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WEAKLY GOLDIE EXTENDING MODULES

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Abstract: As a proper generalization of Goldie extending module, we introduce the concept of weakly $\mathscr{G}$-extending module. Let $X, Y \leq M$. Then $X\gamma Y$ if and only if there exist submodules A and A' of M such that $A \cong A'$, $A \leq_e X$ and $A' \leq_e Y$. M is called a weakly $\mathscr{G}$-extending module if for every submodule X of M, there exists a direct summand D of M such that $X\gamma D$. Let $M = M_1 \oplus M_2$, where M_1 and M_2 are weakly $\mathscr{G}$-extending. It is shown that if M_1 is M_2-ejective (or M_2 is M_1-ejective), then M is weakly $\mathscr{G}$-extending. Finally, we show that if M is weakly $\mathscr{G}$-extending, then so is its rational hull.

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1. Introduction

Throughout this paper all rings are associative with unity and all modules will be unital right R-modules. Let M be a module and $A \leq M$. A is called essential in M (denoted by $A \leq_e M$) if for any $X \leq M, A \cap X = 0$ implies $X = 0$. For $A \leq B \leq M$, if A is essential in B, then B is called an essential extension of A in M. A submodule A is said to be closed in M (denoted by $A \leq_c M$), if A has no proper essential extension in M. For $A \leq B \leq M$, A is said to be a closure of B if $B \leq_e A$ and A is closed in M. A module M is $\mathscr{G}$-extending (i.e., Goldie extending) if for every submodule X of M, there exists a direct summand D of M such that $X\beta D$. In this paper, as a proper generalization of $\mathscr{G}$-extending module, we introduce the concept of weakly $\mathscr{G}$-extending module.

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Some properties of weakly $\mathscr{G}$-extending modules is also weakly $\mathscr{G}$-extending are given.

2. Results

Definition 2.1. Let M be a module. We consider the following relations on the set of submodules of M:

(i) $X\alpha Y$ if and only if there exists $A \leq M$ such that $X \leq_e A$ and $Y \leq_e A$;

(ii) $X\beta Y$ if and only if there exists $A \leq M$ such that $A \leq_e X$ and $A \leq_e Y$;

(iii) $X\gamma Y$ if and only if there exist submodules A and A' of M such that $A \cong A'$, $A \leq_e X$ and $A' \leq_e Y$.

Note that if $X, Y \leq M$ with $X\alpha Y$, then $X\beta Y$. If $X, Y \leq M$ with $X\beta Y$, then $X\gamma Y$. A module M is extending if for every submodule X of M, there exists a direct summand D of M such that $X\alpha D$. A module M is $\mathscr{G}$-extending (i.e., Goldie extending) if for every submodule X of M, there exists a direct summand D of M such that $X\beta D$. In this note, we introduce the following concept.

Definition 2.2. A module M is weakly $\mathscr{G}$-extending if for every submodule X of M, there exists a direct summand D of M such that $X\gamma D$.

Clearly, every $\mathscr{G}$-extending module is weakly $\mathscr{G}$-extending.

Definition 2.3. (see [1]) Let M, N be modules. Then N is M-ejective if, for each $K \leq M$ and each homomorphism $f : K \to N$, there exists a homomorphism $g : M \to N$ and an $X \leq_e K$ such that $g(x) = f(x)$, for all $x \in X$.

Lemma 2.4. *(see [1]) Let M_1 and M_2 be modules such that $M = M_1 \oplus M_2$. Then M_1 is M_2-ejective if and only if for every submodule K of M with $K \cap M_1 = 0$ there exists a submodule M_3 of M such that $M = M_1 \oplus M_3$ and $K \cap M_3 \leq_e K$.*

Proposition 2.5. *Let N be a semisimple module and M a module with zero socle. If $L = N \oplus M$ is $\mathscr{G}$-extending, then N is M-ejective.*

Proof. Let K be any submodule of L with $K \cap N = 0$. Since L is $\mathscr{G}$-extending, there exists a direct summand D of L such that $K \cap D \leq_e K$ and $K \cap D \leq_e D$. Write $L = D \oplus D', D' \leq L$. Clearly, $D \cap N = 0$. Then $\mathrm{Soc}(D) = D \cap \mathrm{Soc}(L) = D \cap \mathrm{Soc}(N) = 0$. Hence $N =\mathrm{Soc}(L) =\mathrm{Soc}((D') \leq$

D'. Then $D' = N \oplus (D' \cap M)$. Now $L = N \oplus D \oplus (D' \cap M)$. Obviously, $K \cap (D \oplus (D' \cap M)) \leq_e K$. Therefore, N is M-ejective by Lemma 2.4. □

Theorem 2.6. *Let $M = M_1 \oplus M_2$, where M_1 and M_2 are weakly $\mathscr{G}$-extending. If M_1 is M_2-ejective (or M_2 is M_1-ejective), then M is weakly $\mathscr{G}$-extending.*

Proof. Let Y be a submodule of M. If $Y \cap M_1 = 0$, by Lemma 2.4, there exists a submodule M_3 of M such that $M = M_1 \oplus M_3$ and $Y \cap M_3 \leq_e Y$. Since $M_3 \cong M_2$, M_3 is weakly $\mathscr{G}$-extending. Then there exists a direct summand D of M such that $(Y \cap M_3)\gamma D$, i.e., there exist submodules A and A' of M such that $A \cong A'$, $A \leq_e Y \cap M_3$ and $A' \leq_e D$. Now assume that $Y \cap M_1 \neq 0$. Then there exists a submodule K of Y such that $(Y \cap M_1) \oplus K \leq_e Y$. Hence $K \cap M_1 = 0$. Thus there exists a submodule M_4 of M such that $M = M_1 \oplus M_4$ and $K \cap M_4 \leq_e K$. Since $M_4 \cong M_2$, M_4 is also weakly $\mathscr{G}$-extending. Then there exists a direct summand D_1 of M_1 such that $(Y \cap M_1)\gamma D_1$, i.e., there exist submodules B and B' of M_1 such that $B \cong B'$, $B \leq_e Y \cap M_1$ and $B' \leq_e D_1$. There exists a direct summand D_2 of M_4 such that $(K \cap M_4)\gamma D_2$, i.e., there exist submodules C and C' of M_4 such that $C \cong C'$, $C \leq_e K \cap M_4$ and $C' \leq_e D_2$. Hence $B \oplus C \leq_e Y$ and $B' \oplus C' \leq_e D_1 \oplus D_2$. Clearly, $B \oplus C \cong B' \oplus C'$ and $D_1 \oplus D_2$ is a direct summand of M. Therefore, M is weakly $\mathscr{G}$-extending. □

Corollary 2.7. *Let $M = M_1 \oplus M_2$, where M_1 and M_2 are weakly $\mathscr{G}$-extending. If M_1 is M_2-injective (or M_2 is M_1-injective), then M is weakly $\mathscr{G}$-extending.*

By analogy with the proof of [1, Corollary 3.2], we can obtain

Corollary 2.8. *Let $M = M_1 \oplus M_2 \oplus \cdots \oplus M_n$ be a finite direct sum. Then if M_i is M_j-ejective for all $j > i$ and each M_i is weakly $\mathscr{G}$-extending, then M is weakly $\mathscr{G}$-extending.*

Corollary 2.9. *Let $M = M_1 \oplus M_2$. Then:*

(i) If M_1 is injective and M_2 is weakly $\mathscr{G}$-extending, then M is weakly $\mathscr{G}$-extending.

(ii) If M_1 is weakly $\mathscr{G}$-extending and M_2 is semisimple, then M is weakly $\mathscr{G}$-extending.

Proposition 2.10. *Let $M = M_1 \oplus M_2$, where M_1 and M_2 are weakly $\mathscr{G}$-extending. If M is a duo module, then M is weakly $\mathscr{G}$-extending.*

Proof. Let Y be a submodule of M. Then $Y = (Y \cap M_1) \oplus (Y \cap M_2)$. Since M_1 is weakly $\mathscr{G}$-extending, there exists a direct summand D_1 of M_1 such that

$(Y \cap M_1)\gamma D_1$, i.e., there exist submodules A and A' of M_1 such that $A \cong A'$, $A \leq_e Y \cap M_1$ and $A' \leq_e D_1$. Since M_2 is weakly $\mathscr{G}$-extending, there exists a direct summand D_2 of M_2 such that $(Y \cap M_2)\gamma D_2$, i.e., there exist submodules B and B' of M_2 such that $B \cong B'$, $B \leq_e Y \cap M_2$ and $B' \leq_e D_1$. Hence $A \oplus B \leq_e Y$ and $A' \oplus B' \leq_e D_1 \oplus D_2$. Therefore, M is weakly $\mathscr{G}$-extending. □

Proposition 2.11. *Let $K \leq_e M$ such that K is weakly $\mathscr{G}$-extending and for each $e^2 = e \in$End(K) there exists $\bar{e}^2 = \bar{e} \in$End(M) such that $\bar{e}|_K = e$. Then M is weakly $\mathscr{G}$-extending.*

Proof. Assume K is weakly $\mathscr{G}$-extending. Let $X \leq M$. There exists $e^2 = e \in$End(K) such that $(X \cap K)\gamma(eK)$, i.e., there exist submodules A and A' of K such that $A \cong A'$, $A \leq_e X \cap K$ and $A' \leq_e eK$. By hypothesis, there exists $\bar{e}^2 = \bar{e} \in$End(M) such that $\bar{e}|_K = e$. Let $0 \neq \bar{e}m \in \bar{e}M$. There exists $r \in R$ such that $0 \neq \bar{e}mr \in K$. Then $0 \neq \bar{e}mr = \bar{e}(\bar{e}mr) = e(\bar{e}mr) \in eK$. Hence $eK \leq_e \bar{e}M$. Then $A \leq_e X$ and $A' \leq_e (\bar{e}M)$. Thus $X\gamma(\bar{e}M)$. Therefore, M is weakly $\mathscr{G}$-extending. □

By analogy with the proof of [2, Corollary 3.14], we can obtain

Corollary 2.12. *Let M be a module. If M is weakly $\mathscr{G}$-extending, then so is the rational hull of M.*

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