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ON THE GEOMETRY OF A FOUR PARAMETER MIXED RATIONAL/LINEAR SYSTEM OF PLANAR DIFFERENCE EQUATIONS

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Abstract: Motivated by questions posed by Ladas et al, we provide a concise geometric analysis of the rational, planar system of difference equations

$$\begin{aligned} x_{n+1} &= \frac{a+by_n}{x_n}, \\ y_{n+1} &= cx_n + dy_n, \end{aligned}$$

with initial values $x_0 > 0$ and $y_0 \geq 0$ and non-negative real parameters a, b, c, and d. This is accomplished by associating a planar mapping F to the system in such a way that the orbit structure of F gives information about solutions to the system. Consequently, we are able to classify the behavior of the system across its 4-dimensional parameter space for all possible initial data.

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1. Introduction

Consider the planar system of rational difference equations

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$$x_{n+1} = \frac{a + by_n}{x_n},$$
$$y_{n+1} = cx_n + dy_n, \tag{1}$$

with real initial values $x_0 > 0$, $y_0 \geq 0$, and non-negative real parameters a, b, c, and d. This system is a specific instance of the more general system

$$x_{n+1} = \frac{\alpha_1 + \beta_1 x_n + \gamma_1 y_n}{A_1 + B_1 x_n + C_1 y_n},$$
$$y_{n+1} = \frac{\alpha_2 + \beta_2 x_n + \gamma_2 y_n}{A_2 + B_2 x_n + C_2 y_n},$$

about which several open questions have been posed (see [3]). In this paper, we provide an analysis of the behavior of (1) across its 4-dimensional parameter space using geometric methods. More specifically, by studying the related planar mapping

$$F(x, y) = \left(\frac{a + by}{x}, cx + dy\right), \tag{2}$$

we develop a qualitatively accurate picture of (1) and provide information concerning the boundedness character and periodicity of solutions as well as the characterization of equilibrium points.

The mapping F is related to the system (1) through iteration. Let $F^0(x, y) = F(x, y)$ and define $F^k(x, y) = F(F^{k-1}(x, y))$ for $k \geq 1$. By the *orbit of* (x, y), we mean the set

$$\{F^k(x, y) | k = 0, 1, ...\},$$

which we denote in the sequel by $\mathcal{O}_F(x, y)$. With this notation, a solution to (1) with initial data x_0, y_0 is equivalent to $\mathcal{O}_F(x_0, y_0)$, equilibrium points of (1) correspond to the fixed points of the mapping F, stability of equilibria to dynamic behavior at fixed points, and so on. Rather than translating results back and forth between the two points of view, we will adhere to the later in what follows, characterizing F and orbits under F rather the system (1) and solutions to the system.

2. Properties of $F(x, y)$

We begin with several observations about the function F. Fixing some notation first, let $\mathbb{Q}_1^o = (0, \infty) \times (0, \infty)$, $\mathbb{Q}_1 = [0, \infty) \times [0, \infty)$, and $D = (0, \infty) \times [0, \infty)$. Then F could be most generally viewed as a continuous mapping from D in to $\mathbb{Q}_1$, with further refinements to the domain and codomain possible depending on the actual parameter values.

To most quickly simplify the situation, we first consider orbits of points of the form $(0, y)$ and $(x, 0)$ where x and y are assumed to be positive. In the first case, we must have $a = b = 0$, thus making F equivalent to the zero mapping in the first coordinate. It then follows that

$$\mathcal{O}_F(0, y) = \{(0, y), (0, dy), (0, d^2 y), ..., (0, d^k y), ...\},$$

whose boundedness depends in the obvious way on the value of the parameter d. In the case of $(x, 0)$, the situation is more complicated. Behavior in this case can be organized into five general types by zero/non-zero parameter combinations. These are summarized below where, unless stated otherwise, a parameter is assumed to be positive. Note that the assertions (a)-(d) can be verified directly by simple computation.

a. If $a = b = c = d = 0$, $a = b = c = 0$, or $a = b = d = 0$, then $\mathcal{O}_F(x, 0)$ is eventually the *zero orbit*, i.e., $\mathcal{O}_F(x, 0) = \{..., (0, 0), (0, 0), ...\}$.

b. If $a = c = d = 0$, $a = c = 0$, $a = d = 0$, or $a = 0$, then $\mathcal{O}_F(x, 0)$ is *undefined*, i.e., it is impossible to evaluate $F^k(x, 0)$ because of division by zero for some $k \geq 0$.

c. If $a = b = 0$, then $\mathcal{O}_F(x, 0) = \{(x, 0), (0, cx), (0, cdx), ..., (0, cd^k x), ...\}$ and the boundedness of this orbit again depends on the value of d.

d. If $b = c = d = 0$, $b = c = 0$, $c = d = 0$, or $c = 0$, then $\mathcal{O}_F(x, 0)$ is eventually periodic of the form

$$\{..., (a/x, 0), (x, 0), (a/x, 0), ...\}.$$

If $b = d = 0$, then $\mathcal{O}_F(x, 0)$ is also eventually periodic with form

$$\{..., (a/x, cx), (x, ac/x), (a/x, cx), ...\}.$$

e. If $b = 0$, $d = 0$, or all parameters are positive, then $F^k(x, 0) \in \mathbb{Q}_1^o$ for all $k > 0$.

Note that, by *eventually* above, we mean *in a finite number of iterations*.

As a consequence of these considerations, we can assume that our initial data x_0, y_0 is *positive* in what follows. Thus, we will consider F as a mapping with domain $\mathbb{Q}_1^o$ from here on, with a single exception: the situation (e) above, whose analysis is included in the cases $b = 0$, $d = 0$ of Theorem 5 and the all parameters positive scenario of Section 3 with *positive* initial data by merit of the fact that, with these parameter values, $F^k(x, 0) \in \mathbb{Q}_1^o$ for all $k > 0$.

After some straightforward algebra, we can derive a fixed point for F, namely

$$(u,v)=\left(\frac{bc+\sqrt{b^2c^2+4a(1-d)^2}}{2(1-d)},\frac{c}{1-d}\cdot\frac{bc+\sqrt{b^2c^2+4a(1-d)^2}}{2(1-d)}\right). \quad (3)$$

This fixed point is defined as long as $0 \le d < 1$. Addressing the question of the behavior of F under iteration in the cases $d = 1$ and $d > 1$ allows an additional refinement to the range of parameter values to be considered, further simplifying our subsequent analysis.

Theorem 1. *Suppose* $(x,y) \in \mathbb{Q}_1^o$. *Then:*

a. If $d = 1$, *the character of* $\mathcal{O}_F(x,y)$ *depends on the parameters* a, b, *and* c *in the following manner:*

i. If $a = b = 0$ *or* $a = b = c = 0$, *then* $\mathcal{O}_F(x,y)$ *is eventually undefined.*

ii. If $a = c = 0$, $b = c = 0$, *or* $c = 0$, *then* $\mathcal{O}_F(x,y)$ *is eventually periodic with prime period 2.*

iii. If $a = 0$ *or* $b = 0$, *then* $\mathcal{O}_F(x,y)$ *is unbounded.*

b. If $d > 1$, *then* $\mathcal{O}_F(x,y)$ *is unbounded.*

Proof. For part (a), (i) and (ii) can be verified by direct computation. We note that under the parameter assumptions in (ii), F fixes an entire curve in $\mathbb{Q}_1^o$: $y = (1/b)x^2$ if $a = c = 0$, $x = \sqrt{a}$ if $b = c = 0$, and $y = (1/b)(x^2 - a)$ if $c = 0$.

The proof of (iii) in part (a) requires some effort. For initial data $(x,y) \in \mathbb{Q}_1^o$, let $(x_k, y_k) = F^k(x,y)$ and $(x_0, y_0) = (x,y)$. Then

$$y_k = \left[\sum_{i=0}^{k-1} c x_i\right] + y_0.$$

Hence, $\{y_0, y_1, ...\}$ is a strictly increasing sequence of of positive real numbers. If this sequence is bounded, then it must have a least upper bound β, and

$$\lim_{k\to\infty} y_k = \beta = \lim_{k\to\infty}\left[\sum_{i=0}^{k-1} c x_i\right] + y_0.$$

Therefore,

$$\lim_{k\to\infty}\left[\sum_{i=0}^{k-1} x_i\right] = \frac{\beta - y_0}{c},$$

from which it follows that

$$\lim_{i\to\infty} x_i = 0.$$

If $b = 0$, pick $m \geq 0$ such that $\frac{1}{2^m} < a$, and let $n > m + 1$. Since the above limit is zero, there must be an index value j such that $i \geq j$ implies $x_i < \frac{1}{2^n}$. Computing, we have that

$$x_{i+1} = \frac{a}{x_i} > \frac{a}{1/2^n} = a \cdot 2^n > a \cdot 2^{m+1} > \frac{1}{2^m} \cdot 2^{m+1} = 2,$$

an obvious contradiction. We can derive a similar contradiction in the case $a = 0$ by picking $m \geq 0$ such that $\frac{1}{2^m} < by_0$ and proceeding as above. Thus, in either case, the assumption that the sequence $\{y_0, y_1, ...\}$ is bounded leads to a contradiction. Consequently, $\mathcal{O}_F(x, y)$ must be unbounded as claimed.

For (b), we again let $(x_k, y_k) = F^k(x, y)$ with $(x_0, y_0) = (x, y) \in \mathbb{Q}_1^o$. A simple induction argument suffices to show that $y_k > d^k y_0$. Consequently, $\mathcal{O}_F(x, y)$ must be unbounded since we are assuming $d > 1$. □

The preceding discussion and Theorem 1 simplify both the domain of F and the parameter space, thus allowing for a more general analysis in the remaining cases. We therefore make the following standing assumptions:

A1. All initial data are positive. Said differently, the domain of F is restricted to $\mathbb{Q}_1^o$.

A2. The value of the parameter d is constrained to the interval $[0, 1)$.

The assumptions A1 and A2 will remain in effect in the sequel unless noted otherwise.

Finally, by way of summary, the following theorem codifies the important characteristics of F under the standing assumptions A1 and A2:

Theorem 2. *Assuming A1 and A2 above, F is a continuous mapping from $\mathbb{Q}_1^o$ to $\mathbb{Q}_1$. If, in addition to A1 and A2, we assume $c \neq 0$ and that a and b are not* both *zero, then F has a fixed point (u, v) given by (3) which also belongs to $\mathbb{Q}_1^o$ and which is a continuous function of the parameters a, b, c, and d.*

The proof of Theorem 2 is immediate and will be omitted. We conclude by noting that the expression (3) for (u, v) actually holds for a wider range of parameters than allowed in the preceding theorem. However, in these cases, it may be that (u, v) falls outside of $\mathbb{Q}_1^o$. Such situations will be dealt with when zero parameter cases are consider below.

3. Orbits in Non-Zero Parameter Cases

In this section, we assume all parameters are positive and classify the fixed point (3) (which we will continue to refer to as (u, v)). This is done in the context of the standing assumptions A1 and A2 and leads to a global description of the system (1) under the assumptions on the parameters.

An elementary computation yields the Jacobian matrix for F:

$$J_F(x, y) = \begin{bmatrix} -\dfrac{a + by}{x^2} & \dfrac{b}{x} \\ c & d \end{bmatrix}. \tag{4}$$

Since each of the first order partial derivatives in $J_F(x, y)$ is continuous on $\mathbb{Q}_1^o$, it follows that F itself is differentiable there (see [1], Theorem 20.7, for example). In particular, F is differentiable and therefore locally linearizable at the fixed point (u, v). Evaluating $J_F(x, y)$ at (u, v) yields

$$J_F(u, v) = \begin{bmatrix} -\dfrac{a + bv}{u^2} & \dfrac{b}{u} \\ c & d \end{bmatrix} = \begin{bmatrix} -1 & \dfrac{b}{u} \\ c & d \end{bmatrix},$$

where

$$-\frac{a + bv}{u^2} = -\frac{1}{u} \cdot \frac{a + bv}{u} = -\frac{1}{u} \cdot u,$$

since F fixes (u, v). The eigenvalues for $J_F(u, v)$ are

$$\lambda^+ = \frac{-u(1 - d) + \sqrt{[u(1 - d)]^2 + 4u(du + bc)}}{2u},$$

and

$$\lambda^- = \frac{-u(1 - d) - \sqrt{[u(1 - d)]^2 + 4u(du + bc)}}{2u}.$$

Since the radicand in both of the preceding equations always exceeds $u(1 - d)$, it follows that

$$\lambda^- < 0 < \lambda^+. \tag{5}$$

We can take this analysis a step further, as follows:

Theorem 3. *Suppose that all parameters are positive. Then the eigenvalues for J_F satisfy the inequality*

$$\lambda^- < -1 < 0 < \lambda^+ < 1.$$

Proof. To obtain the inequality, we compute $\lambda^- + 1$ and $\lambda^+ - 1$ and compare the results to 0. After some simplification, we have

$$\lambda^- + 1 = \frac{u(1+d) - \sqrt{[u(1+d)]^2 + 4bcu}}{2u},$$

while

$$\lambda^+ - 1 = -\frac{3}{2} + \frac{d}{2} + \sqrt{\frac{1}{4}(1+d)^2 + \frac{b}{u} \cdot c}.$$

Working with $\lambda^- + 1$ first, we have

$$\lambda^- + 1 = \frac{u(1+d) - \sqrt{[u(1+d)]^2 + 4bcu}}{2u} < \frac{u(1+d) - \sqrt{[u(1+d)]^2}}{2u} = 0;$$

consequently, $\lambda^- < -1$.

To evaluate $\lambda^+ - 1$, the bound

$$\frac{b}{u} \leq \frac{1-d}{c}$$

is helpful (this can be obtained by a straightforward comparison). Then

$$\begin{aligned}\lambda^+ - 1 = -\frac{3}{2} + \frac{d}{2} + \sqrt{\frac{1}{4}(1+d)^2 + \frac{b}{u} \cdot c} &\leq -\frac{3}{2} + \frac{d}{2} + \sqrt{\frac{1}{4}(1+d)^2 + 1 - d} \\ &= -\frac{3}{2} + \frac{d}{2} + \frac{1}{2}\sqrt{d^2 - 2d + 5} = h(d).\end{aligned}$$

The function $h(d)$ is defined for all $d \in [0, 1)$, evaluates to $-\frac{3}{2} + \frac{1}{2}\sqrt{5} < 0$ when $d = 0$, limits to 0 as $d \to 1$, and, by calculus, is strictly increasing on $[0, 1)$. Consequently, for all $d \in [0, 1)$, we have $\lambda^+ - 1 < 0$, which implies $\lambda^+ < 1$. This, combined with (5) and the bound on λ^- above, yield the desired result. □

Using this, we can now classify orbits in the non-zero parameter case:

Corollary 4. *Suppose that all parameters are positive. Then (u, v) is a saddle point. Consequently, there exists an unbounded curve S (the* stable *manifold) along which orbits converge to (u, v) and an unbounded curve U (the* unstable *manifold) along which orbits diverge from (u, v). Furthermore, the orbit of any point (x, y) not equal to (u, v) and not belonging to S or U is unbounded and asymptotically approaches U.*

Proof. This follows from the bound on the eigenvalues for (4) evaluated at (u, v) given in Theorem 3 and basic classification results for saddle points (see for example [2], Chapter 4). □

In fact, the bounds on λ^+ and λ^- give even more information about the behavior of F in the case of non-zero parameters. Since $\lambda^- < 0$, it follows that

F reflects across S with each iteration. This situation is pictured in Figure 1.

4. Orbits in Zero Parameter Cases

We now turn to the question of the behavior of F when one or more of the parameter values is zero. In some cases, the character of F is radically simplified, while in others the situation is equivalent to that discussed in Section 3. We will continue with the notational convention established in Section 2 concerning parameter values: unless stated otherwise, a parameter will be assumed to be positive. Furthermore, assumptions A1 and A2 will be assumed to be in effect unless an indication to the contrary is given.

The following theorem summarizes the results in the zero parameter cases:

Theorem 5. *Assuming A1 and A2 above, suppose that one or more of the parameters a, b, c, or d is zero. Then:*

a. If $a = b = 0$, $a = c = 0$, or $a = b = c = 0$, then all orbits converge to $\vec{0}$.

b. If $b = d = 0$, $c = d = 0$, or $b = c = d = 0$, then all orbits are eventually 2-cycles.

c. If $a = b = d = 0$ or $a = b = c = d = 0$, then all orbits are eventually constant.

d. If $b = c = 0$ or $c = 0$, then all orbits are attracted to the 2-cycle

$$\{..., (x', 0), (a/x', 0), (x', 0), ...\},$$

where the value of x' depends on which parameter combination is zero.

e. If $a = 0$, $b = 0$, $d = 0$, or $a = d = 0$, then F has a saddle point (u, v) of the form (3) and the results of Section 3 apply.

f. If $a = c = d = 0$, then all orbits are eventually undefined due to a zero division.

Proof. Cases (a), (b), (c), and (f) are immediate consequences of direct computation and so their proofs will be omitted.

For (d), direct computation in the case $b = c = 0$ yields

$$\mathcal{O}_F(x, y) = \{(x, y), (a/x, dy), (x, d^2y), (a/x, d^3y), (x, d^4y), (a/x, d^5y), ...\},$$

from which the claim follows since $0 \leq d < 1$. The remaining case $c = 0$ can be verified by considering the action of

$$F^2(x, y) = \left(\frac{a + bdy}{a + by} \cdot x, d^2y\right)$$

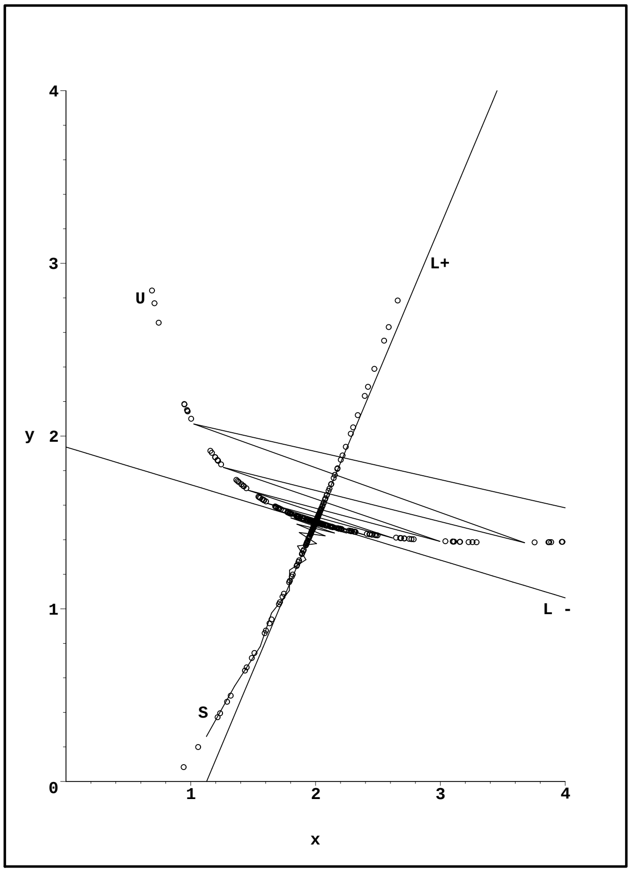

Figure 1: Manifolds, eigenlines, and orbits for a typical system with all positive parameters. In this image, $a = 1.0$, $b = 2.0$, $c = 0.375$, and $d = 0.5$. Approximations of S and U (indicated by open dots) are obtained by randomly sampling points and computing partial orbits. Those points whose orbits tend toward the fixed point (u, v) belong to S, while those points whose orbits tend toward infinity belong to U. Note the tangential agreement at (u, v) between the eigenlines L^+ and L^- (i.e., the lines through (u, v) with directions parallel to the eigenvectors for λ^+ and λ^-) and the stable and unstable manifolds S and U, respectively. For $P = (1.125, 0.26)$ (selected for purposes of illustration), observe that $\mathcal{O}_F(P)$ (the PL-curve in the image) reflects across S with each iteration.

on $\mathbb{Q}_1^o$. To this end, let

$$\Delta_x = \frac{a+bdy}{a+by} \cdot x - x,$$

and

$$\Delta_y = d^2y - y.$$

Then Δ_x and Δ_y are both negative, indicating that, relative to the argument (x, y), F^2 moves points in a *southwest* direction (i.e., towards $\vec{0}$).

Let $(x_k, y_k) = F^{2k}(x, y)$, with $(x_0, y_0) = (x, y)$. Then

$$y_k = d^{2k}y$$

and, after a little more work,

$$x_k = \left(\prod_{i=0}^{k-1} \frac{a+bd^{2i+1}y}{a+bd^{2i}y}\right) \cdot x.$$

To determine the boundedness of $\mathcal{O}_{F^2}(x, y)$ it is necessary to find $\lim_{k\to\infty} y_k$ and $\lim_{k\to\infty} x_k$. For y_k, we have

$$\lim_{k\to\infty} y_k = \lim_{k\to\infty} d^{2k}y = 0$$

since $0 < d < 1$. Thus, $\mathcal{O}_{F^2}(x, y)$ approaches the x-axis. For x_k,

$$\lim_{k\to\infty} x_k = \lim_{k\to\infty} \left(\prod_{i=0}^{k-1} \frac{a+bd^{2i+1}y}{a+bd^{2i}y}\right) \cdot x = \left(\prod_{i=0}^{\infty} \frac{a+bd^{2i+1}y}{a+bd^{2i}y}\right) \cdot x.$$

Consequently, the behavior of $\mathcal{O}_{F^2}(x, y)$ with respect to the first coordinate is determined by the infinite product

$$\prod_{i=0}^{\infty} \frac{a+bd^{2i+1}y}{a+bd^{2i}y}. \tag{6}$$

If we write

$$\alpha_i = \frac{bd^{2i}y(1-d)}{a+bd^{2i}y},$$

then

$$\frac{a+bd^{2i+1}y}{a+bd^{2i}y} = 1 - \alpha_i$$

and (6) converges absolutely to a positive real number if and only if

$$\sum_{i=0}^{\infty} |\alpha_i| \tag{7}$$

converges (see [4] Chapter 8, for example). Since $|\alpha_i| = \alpha_i$ and

$$\lim_{i\to\infty} \frac{\alpha_{i+1}}{\alpha_i} = \lim_{i\to\infty} \frac{ad^2 + bd^{2i+2}y}{a + bd^{2i+2}y} = d^2 < 1,$$

it follows that (7) and, consequently, (6), converge. If we let p_∞ denote the value of (6), then

$$\mathcal{O}_{F^2}(x, y) \to (p_\infty \cdot x, 0).$$

Furthermore, since F^2 moves points strictly to the southwest, it follows that $0 < p_\infty \cdot x < x$. Finally,

$$(p_\infty \cdot x, 0) \xrightarrow{F} \left(\frac{a}{p_\infty \cdot x}, 0\right) \xrightarrow{F} (p_\infty \cdot x, 0)\,,$$

concluding the proof of part (d).

Finally, for (e), observe that the fixed point formula given in (3) yields a fixed point $(u, v) \in \mathbb{Q}_1^o$ in this case and that the classification of (u, v) completed in Section 3 remains valid. □

5. Conclusions and Open Questions

The analysis completed in Section 2 and Section 3 gives a complete picture of the behavior of F and, consequently, the system (1), across the 4-dimensional parameter space defined by the non-negative real parameters a, b, c, and d. Under suitable assumptions on the parameters and the domain of F, orbits exhibit all possible behaviors alluded to in Section 1: bounded, unbounded, and periodic. Solutions to the system will have the same essential characteristics under the same assumptions.

It would be interesting to know whether the curves S and U from Corollary 4 can be expressed in terms of elementary functions. Some progress along these lines can be made in the case of positive initial data with $a = d = 0$ (case (e) from Theorem 5). In this situation, an exponential change of variables

$$E(x, y) = (ue^x, ve^y)$$

yields a function $H(x, y) = (E^{-1} \circ F \circ E)(x, y) = (y - x, x)$ defined on the entire plane with fixed point at the origin $\vec{0}$. Speculatively solving

$$H(x, mx) = (mx - x, x) = (mx - x, m(mx - x))$$

for a slope m corresponding to a line fixed by H yields two values for m: $\phi^+ = 1/2 + (1/2)\sqrt{5}$ and $\phi^- = 1/2 - (1/2)\sqrt{5}$. In this case, these lines correspond exactly with the eigenlines through $\vec{0}$ (i.e, the lines through the fixed point

$\vec{0}$ with slopes parallel to the eigenvectors corresponding to the eigenvalues for $J_H(\vec{0})$).

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