n$, each $(\mathcal{J}_n)$open refinement of $\mathcal{U}$ has a $(\mathcal{J}_m)$open refinement $\mathcal{U}_0$ and $\mathcal{U}_0$ has a $(\mathcal{J}_m)$closure preserving refinement.*

*(v) For every $(\omega)$open cover $\mathcal{U}$ of $X$, there exists an $m$ such that for any $n$, each $(\mathcal{J}_n)$open refinement of $\mathcal{U}$ has a $(\mathcal{J}_m)$open refinement $\mathcal{U}_0$ and $\mathcal{U}_0$ has a $(\mathcal{J}_m)$closure preserving $(\mathcal{J}_m)$closed refinement.*

Since a $(\mathcal{J}_m)$locally finite collection of subsets of $X$ is hereditarily $(\mathcal{J}_m)$closure preserving, $(i) \Rightarrow (ii)$ follows. $(ii) \Rightarrow (iii)$ is trivial.

*Proof of* $(iii) \Rightarrow (iv)$. Let $\mathcal{U}$ be an $(\omega)$open cover of $X$ and let $\mathcal{V}$ be a $(\mathcal{J}_n)$open refinement of $\mathcal{U}$. By $(iii)$, there exists a $(\mathcal{J}_m)$open refinement $\mathcal{G} = \cup_{i=1}^{\infty}\mathcal{G}_i$ of $\mathcal{V}$ such that each $\mathcal{G}_i$ is hereditarily $(\mathcal{J}_m)$closure preserving. Let

$$G_i = \cup\{G \mid G \in \mathcal{G}_i\}, \quad K_1 = X, \quad K_i = X - \cup_{j=1}^{i-1} G_j, \ i = 2, 3, ...$$

Then $\{K_i \mid i \in N\}$ is $(\mathcal{J}_m)$locally finite. Let

$$\mathcal{B}_i = \{G \cap K_i | G \in \mathcal{G}_i\}.$$

Since each $\mathcal{G}_i$ is hereditarily $(\mathcal{J}_m)$closure preserving, each $\mathcal{B}_i$ is $(\mathcal{J}_m)$closure preserving. Therefore by Lemma 8, it follows that $\mathcal{B} = \cup\{\mathcal{B}_i \mid i \in N\}$ is $(\mathcal{J}_m)$closure preserving. Also $\mathcal{B}$ is a refinement of $\mathcal{G}$. Taking $\mathcal{G}$ as $\mathcal{U}_0$, $(iv)$ follows. □

*Proof of* $(iv) \Rightarrow (v)$. Let $\mathcal{U}$ be an $(\omega)$open cover of $X$ and let $\mathcal{U}_0$ be a $(\mathcal{J}_m)$open refinement of a $(\mathcal{J}_n)$open refinement of $\mathcal{U}$. Consider $x \in X$ and $U_x \in \mathcal{U}_0$ such that $x \in U_x$. By the $(\mathcal{J}_m)$regularity of $X$, we get a $V_x \in \mathcal{J}_m$ with $x \in V_x \subset (\mathcal{J}_m)clV_x \subset U_x$. Then $\mathcal{V} = \{V_x \mid x \in X\}$ is a $(\mathcal{J}_m)$open refinement of $\mathcal{U}$. By $(iv)$, there is a refinement $\mathcal{B}$ of $\mathcal{V}$, which is $(\mathcal{J}_m)$closure preserving. For $B \in \mathcal{B}$, there exists a $V_x$ such that $B \subset V_x$ and $(\mathcal{J}_m)clB \subset (\mathcal{J}_m)clV_x \subset U_x$. Therefore $\{(\mathcal{J}_m)clB \mid B \in \mathcal{B}\}$ is the required refinement of $\mathcal{U}_0$. □

To prove $(v) \Rightarrow (i)$, we require the following lemma. We assume $(v)$ holds.

**Lemma 14.** *If $\mathcal{U}_0 = \{U_\alpha\}_{\alpha \in A}$ is the $(\mathcal{J}_m)$open refinement of a $(\mathcal{J}_n)$open refinement of the $(\omega)$open cover $\mathcal{U}$ of $X$, then there exists a $(\mathcal{J}_m)$closure preserving $(\mathcal{J}_m)$closed refinement $\{K_\alpha \mid \alpha \in A\}$ of $\mathcal{U}_0$ such that $K_\alpha \subset U_\alpha$, $\alpha \in A$.*

*Proof.* Let $\mathcal{B}$ be a $(\mathcal{J}_m)$closure preserving $(\mathcal{J}_m)$closed refinement of $\mathcal{U}_0$. We write $K_\alpha = \cup\{B \in \mathcal{B} \mid B \subset U_\alpha\}$ for all $\alpha$. Then for any $\alpha$, $K_\alpha \subset U_\alpha$ and

$$(\mathcal{J}_m)clK_\alpha = (\mathcal{J}_m)cl\left(\cup\{B \in \mathcal{B} \mid B \subset U_\alpha\}\right)$$

$$= \cup\{(\mathcal{J}_m)clB \mid B \in \mathcal{B}, B \subset U_\alpha\} \text{ (since } \mathcal{B} \text{ is } (\mathcal{J}_m)\text{closure preserving)}$$
$$= \cup\{B \in \mathcal{B} \mid B \subset U_\alpha\} \text{ (since each } B \text{ is } (\mathcal{J}_m)\text{closed)} = K_\alpha.$$

Therefore $K_\alpha$ is ($\mathcal{J}_m$)closed for each $\alpha \in A$. Since $\mathcal{B}$ is ($\mathcal{J}_m$)closure preserving, it follows that $\{K_\alpha \mid \alpha \in A\}$ is ($\mathcal{J}_m$)closure preserving. □

*Proof of* $(v) \Rightarrow (i)$. Suppose $\mathcal{U}_0 = \{U_\alpha \mid \alpha \in A\}$ is the ($\mathcal{J}_m$)open refinement of a ($\mathcal{J}_n$)open refinement of $\mathcal{U}$. Let the index set $A$ be well ordered.

For each positive integer $i$, we first construct a collection $\{K_{\alpha,i} \mid \alpha \in A\}$ of subsets of $X$ satisfying the following conditions:

(1) $\{K_{\alpha,i} \mid \alpha \in A\}$ is a ($\mathcal{J}_m$)closure preserving ($\mathcal{J}_m$)closed cover of $X$ with $K_{\alpha,i} \subset U_\alpha$.

(2) $K_{\alpha,i+1} \cap K_{\beta,i} = \emptyset$ for all $\alpha > \beta$.

Using Lemma 14, we get the cover $\{K_{\alpha,1} \mid \alpha \in A\}$ satisfying (1) for $i = 1$. Suppose the covers $\{K_{\alpha,i} \mid \alpha \in A\}$ have been constructed for $i = 1, 2, ..., r$. Let

$$U_{\alpha,r+1} = U_\alpha - \cup_{\beta<\alpha} K_{\beta,r}, \text{ for all } \alpha \in A.$$

Then $U_{\alpha,r+1}$ is ($\mathcal{J}_m$)open. It then follows that $\{U_{\alpha,r+1} \mid \alpha \in A\}$ is a ($\mathcal{J}_m$)open refinement of $\mathcal{U}_0$ and hence of $\mathcal{U}$. By Lemma 14, we get, a ($\mathcal{J}_m$)closure preserving ($\mathcal{J}_m$)closed refinement $\{K_{\alpha,r+1} \mid \alpha \in A\}$ such that $K_{\alpha,r+1} \subset U_{\alpha,r+1}$ for all $\alpha$. Then the condition (1) for $i = r + 1$ and the condition (2) for $i = r$ are obviously satisfied.

Now we write

$$V_{\alpha,i} = X - \cup_{\beta\neq\alpha} K_{\beta,i}, \text{ for all } \alpha, i$$

and show that:

$(a)$ $\{V_{\alpha,i} \mid \alpha \in A,\, i \in N\}$ is a ($\mathcal{J}_m$)open cover of $X$ and $V_{\alpha,i} \subset U_\alpha$ for all $\alpha$ and $i$.

$(b)$ $V_{\alpha,i} \cap V_{\beta,i} = \phi$ whenever $\alpha \neq \beta$.

Since $\{K_{\beta,i} \mid \beta \in A\}$ is ($\mathcal{J}_m$)closure preserving and each $K_{\beta,i}$ is ($\mathcal{J}_m$)closed, the ($\mathcal{J}_m$)openness of each $V_{\alpha,i}$ follows. Also since $\{K_{\alpha,i} \mid \alpha \in A\}$ forms a cover of $X$, we get

$$V_{\alpha,i} \subset K_{\alpha,i} \subset U_\alpha, \text{ for all } \alpha \text{ and } i,$$

and hence $(b)$ follows. It remains to show that $\{V_{\alpha,i} \mid \alpha \in A,\, i \in N\}$ is a cover of $X$.

Let $x \in X$ be any point. If $\alpha_l = \min\{\alpha_i \mid i \in N\}$ where, for all $i \in N$, $\alpha_i = \min\{\alpha \in A \mid x \in K_{\alpha,i}\}$, then

$$x \in V_{\alpha_l,l+1} = X - (\cup_{\beta\neq\alpha_l} K_{\beta,l+1}).$$

In fact, if $\alpha < \alpha_l$, then $x \notin K_{\alpha,l+1}$ by the definition of $\alpha_l$ and if $\alpha > \alpha_l$, then $x \notin K_{\alpha,l+1}$ by the relation (2) with $i = l$ and $\beta = \alpha_l$ and by the fact that $x \in K_{\alpha_l,l}$. Thus $\{V_{\alpha,i} \mid \alpha \in A, i \in N\}$ forms a cover of $X$. Therefore it is a $(\mathcal{J}_m)$open refinement of $\mathcal{U}_0$ and hence of $\mathcal{U}$.

By Lemma 14, we get a $(\mathcal{J}_m)$closure preserving $(\mathcal{J}_m)$closed refinement $\{F_{\alpha,i} \mid \alpha \in A, i \in N\}$ of $\{V_{\alpha,i} \mid \alpha \in A, i \in N\}$ such that $F_{\alpha,i} \subset V_{\alpha,i}$ for all $\alpha$ and $i$.

Again we see that the topological space $(X, \mathcal{J}_m)$ is $(\mathcal{J}_m)$normal: Let $A$ and $B$ be two disjoint $(\mathcal{J}_m)$closed subsets of $X$. Then $\{X-A, X-B\}$ is a $(\mathcal{J}_m)$open cover of $X$. Therefore there exists a $(\mathcal{J}_m)$closed cover $\{F, K\}$ of $X$ such that $F \subset X - A$ and $K \subset X - B$. Then obviously $A \subset X - F$, $B \subset X - K$ and $(X - F) \cap (X - K) = \emptyset$. So $X$ is $(\mathcal{J}_m)$normal.

Further, since $\{F_{\alpha,i} \mid \alpha \in A, i \in N\}$ is a $(\mathcal{J}_m)$closure preserving $(\mathcal{J}_m)$closed refinement so $\cup_{\alpha\in A} F_{\alpha,i}$ is $(\mathcal{J}_m)$closed. Then using Lemma 9, we obtain for each $i \in N$, a $(\mathcal{J}_m)$discrete collection $\{W_{\alpha,i} \mid \alpha \in A\}$ of $(\mathcal{J}_m)$open sets such that

$$F_{\alpha,i} \subset W_{\alpha,i} \subset V_{\alpha,i}, \text{ for all } \alpha.$$

The collection $\{W_{\alpha,i} \mid \alpha \in A, i \in N\}$ is then $\sigma - (\mathcal{J}_m)$discrete and so $\sigma - (\mathcal{J}_m)$locally finite $(\mathcal{J}_m)$open refinement of $\mathcal{U}_0$. Therefore by Theorem 12, $X$ is $(\omega^s)$paracompact. □

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