International Journal of Pure and Applied Mathematics
Volume 59 No. 4 2010, 469-488

ON THE EXISTENCE OF EXACTLY N LIMIT CYCLES IN LIENARD SYSTEMS

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Abstract: A theorem on the existence of exactly N limit cycles around a critical point for the Lienard system $\ddot{x} + f(x)\dot{x} + g(x) = 0$ is proved. An alogrithm on the determination of a desired number of limit cycles for this system has been considered which might become relevant for a Lienard system with incomplete data.

AMS Subject Classification: 34A34, 70K05
Key Words: autonomous system, Lienard equation, limit cycle

1. Introduction

Limit cycles are isolated closed curves in an autonomous system in a phase plane. Determination of shape and number of limit cycles has been a challenging problem in the theory of autonomous systems. Lienard system has been a field of active interest in recent past because of its relevance in various physical and mathematical problem [4]-[1]. Recently non-smooth Lienard systems even allowing discontinuities [5] are also being studied.

Received: February 6, 2010

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Here we consider the Lienard equation of the type

$$\ddot{x} + f(x)\,\dot{x} + g(x) = 0. \tag{1}$$

The Lienard equation (1) can be written as a non-standard autonomous system

$$\dot{x} = y - F(x), \quad \dot{y} = -g(x), \tag{2}$$

where $F(x) = \int_0^x f(u)\,du$. The phase plane defined by (2) is known as Lienard plane. Lienard gave a uniqueness theorem [4], [9] for periodic cycles for a general class of equations when $F(x)$ is an odd function and satisfies a monotonicity condition as $x \to \infty$. A challenging problem is the determination of the number of limit cycles for a given polynomial $F(x)$ of degree (m) for the system (2), see [8], [5], [1]. Recently we have presented a new method for proving the existence of exactly two limit cycles of a Lienard system [8]. Recall that the proof of Lienard Theorem depends on the existence of an odd function $F(x)$ with zeros at $x = 0$ and $x = \pm a$ $(a > 0)$ and that $F(x) > 0$ for $x > a$ and tends to ∞ as $x \to \infty$. To weaker this assumption, we note at first that the existence of a limit cycle is still assured if there exists a value $\bar{\alpha} > a$ (called an efficient upper estimate of the amplitude of the limit cycle) such that $F(x)$ is increasing for $a \leq x < \bar{\alpha} < L_1$, where L_1 is the first extremum of $F(x)$, $x > a$. Based on this observation we are then able to generalize the standard theorem for the existence of exactly two limit cycles. Our theorem not only extends the class of $F(x)$ considered by Odani [6], [7], but also that of the more recent work of Chen et al [1] (see [8] for more details).

In the present paper we prove the theorem for the existence of exactly N limit cycles for the system (1). In the second part of the paper we present an algorithm to generate any desired number of limit cycles around the origin, which is the only critical point for the system (1). Limit cycles represent an important class of nonlinear periodic oscillations. Existence of such nonlinear periodic cycles have been established in various natural and biological systems [4], [9], [3]. It is well known that mammalian heartbeats may follow a non-linear oscillatory patterns under certain (physiological) constraints [3]. However, sometimes it becomes very difficult to obtain total information about a nonlinear system due to various natural constraints, as a result of which we obtain only a partial or incomplete data [2]. Our objective is to fill up those gaps and construct a Lienard system that may be considered to model the dynamics of the missing part of the phenomena in an efficient manner.

To state this in another way, let us suppose that the Lienard system is defined only on a bounded region $[-a_1, a_1]$, $a_1 > 0$ having one (or at most a finite number of) limit cycles in that region. Our aim is to develop an algorithm to extend the Lienard system minimally throughout the plane accommodating a given number of limit cycles in the extended region. By minimal extension we mean that the graph $(x, F(x))$, of the function F which is initially defined only in $|x| < a_1$ is extended beyond the line $x = a_1$ iteratively as an action induced by two suitably chosen functions $\phi(x)$ and $H(x)$ so that ϕ acts on the abscissa x and H acts on the ordinate $F(x)$ respectively. Accordingly the desired extension $\tilde{F}(x)$ of $F(x)$, $x > a_1$ is realized as $H \circ F(x) = \tilde{F} \circ \phi(x)$. The choice of ϕ and H is motivated by Theorem 2 so that the extension $\tilde{F}$ satisfies the conditions of the said theorem. It turns out that ϕ can simply be a bijective function, while H may be any monotonic function admitting $\bar{\alpha} < L$ (cf. equation (9)), L being the unique extremum of $\tilde{F}(x)$, $x \in [a_1, a_2]$, $\tilde{F}(a_1) = \tilde{F}(a_2) = 0$.

The paper is organized as follows. In Section 2 we introduced our notations. In Section 3 we have proved an extension of the theorem in [8] for existence of exactly N limit cycles in the Lienard equation. In Section 4 we present the construction by which we can get a system of the form (1) having any desired number of limit cycles around a single critical point. Examples in support of this algorithm are studied in Section 5.

2. Notations

We recall that [4] by symmetry of paths, a typical phase path YQY' of the system (2) becomes a limit cycle iff $OY = OY'$.

We consider,

$$v(x, y) = \int_0^x g(u)\, du + \frac{1}{2}y^2 \tag{3}$$

and

$$v_{YQY'} = v_{Y'} - v_Y = \int\limits_{YQY'} dv. \tag{4}$$

It follows that

$$dv = ydy + gdx = Fdy, \tag{5}$$

so that

$$OY = OY' \Longleftrightarrow V_{YQY'} = 0. \tag{6}$$

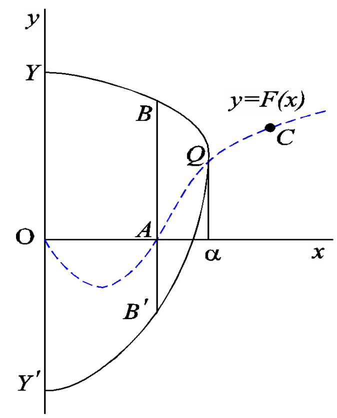

Figure 1: Typical path for the Lienard Theorem

We define,

$$G(x) = \int_0^x g(u)\, du,$$
$$y_+(0) = OY, \quad y_-(0) = OY',$$

and let Q has coordinates $(\alpha, F(\alpha))$. Let α' and α'' be respectively two positive roots of the equations

$$G(\alpha) = \frac{1}{2}y_+^2(0) - \frac{1}{2}F^2(\alpha) \tag{7}$$

$$\text{and} \quad G(\alpha) = \frac{1}{2}y_-^2(0) - \frac{1}{2}F^2(\alpha). \tag{8}$$

Also let,

$$\bar{\alpha} = \max\left\{\alpha', \alpha''\right\}. \tag{9}$$

We show that $V_{YQY'}$ has a simple zero at an $\alpha \leq \bar{\alpha}$ [8] for the system (2) in Lienard Theorem. It turns out that $\bar{\alpha}$ provides an efficient estimate of the amplitude of the unique limit cycle of the Van der Pol equation [8]. This result has been extended in [8] for the existence of exactly two limit cycles as stated in the following theorem.

Theorem 1. *Let f and g be two functions satisfying the following properties:*

(i) f and g are continuous;

(ii) F and g are odd functions and $g(x) > 0$ for $x > 0$;

(iii) F has $+ve$ simple zeros only at $x = a_1$, $x = a_2$ for some $a_1 > 0$ and some $a_2 > \bar{\alpha}$, $\bar{\alpha}$ being defined by 9 and $\bar{\alpha} < L$, where L is the first local maxima of $F(x)$ in $[a_1, a_2]$;

(iv) F is monotonic increasing in $a_1 < x \leq \bar{\alpha}$ and $F(x) \to -\infty$ as $x \to \infty$ monotonically for $x > a_2$.

Then the equation (1) has exactly two limit cycles around the origin.

It has been shown [8] that these two limit cycles are simple in the sense that neither can bifurcate under any small C^1 perturbation satisfying the conditions of Theorem 1. The existence of $\bar{\alpha}$ satisfying an equation of the form (9) ensures the existence of two distinct limit cycles.

3. Existence of Exactly N Limit Cycles for Lienard System

We generalize Theorem 1 as follows.

Theorem 2. *Let f and g be two functions satisfying the following properties:*

(i) f and g are continuous;

(ii) F and g are odd functions and $g(x) > 0$ for $x > 0$;

(iii) F has N number of $+ve$ simple zeros only at $x = a_i$, $i = 1, 2, \ldots, N$, where $0 < a_1 < a_2 < \ldots < a_N$ such that in each interval $I_i = [a_i, a_{i+1}]$, $i = 1, 2, \ldots, N-1$, there exists $\bar{\alpha}_i$, satisfying properties given by (9), such that $\bar{\alpha}_i < L_i$ where L_i is the unique extremum in I_i, $i = 1, \ldots, N-2$ and L_{N-1}, the first local extremum in $[a_{N-1}, a_N]$.

(iv) F is monotonic in $a_i < x \leq \bar{\alpha}_i$ $\forall$ i and $|F(x)| \to \infty$ as $x \to \infty$ monotonically for $x > a_N$.

Then the equation (1) has exactly N limit cycles around the origin, all are simple.

Proof. We shall prove the theorem by showing the result that each limit cycle intersects the $x-$axis at a point lying in the open interval $(\bar{\alpha}_i, \bar{\alpha}_{i+1}]$, $i = 0, 1, 2, \ldots, N-1$, where $\bar{\alpha}_0 = L_0$ is the local minima of $F(x)$ in $[0, a_1]$. By Lienard Theorem and Theorem 1 it follows that the result is true for $N = 1$ and $N = 2$. We shall now prove the theorem by induction. We assume that the theorem is true for $N = n-1$ and we shall prove that it is true for $N = n$. We prove the theorem by taking n as an odd $+ve$ integer so that $(n-1)$ is

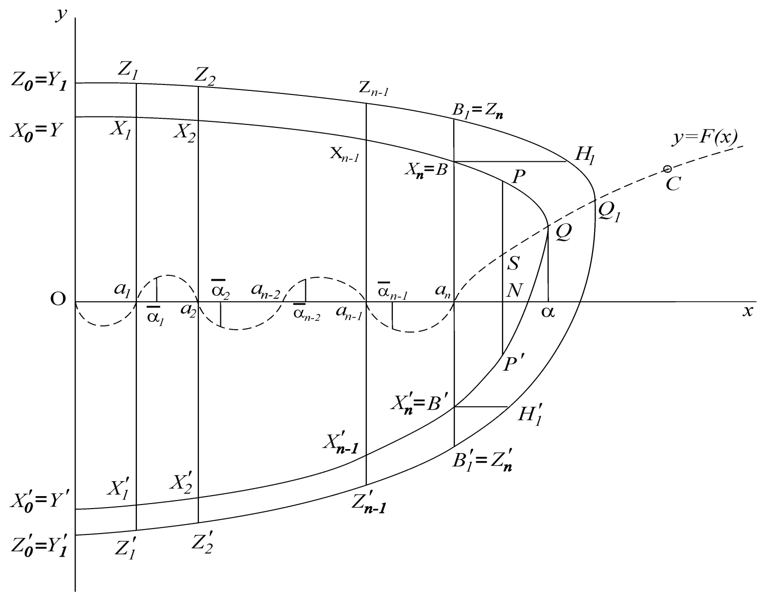

Figure 2:

even. The case for which n is even can similarly be proved and so is omitted. It can be shown that [4], $V_{YQY'}$ changes its sign from $+ve$ to $-ve$ as Q moves out of $A_1\,(a_1, 0)$ along the curve $y = F\,(x)$ and hence vanishes there due to its continuity and generates the first limit cycle around the origin. Next, in [8] we see $V_{YQY'}$ again changes its sign from $-ve$ to $+ve$ and generates the second limit cycle around the first. Also, we see that for existence of second limit cycle we need the existence of the point $\bar{\alpha}$, which we denote here as $\bar{\alpha}_1$.

Since by induction hypothesis the theorem is true for $N = n-1$, so it follows that in each and every interval $(\bar{\alpha}_k, \bar{\alpha}_{k+1}]$, $k = 0, 1, 2, \ldots, n-2$ the system (2) has a limit cycle and the outermost limit cycle cuts the x-axis somewhere in $(\bar{\alpha}_{n-1}, \infty)$. Also $V_{YQY'}$ changes its sign alternately as the point Q moves out of a_i's, $i = 1, 2, \ldots, n-1$. Since $(n-1)$ is even, it follows that $V_{YQY'}$ changes its sign from $+ve$ to $-ve$ as Q moves out of a_{n-2} along the curve $y = F\,(x)$. Since there is only one limit cycle in the region $(\bar{\alpha}_{n-1}, \infty)$, so it is clear that $V_{YQY'}$ must change its sign from $-ve$ to $+ve$ once and only once as Q moves out of $A_{n-1}\,(a_{n-1}, 0)$ along the curve $y = F\,(x)$. Also it follows that once $V_{YQY'}$ becomes $+ve$ so that it does not vanish further, otherwise we would get one more limit cycle, contradicting the hypothesis so that total number of limit cycle become n. We now try to find an estimate of α for which $V_{YQY'}$ vanishes for the last time.

We shall now prove that the result is true for $N = n$ and so we assume that all the hypotheses or conditions of this theorem are true for $N = n$. So, we get one more point $\bar{\alpha}_n$ and another root a_n, ensuring the fact that $V_{YQY'}$ vanishes as Q moves out of A_{n-1} through the curve $y = F(x)$, thus accommodating a unique limit cycle in the interval $(\bar{\alpha}_{n-1}, \bar{\alpha}_n]$.

By the result discussed so far it follows that $V_{YQY'} > 0$ when α lies in certain suitable small right neighbourhood of $\bar{\alpha}_{n-1}$. We shall prove that $V_{YQY'}$ ultimately becomes $-ve$ and remains $-ve$ as Q moves out of $A_n(a_n, 0)$ along the curve $y = F(x)$ generating the unique limit cycle and hence proving the required result for $N = n$.

We draw straight line segments $X_k X_k'$, $k = 1, 2, 3, \ldots, n$, passing through A_k and parallel to y-axis as shown in Figure 2. For convenience, we shall call the points X_n, X_n', Y, Y' as B, B', X_0, X_0' respectively. We write the curves

$$\Gamma_k = X_{k-1}X_k, \quad \Gamma_k' = X_k' X_{k-1}', \quad k = 1, 2, 3, \ldots, n,$$

so that

$$YQY' = X_0 Q X_0' = \sum_{k=1}^{n} \Gamma_k + X_n Q X_n' + \sum_{k=1}^{n} \Gamma_k' = \sum_{k=1}^{n} (\Gamma_k + \Gamma_k') + BQB'$$

and

$$V_{YQY'} = \sum_{k=1}^{n} \left(V_{\Gamma_k} + V_{\Gamma_k'} \right) + V_{BQB'}. \tag{10}$$

We shall prove the result through the following steps.

Step (A). As Q moves out of A_n along A_nC, $V_{\Gamma_k} + V_{\Gamma_k'}$ is $+ve$ and monotonic decreasing for odd k.

We choose two points $Q(\alpha, F(\alpha))$ and $Q_1(\alpha_1, F(\alpha_1))$ on the curve of $F(x)$, where $\alpha_1 > \alpha > a_n$. Let YQY' and $Y_1 Q_1 Y_1'$ be two phase paths through Q and Q_1 respectively. We have already taken $Y = X_0$, $Y' = X_0'$, $B = X_n$ and $B' = X_n'$. We now take $Y_1 = Z_0$, $Y_1' = Z_0'$, $B_1 = Z_n$, $B_1' = Z_n'$ and $Z_k Z_k'$ as the extension of the line segment $X_k X_k'$ $\forall\, k$. Also we write $Z_{k-1}Z_k = \Lambda_k$ and $Z_k' Z_{k-1}' = \Lambda_k'$. If k is odd, then on the segments Γ_k and Λ_k we have $y > 0$, $F(x) < 0$ and $y - F(x) > 0$. Now,

$$0 < [y - F(x)]_{\Gamma_k} < [y - F(x)]_{\Lambda_k}.$$

Since $g(x) > 0$ for $x > 0$ so we have

$$\left[\frac{-g(x)}{y - F(x)}\right]_{\Gamma_k} < \left[\frac{-g(x)}{y - F(x)}\right]_{\Lambda_k} < 0.$$

So, by (2) we get

$$\left[\frac{dy}{dx}\right]_{\Gamma_k} < \left[\frac{dy}{dx}\right]_{\Lambda_k} < 0. \tag{11}$$

Therefore by (11) we have

$$V_{\Gamma_k} = \int\limits_{\Gamma_k} F\ dy = \int\limits_{\Gamma_k} (-F)\left(-\frac{dy}{dx}\right) dx > \int\limits_{\Lambda_k} (-F)\left(-\frac{dy}{dx}\right) dx = \int\limits_{\Lambda_k} F\ dy = V_{\Lambda_k}.$$

Since $F(x)$ and $dy = \dot{y}dt = -g(x)\,dt$ are both $-ve$ along Λ_k for odd k, so we have

$$V_{\Gamma_k} > V_{\Lambda_k} = \int\limits_{\Lambda_k} F\ dy > 0. \tag{12}$$

Next, on the segments Γ'_k and Λ'_k we have $y < 0$, $F(x) < 0$ and $y - F(x) < 0$. Now,

$$0 > [y - F(x)]_{\Gamma'_k} > [y - F(x)]_{\Lambda'_k}.$$

So, by (2) we get

$$\left[\frac{dy}{dx}\right]_{\Gamma'_k} > \left[\frac{dy}{dx}\right]_{\Lambda'_k} > 0. \tag{13}$$

Therefore by (13) we have

$$V_{\Gamma'_k} = \int\limits_{\Gamma'_k} F\ dy = \int\limits_{-\Gamma'_k} (-F)\frac{dy}{dx}dx > \int\limits_{-\Lambda'_k} (-F)\frac{dy}{dx}dx = \int\limits_{\Lambda'_k} F\ dy = V_{\Lambda'_k}.$$

Since $F(x)$ and $dy = \dot{y}dt = -g(x)\,dt$ are both $-ve$ along Λ'_k for odd k, so we have

$$V_{\Gamma'_k} > V_{\Lambda'_k} = \int\limits_{\Lambda'_k} F\ dy > 0. \tag{14}$$

From (12) and (14) we have

$$V_{\Gamma_k} + V_{\Gamma'_k} > V_{\Lambda_k} + V_{\Lambda'_k} > 0.$$

Therefore $V_{\Gamma_k} + V_{\Gamma'_k}$ is $+ve$ and monotone decreasing as the point Q moves out of A_n along A_nC.

Step (B). As Q moves out from A_n along A_nC, $V_{\Gamma_k} + V_{\Gamma'_k}$ is $-ve$ and monotonic increasing for even k.

On the segments Γ_k and Λ_k we have $y > 0$, $F(x) > 0$ and $y - F(x) > 0$. Now,

$$0 < [y - F(x)]_{\Gamma_k} < [y - F(x)]_{\Lambda_k}.$$

Since $g(x) > 0$ for $x > 0$ so we have

$$\left[\frac{-g(x)}{y - F(x)}\right]_{\Gamma_k} < \left[\frac{-g(x)}{y - F(x)}\right]_{\Lambda_k} < 0.$$

So, by (2) we get

$$\left[\frac{dy}{dx}\right]_{\Gamma_k} < \left[\frac{dy}{dx}\right]_{\Lambda_k} < 0. \tag{15}$$

Therefore by (15) we have

$$V_{\Gamma_k} = \int_{\Gamma_k} F\ dy = \int_{\Gamma_k} F\frac{dy}{dx}dx < \int_{\Lambda_k} F\frac{dy}{dx}dx = \int_{\Lambda_k} F\ dy = V_{\Lambda_k}.$$

Since $F(x) > 0$ and $dy = \dot{y}dt = -g(x)\,dt < 0$ along Λ_k for even k, so we have

$$V_{\Gamma_k} < V_{\Lambda_k} = \int_{\Lambda_k} F\ dy < 0. \tag{16}$$

Next, on the segments Γ'_k and Λ'_k we have $y < 0$, $F(x) > 0$ and $y - F(x) < 0$. Now,

$$0 > [y - F(x)]_{\Gamma'_k} > [y - F(x)]_{\Lambda'_k},$$

so that by (2) we get

$$\left[\frac{dy}{dx}\right]_{\Gamma'_k} > \left[\frac{dy}{dx}\right]_{\Lambda'_k} > 0. \tag{17}$$

Therefore by (17) we have

$$V_{\Gamma'_k} = \int_{\Gamma'_k} F\ dy = \int_{-\Gamma'_k} F\left(-\frac{dy}{dx}\right)dx < \int_{-\Lambda'_k} F\left(-\frac{dy}{dx}\right)dx = \int_{\Lambda'_k} F\ dy = V_{\Lambda'_k}.$$

Since $F(x) > 0$ and $dy = \dot{y}dt = -g(x)\,dt < 0$ along Λ'_k for even k, so we have

$$V_{\Gamma'_k} < V_{\Lambda'_k} = \int_{\Lambda'_k} F\ dy < 0. \tag{18}$$

From (16) and (18) we have

$$V_{\Gamma_k} + V_{\Gamma'_k} < V_{\Lambda_k} + V_{\Lambda'_k} < 0.$$

Therefore $V_{\Gamma_k} + V_{\Gamma'_k}$ is $-ve$ and monotone increasing as the point Q moves out of A_n along A_nC.

Step (C). $V_{BQB'}$ is $-ve$ and monotone decreasing and tends to $-\infty$ as Q tends to infinity along A_nC.

On BQB' and $B_1Q_1B'_1$ we have $F(x) > 0$. We draw BH_1 and $B'H'_1$ parallel to x-axis. Since $F(x) > 0$ and $dy = \dot{y}dt = -g(x)\,dt < 0$ along $B_1Q_1B_1'$ so

$$V_{B_1Q_1B_1'} = \int\limits_{B_1Q_1B_1'} F\ dy \leq \int\limits_{H_1Q_1H_1'} F\ dy.$$

Since $[F(x)]_{H_1Q_1H_1'} \geq [F(x)]_{BQB'}$ and $F(x) > 0$, $dy = \dot{y}dt = -g(x)\,dt < 0$ along BQB' so we have

$$V_{B_1Q_1B_1'} \leq \int\limits_{H_1Q_1H_1'} F\ dy \leq \int\limits_{BQB'} F\ dy = V_{BQB'} < 0. \tag{19}$$

Let S be a point on $y = F(x)$, to the right of A_n and let BQB' be an arbitrary path, with Q to the right of S. The straight line $PSNP'$ is parallel to the y-axis. Since $F(x) > 0$ and $dy = \dot{y}dt = -g(x)\,dt < 0$ along BQB' and PQP' is a part of BQB' so we have

$$V_{BQB'} = \int\limits_{BQB'} F\ dy = -\int\limits_{B'QB} F\ dy \leq -\int\limits_{P'QP} F\ dy. \tag{20}$$

By hypothesis F is monotone increasing on A_nC and so $F(x) \geq NS$ on PQP' and hence (20) gives

$$V_{BQB'} \leq -\int\limits_{P'QP} NS \cdot dy = -NS \int\limits_{P'QP} dy = -NS \cdot PP' \leq NS \cdot NP.$$

But as Q goes to infinity towards the right, so $NP \to \infty$ and hence by the above relation it follows that $V_{BQB'} \to -\infty$.

Step (D). From steps (A) and (B) it follows that $\sum\limits_{k=1}^{n} \left(V_{\Gamma_k} + V_{\Gamma_k'}\right)$ in (10) is bounded. Therefore as Q moves to infinity from the right of A_n ultimately the quantity $V_{BQB'}$ dominates and hence $V_{YQY'}$ monotonically decreases to $-\infty$ to the right of A_n. The monotone decreasing nature of $V_{BQB'}$ inherits the same nature to $V_{YQY'}$ as Q moves out of A_n along the curve $y = F(x)$.

By the construction of $\bar{\alpha}_n$ it is clear that $V_{YQY'} > 0$ at a point on the left of A_n and ultimately it becomes $-ve$ when the point Q is at the right of A_n. So, by monotonic decreasing nature of $V_{YQY'}$ it can vanish only once as the point Q moves out of A_n along the curve $y = F(x)$. Thus, there is a unique path for which $V_{YQY'} = 0$. By (6) and symmetry of the path it follows that the path is closed and the proof is complete. □

4. Construction of a Lienard System with Desired Number of Limit Cycles

We now present an algorithm by which we can form a Lienard system with as many limit cycles as required. We present the technique for two limit cycles around a single critical point. This technique can similarly be extended for n number of limit cycles. As stated in the introduction, this algorithm is expected to become relevant in a physical model with partial or incomplete data.

Suppose in a given physical or dynamical problem, the function F of the Lienard equation (1) is well defined only with a finite interval $[-a_1, a_1]$ denoting $F(x) = f_1(x)$ for $x \in [-a_1, a_1]$ and satisfying the conditions:

(i) f_1 is a continuous odd function having only one $+ve$ zero a_1.

(ii) $xf_1(x) < 0 \ \forall \ x \in [-a_1, a_1]$.

(iii) f_1 has a unique local minimum at the point L_0 within (a_0, a_1), $a_0 = 0$.

Suppose it is also known that the system has a limit cycle just outside the interval $[-a_1, a_1]$. We have no information of $F(x)$ beyond the interval. Our aim is to develop an algorithm to determine a function f_2 as a restriction of F in an interval of the form $[a_1, a_2]$ so that it satisfies the conditions of Theorem 2, ensuring the second limit cycle outside $[a_1, a_2]$. Now to determine f_2 precisely from the information of f_1 in $[a_0, a_1]$ we need to define two functions ϕ_1 and H_1 so that we get the abscissa and ordinates of f_2 in the interval $[a_1, a_2]$ respectively. The choice of ϕ_1 is motivated by Odani's Choice function [6] (cf. Remarks 1 for further details). The functions ϕ_1 and H_1 are defined as follows: The functions ϕ_{1L} and ϕ_{1R} are bijective such that

$$\phi_{1L} : [a_0, L_0] \to [a_1, L_1], \quad \phi_{1L}(a_0) = a_1, \phi_{1L}(L_0) = L_1,$$
$$\phi_{1R} : [L_0, a_1] \to [L_1, a_2], \quad \phi_{1R}(L_0) = L_1, \phi_{1R}(a_1) = a_2,$$

$$\phi_1(s) = \begin{cases} \phi_{1L}(s), & s \in [a_0, L_0], \\ \phi_{1R}(s), & s \in [L_0, a_1], \end{cases}$$

and H_1 is monotone decreasing on $[0, f_1(L_0)]$ such that

$$H_1 \circ f_1 := f_2 \circ \phi_1. \tag{21}$$

To make the definition (21) explicit we define at first two monotone functions f_{2L}^* and f_{2R}^* and then introduce H_1 parametrically by the help of two monotone decreasing functions H_{1L} and H_{1R} on $[0, f_1(L_0)]$ as

$$H_{1L} : f_{1L}(s) \to f_{2L}^*(\phi_{1L}(s)), \quad s \in [a_0, L_0],$$
$$H_{1R} : f_{1R}(s) \to f_{2R}^*(\phi_{1R}(s)), \quad s \in [L_0, a_1],$$

$$H_1(x) = \begin{cases} H_{1L}(x), & \text{if } x = f_{1L}(s),\ s \in [a_0, L_0], \\ H_{1R}(x), & \text{if } x = f_{1R}(s),\ s \in [L_0, a_1]. \end{cases}$$

The choice of f_2^* is made on the basis of $f_1(x)$ defined on $[-a_1, a_1]$ and the second zero a_2 of $F(x)$ that must lie close to but nevertheless, less than the expected amplitude of the second limit cycle. We define the functions f_{2L} and f_{2R} as

$$f_{2L} : \phi_{1L}(s) \to H_{1L}(f_{1L}(s)), \quad s \in [a_0, L_0],$$
$$f_{2R} : \phi_{1R}(s) \to H_{1R}(f_{1R}(s)), \quad s \in [L_0, a_1].$$

We should note that in the definition of ϕ_{1L} and ϕ_{1R} we have used the conditions $\phi_{1L}(a_0) = a_1, \phi_{1L}(L_0) = L_1$ and $\phi_{1R}(L_0) = L_1, \phi_{1R}(a_1) = a_2$. We could also have used the conditions $\phi_{1L}(a_0) = L_1, \phi_{1L}(L_0) = a_1$ and $\phi_{1R}(L_0) = a_2, \phi_{1R}(a_1) = L_1$ instead, but in that case the function H_1 and H_2 must be monotone increasing.

If $x \in [a_1, L_1]$, then $x = \phi_{1L}(s)$ for some $s \in [a_0, L_0]$. Therefore,

$$f_{2L}(x) = f_{2L}(\phi_{1L}(s)) = H_{1L}(f_{1L}(s)) = f_{2L}^*(\phi_{1L}(s)) = f_{2L}^*(x).$$

So,

$$f_{2L} = f_{2L}^*.$$

Next, if $x \in [L_1, a_2]$, then $x = \phi_{1R}(s)$ for some $s \in [L_0, a_1]$. Therefore,

$$f_{2R}(x) = f_{2R}(\phi_{1R}(s)) = H_{1R}(f_{1R}(s)) = f_{2R}^*(\phi_{1R}(s)) = f_{2R}^*(x).$$

So,

$$f_{2R} = f_{2R}^*.$$

Thus the unknown functions f_{2L} and f_{2R} can be expressed by known functions f_{2L}^* and f_{2R}^* so that we have

$$f_2(x) = \begin{cases} f_{2L}(x), & x \in [a_1, L_1], \\ f_{2R}(x), & x \in [L_1, a_2], \end{cases} = \begin{cases} f_{2L}^*(x), & x \in [a_1, L_1], \\ f_{2R}^*(x), & x \in [L_1, a_2]. \end{cases}$$

Next we construct the restriction f_3 of the function F in $[a_2, a_3]$ having unique local maximum at L_2 (say) in (a_2, a_3). We assume two bijective functions

$$\phi_{2L} : [a_1, L_1] \to [a_2, L_2], \quad \phi_{2L}(a_1) = a_2, \phi_{2L}(L_1) = L_2,$$
$$\text{and } \phi_{2R} : [L_1, a_2] \to [L_2, a_3], \quad \phi_{2R}(L_1) = L_2, \phi_{2R}(a_2) = a_3,$$

and two more functions f_{3L}^* and f_{3R}^*. We define two monotone decreasing functions H_{2L} and H_{2R} on $[0, f_2(L_1)]$ parametrically as

$$H_{2L} : f_{2L}(s) \to f_{3L}^*(\phi_{2L}(s)), \quad s \in [a_1, L_1],$$
$$H_{2R} : f_{2R}(s) \to f_{3R}^*(\phi_{2R}(s)), \quad s \in [L_1, a_2].$$

We define

$$f_{3L} : \phi_{2L}(s) \to H_{2L}(f_{2L}(s)), \quad s \in [a_1, L_1],$$
$$f_{3R} : \phi_{2R}(s) \to H_{2R}(f_{2R}(s)), \quad s \in [L_1, a_2],$$

so that as shown above we have

$$f_{3L} = f_{3L}^* \text{ and } f_{3R} = f_{3R}^*.$$

Therefore,

$$f_3(x) = \begin{cases} f_{3L}(x), & x \in [a_2, L_2], \\ f_{3R}(x), & x \in [L_2, a_3], \end{cases} = \begin{cases} f_{3L}^*(x), & x \in [a_2, L_2], \\ f_{3R}^*(x), & x \in [L_2, a_3]. \end{cases}$$

We observe that

$$f_{3L} : \phi_{2L}(\phi_{1L}(s)) \to H_{2L}(f_{2L}(\phi_{1L}(s))), \quad s \in [a_0, L_0],$$
$$f_{3R} : \phi_{2R}(\phi_{1R}(s)) \to H_{2R}(f_{2R}(\phi_{1R}(s))), \quad s \in [L_0, a_1].$$

We can similarly proceed and construct all the restrictions f_k of the function F in $[a_{k-1}, a_k]$ for $k = 4, 5, 6, \ldots, N$ so that the corresponding Lienard system have exactly N limit cycles. Thus an incomplete Lienard system can be extended iteratively over larger and larger intervals of x, having as many (simple) limit cycles as desired. We note, however, that the choice of iterated functions has as large arbitrariness except for the required minimal conditions of monotonicity satisfying Theorem 2. The number of limit cycles for each such choices remain invariant. The problem of reconstructing data with a given number of limit cycles and having specified shapes is left for future study. We now illustrate the above construction by the following examples.

5. Examples

We now present some examples following the construction described in Section 4. Here the figures has been drawn using *Mathematica 5.1*.

Example 1. Let $a_1 = 0.2$, $a_2 = 0.5$ and

$$f_1(x) = 0.15 - 0.25\sqrt{1 - \frac{(x - 0.1)^2}{0.125^2}}, \quad -0.2 \le x \le 0.2.$$

Here, $L_0 = 0.1$. Let $L_1 = 0.3$. Let us choose

$$f_{2L}^*(x) = -0.15 + 0.25\sqrt{1 - \frac{(x - 0.3)^2}{(0.125)^2}},$$

$$f_{2R}^{*}(x) = -0.15 + 0.25\sqrt{1 - \frac{(x-0.3)^2}{(0.25)^2}}.$$

Also, let

$$\phi_{1L}(s) = \sqrt{As^2 + B}.$$

To determine the unknown parameters A and B we assume that $\phi_{1L}(a_0) = a_1$, $\phi_{1L}(L_0) = L_1$. Then $A = 5$ and $B = 0.04$. Next, let

$$\phi_{1R}(s) = \sqrt{A's^2 + B'}$$

with $\phi_{1R}(L_0) = L_1$ and $\phi_{1R}(a_1) = a_2$. Then, $A' = \frac{16}{3}$ and $B' = \frac{11}{300}$. Then following the algorithm in Section 4 we have

$$f_{2L} = f_{2L}^{*} \text{ in } [a_1, L_1] \quad \text{and} \quad f_{2R} = f_{2R}^{*} \text{ in } [L_1, a_2],$$

so that

$$f_2(x) = \begin{cases} f_{2L}(x), & x \in [a_1, L_1], \\ f_{2R}(x), & x \in [L_1, a_2]. \end{cases}$$

We now define

$$F_+(x) = \begin{cases} f_1(x), & 0 \le x < a_1, \\ f_2(x), & a_1 \le x < a_2, \\ -\frac{4}{3}(x - 0.5), & x \ge a_2, \end{cases}$$

to make F_+ continuously differentiable in $[0, \infty)$. The last part of the function F_+ is taken to make F_+ monotone decreasing for $x \ge a_2$ so that the function F defined below satisfy the condition that $|F(x)| \to \infty$ as $x \to \infty$ monotonically for $x \ge a_2$. We take

$$F(x) = \begin{cases} F_+(x), & x \ge 0, \\ F_-(x), & x < 0. \end{cases}$$

We find two limit cycles which cross the $+ve$ y-axis at the points $(0, 0.26731065)$ and $(0, 0.5749823)$ respectively. So, $y_+(0) = y_-(0) = 0.26731065$ and $\bar{\alpha}_1 = 0.254219124$. So, the conditions $\bar{\alpha}_1 \le L_1$ are satisfied in this example. Thus the existence of limit cycles are ensured by Theorem 2 with $g(x) = x$ establishing the construction in Section 4. The limit cycles along with the curve of $F(x)$ has been shown in Figure 3.

Remark 1. From condition $(C2)$ in [6] we see that

$$g(\phi_k(x))\,\phi_k'(x) \ge g(x).$$

If $g(x) = x$, then it gives

$$\phi_k(x)\,\phi_k'(x) \ge x.$$

Thus, in Example 1 if we take $\phi_k(s) = \phi_{1L}(s) = \sqrt{As^2 + B}$, then the above

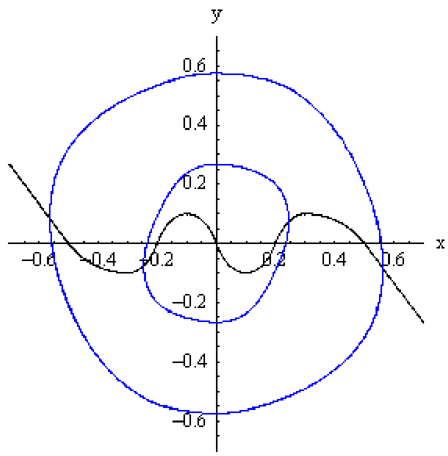

Figure 3: Limit cycles for the system in Example 1 with the curve of $F(x)$

inequality gives

$$\sqrt{As^2+B}\cdot\frac{2As}{2\sqrt{As^2+B}} \geq s,$$
$$\text{i.e.,} \qquad As \geq s,$$
$$\text{i.e.,} \qquad A \geq 1.$$

By the definition of f_{2L}^{*} and f_2 it follows that the remaining part of the condition $(C2)$ is satisfied if

$$|F(\phi_{1L}(s))| \geq |F(s)|, \quad s \in [a_0, L_0],$$

and in particular

$$|F(\phi_{1L}(0))| = |F(0)|.$$

Since, $\phi_{1L}(s) \in [a_1, a_2]$ and $s \in [a_0, L_0]$ so it gives

$$|f_2(\phi_{1L}(s))| \geq |f_1(s)|, \quad s \in [a_0, L_0],$$
$$\text{i.e.,}\ |H_{1L}(f_{1L}(s))| \geq |f_{1L}(s)|, \quad s \in [a_0, L_0]. \tag{22}$$

Next, in particular the equality occurs at $s = a_0 = 0$ and so we have

$$|F(\phi_{1L}(0))| = |F(0)| \quad \Longrightarrow |H_{1L}(f_{1L}(0))| = 0$$
$$\Longrightarrow |H_{1L}(0)| = 0 \quad \Longrightarrow H_{1L}(0) = 0$$

since $F(0) = 0$ and $f_{1L}(0) = 0$. By our construction we also see

$$s \cdot H_1(s) < 0 \quad \forall\ s.$$

Thus, ϕ_{1L} behaves like choice function described by Odani. Here, the condition (22) does not hold for the system discussed in Example 1. In fact, here

$$\left|H_{1L}\left(f_{1L}\left(s\right)\right)\right| \leq \left|f_{1L}\left(s\right)\right|, \quad s \in \left[a_0, L_0\right].$$

However, the conditions (viz. $\bar{\alpha}_i < L_i$, etc.) of Theorem 2 are satisfied ensuring the existence of exactly two limit cycles. This shows that Theorem 2 and the construction presented above covers a larger class of functions F than those covered in [6]. The equality in (22) occurs in Example 1 only at the point $s = a_0 = 0$. However, the equality can occur at points where $s \neq a_0$. We present Example 2 below to show this kind of behaviour.

Remark 2. The function f_2 in Example 1 is obtained from f_1 by reflection and translation along x-axis. However, it is clear from the construction of Section 4, there is a plenty of freedom in the possible extensions of f_1 having a fixed number of limit cycles, as illustrated in Examples 2 and 3. In these examples we consider more general transformations so that the limit cycles are obtained having amplitudes close to those expected from the given physical (dynamical) problem.

Example 2. In this example our target is to construct an example in which

$$F_+\left(x\right) = \begin{cases} 0.055518 - 0.08\sqrt{1 - \dfrac{\left(x - 0.144\right)^2}{0.04}}, & 0 \leq x \leq 0.144, \\ 0.148506 - 0.172988\sqrt{1 - \dfrac{\left(x - 0.144\right)^2}{\left(0.206686\right)^2}}, & 0.144 < x \leq 0.34, \\ 0.0910146 + 0.0209854\sqrt{1 - \dfrac{\left(x - 0.407\right)^2}{\left(0.06751554\right)^2}}, & 0.34 < x \leq 0.407, \\ -0.2280727 + 0.340073\sqrt{1 - \dfrac{\left(x - 0.407\right)^2}{\left(0.125376\right)^2}}, & 0.407 < x \leq 0.5, \\ -3.0000372\left(x - 0.5\right), & x > 0.5, \end{cases}$$

and

$$F\left(x\right) = \begin{cases} F_+\left(x\right), & x \geq 0, \\ F_-\left(x\right), & x < 0. \end{cases}$$

Here, $a_1 = 0.2$, $a_2 = 0.5$, $L_0 = 0.144$ and $L_1 = 0.407$. It is easy to show that $\phi_{1L}\left(s\right) = \sqrt{4.974392361 \cdot s^2 + 0.0625}$ and $\phi_{1R}\left(s\right) = \sqrt{2.019706 \cdot s^2 + 0.12376838}$.

Here,

$$f_1(x) = \begin{cases} 0.055518 - 0.08\sqrt{1 - \dfrac{(x-0.144)^2}{0.04}}, & 0 \le x \le 0.144, \\ 0.148506 - 0.172988\sqrt{1 - \dfrac{(x-0.144)^2}{(0.206686)^2}}, & 0.144 < x \le 0.2, \end{cases}$$

$$f_2(x) = \begin{cases} 0.148506 - 0.172988\sqrt{1 - \dfrac{(x-0.144)^2}{(0.206686)^2}}, & 0.2 < x \le 0.34, \\ 0.0910146 + 0.0209854\sqrt{1 - \dfrac{(x-0.407)^2}{(0.06751554)^2}}, & 0.34 < x \le 0.407, \\ -0.2280727 + 0.340073\sqrt{1 - \dfrac{(x-0.407)^2}{(0.125376)^2}}, & 0.407 < x \le 0.5. \end{cases}$$

The second part of the condition $(C2)$ in [6], i.e., the condition (22) does not hold. In fact,

$$\begin{aligned} &|H_{1L}(f_{1L}(s))| < |f_{1L}(s)| \text{ in } (0, 0.05290111) \\ \text{and } &|H_{1L}(f_{1L}(s))| > |f_{1L}(s)| \text{ in } (0.05290111, 0.144). \end{aligned}$$

The equality occurs at $s = 0$ and $s = 0.05290111$. Here, we get two limit cycles crossing the $+ve$ y-axis at the points $(0, 0.29039755)$ and $(0, 0.567249)$ respectively so that $y_+(0) = y_-(0) = 0.29039755$ and $\bar{\alpha}_1 = 0.2892792083$. Consequently, $\bar{\alpha}_1 \le L_1$ and the other conditions of Theorem 2 with $g(x) = x$ are satisfied justifying the existence of exactly two limit cycles. These two limit cycles along with the curve of $F(x)$ has been shown in Figure 4.

Example 3. We now consider an example involving three limit cycles by taking $a_1 = 0.1$, $a_2 = 0.2$, $a_3 = 0.4$ and

$$f_1(x) = 0.04422166 - 0.08\sqrt{1 - \frac{(x-0.05)^2}{(0.06)^2}}, \quad -0.1 \le x \le 0.1.$$

Here $L_0 = 0.05$ and let $L_1 = 0.15$, $L_2 = 0.3$. We take

$$f_{2L}^*(x) = -0.04422166 + 0.08\sqrt{1 - \frac{(x-0.15)^2}{(0.06)^2}},$$

$$f_{2R}^*(x) = -0.04422166 + 0.08\sqrt{1 - \frac{(x-0.15)^2}{(0.06)^2}}.$$

It is easy to construct

$$\phi_{1L}(s) = \sqrt{5s^2 + 0.01}, \quad s \in [a_0, L_0],$$

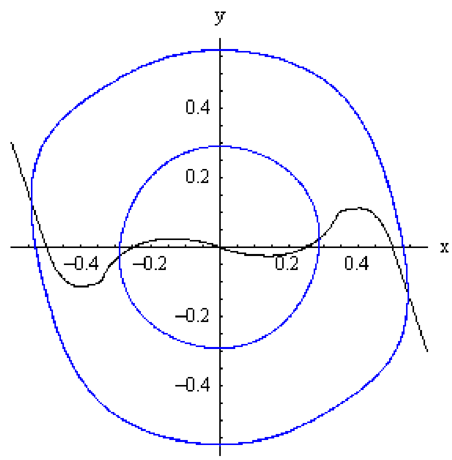

Figure 4: Limit cycles for the system in Example 2 with the curve of $F(x)$

$$\phi_{1R}(s) = \sqrt{\frac{7}{3}s^2 + \frac{5}{300}}, \quad s \in [L_0, a_1].$$

Next, we take

$$f^*_{3L}(x) = 0.0043819183 - 0.03\sqrt{1 - \frac{(x-0.3)^2}{(0.101084111)^2}},$$

$$f^*_{3R}(x) = 0.0043819183 - 0.03\sqrt{1 - \frac{(x-0.3)^2}{(0.101084111)^2}}.$$

We can similarly construct

$$\phi_{2L}(s) = \sqrt{4s^2} = 2s, \quad s \in [a_1, L_1],$$

$$\phi_{2R}(s) = 2s, \quad s \in [L_1, a_2],$$

so that $\phi_{2L}(a_1) = a_2$, $\phi_{2L}(L_1) = L_2$, $\phi_{2R}(L_1) = L_2$ and $\phi_{2R}(a_2) = a_3$. We

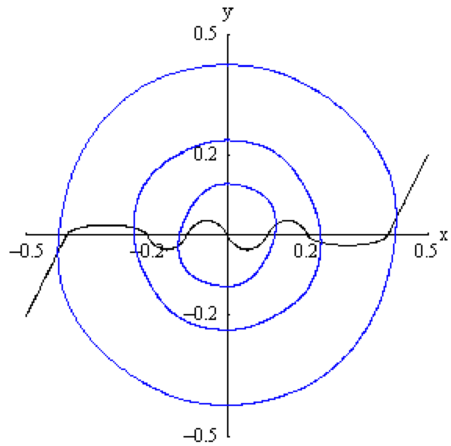

Figure 5: Three limit cycles for the system in Example 3 with the curve of $F(x)$

define

$$F_+(x) = \begin{cases} 0.04422166 - 0.08\sqrt{1 - \dfrac{(x-0.05)^2}{(0.06)^2}}, & 0 \le x < 0.1, \\ -0.04422166 + 0.08\sqrt{1 - \dfrac{(x-0.15)^2}{(0.06)^2}}, & 0.1 \le x < 0.2, \\ 0.0043819183 - 0.03\sqrt{1 - \dfrac{(x-0.3)^2}{(0.101084111)^2}}, & 0.2 \le x < 0.4, \\ 2.0100758\,(x-0.4), & 0.4 \le x. \end{cases}$$

$$F(x) = \begin{cases} F_+(x), & x \ge 0, \\ F_-(x), & x < 0, \end{cases}$$

to make F continuously differentiable. We can easily calculate that $\bar{\alpha}_1 = 0.12418214965$ and $\bar{\alpha}_2 = 0.2354818163$ and consequently $\bar{\alpha}_i < L_i$ for $i = 1, 2$. All the other conditions of Theorem 2 with $g(x) = x$ are satisfied and hence we get three distinct limit cycles as shown in Figure 5 along with the curve of $F(x)$ defined above.

Remark 3. Here the function F is defined in such a manner that $|F(L_0)| > |F(L_2)|$ implying that β_2 mentioned in Theorem 3 of [1] or in Theorem 7.12, Chapter 4 of the book [9] does not exist and hence these theorems are not applicable for the corresponding Lienard system.

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