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A NOTE ON DISCRETE MORSE THEORY AND ITS APPLICATIONS

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Abstract: We investigate the key notions of discrete Morse theory on CW-complexes including discrete Morse functions and discrete gradient vector fields. An application on torus will be presented.

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1. Introduction

Morse Theory describes the relationship between critical points of a function and the homotopy type of the function's domain. It is also a fundamental tool for investigating the topology on the smooth manifold M and the critical points of a smooth function f on the manifold. The classical approach is beautifully written in J. Milnor's book [5]. Discrete Morse theory is combinatorially constructed from a given CW-complex and a second CW-complex that is homotopy equivalent to the first but has fewer cells. It has been shown by the recent work of Robin Forman in [2] and [3]. Almost all concepts about the classical Morse theory have a discrete counterpart for simplical and cell complexes.

In this paper, we first review Forman's discrete Morse theory. Then we give an example about discrete Morse functions and discrete gradient vector fields on a tours. A discussion about further applications of this theory will be presented in the last section.

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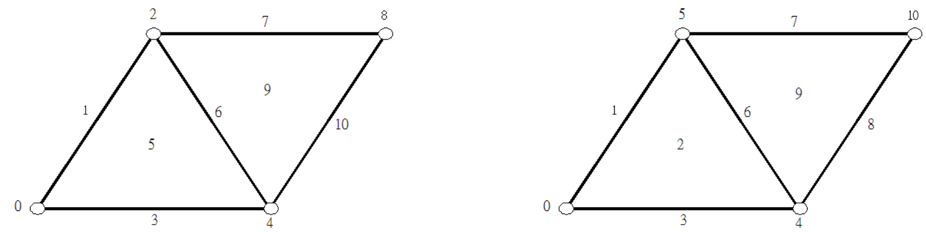

Figure 1: (i) f is a discrete Morse function; (ii) f is not a discrete Morse function

2. Basic Ideas in Discrete Morse Theory

First we introduce the basics in discrete morse theory (c.f. [3]). Recalling that a finite simplicial complex is a finite set of vertices V, along with a set of subsets K of V such that K satisfies the following two properties :

(1) $V \subseteq K$, where V is a finite set of vertices.

(2) If $\alpha \in K$ and $\beta \subseteq \alpha$ then $\beta \in K$.

The elements of K are called simplices. If $\alpha \in K$ and α contains $p+1$ vertices, then we say the dimension of α is p and denote it by $\alpha^{(p)}$. The notation $\alpha \prec \beta$ or $\beta \succ \alpha$ means that α is a subset of β, and we say that α is a face of β.

Definition 1. A function $f : K \to \mathbf{R}$ is a discrete Morse function if for every $\alpha^{(p)} \in K$, the following two conditions hold:

(1) $\#\left\{\beta^{(p+1)} \succ \alpha \mid f(\beta) \leq f(\alpha)\right\} \leq 1$.

(2) $\#\left\{\gamma^{(p-1)} \prec \alpha \mid f(\gamma) \geq f(\alpha)\right\} \leq 1$.

Consider the two complexes as in Figure 1. The function (i) is a discrete Morse function. The function (ii) is not a discrete Morse function because $f^{-1}(2)$ has 2 lower dimensional neighbors on which f takes on high values, and $f^{-1}(10)$ has two higher dimensional neighbors on which f takes on lower values. Note that a discrete Morse function is not a continuous function on K. It is just an assignment of a single number to each complex.

Definition 2. Let f: $K \to \mathbf{R}$ is a discrete morse function. A simplex $\alpha^{(p)}$ is critical if the following two condition hold:

(1)$\#\left\{\beta^{(p+1)} \succ \alpha \mid f(\beta) \leq f(\alpha)\right\} = 0$.

(2)$\#\left\{\gamma^{(p-1)} \prec \alpha \mid f(\gamma) \geq f(\alpha)\right\} = 0$.

Consider the complex in Figure 1(i). The vertex $f^{-1}(0)$ is critical because it has no higher dimensional neighbors on which f takes on lower values, and there are no other critical simplices.

Lemma 3. *If K is a simplicial complex with a discrete Morse function f, then for any simplex α, either*

$$\#\left\{\beta^{(p+1)} > \alpha \mid f(\beta) \leq f(\alpha)\right\} = 0, \qquad \#\left\{\gamma^{(p-1)} < \alpha \mid f(\gamma) \geq f(\alpha)\right\} = 0.$$

Theorem 4. *Suppose that K is a simplicial complices with a discrete morse function f. Then K is homotopy equivalent to a CW-complex with exactly one cell of dimension p for each critical simplex of dimension p.*

For $c \in \mathbf{R}$, define $K(c) = \cup_{f(\alpha)\leq c} \cup_{\tau \preceq \alpha} \tau$. That is, $K(c)$ denotes all cells on which f is $\leq c$, as well as all of their faces. In particular, $K(c)$ is a subcomplex of K. If a cell α with $f(\alpha) > c$ lies in $K(c)$ we must see if there is any τ with $\alpha \prec \tau$ and $f(\tau) \leq c$. In fact, it is enough to consider τ with dimτ =dimα+1

Theorem 4 follows from two basic lemmas. They have analog in smooth Morse theory (c.f. [3]).

Lemma 5. *If there are no critical simplices $\alpha^{(p)}$ with $f(\alpha) \in [a, b]$, then $K(b)$ is a homotopy equivalent to $K(a)$.*

Proof. If $f^{-1}([a, b]) = \emptyset$ then $K(a) = K(b)$, so there is nothing to prove.

Otherwise, by partitioning $[a, b]$ into smaller if necessary, we may assume there is a single noncritical cell α with $f(\alpha) \in [a, b]$. Thus, exactly one of the following holds:

(i) There is a $\tau^{(p+1)} \succ \alpha$ with $f(\tau) \leq f(\alpha)$.

(ii) There is a $\nu^{(p-1)} \prec \alpha$ with $f(\nu) \geq f(\alpha)$.

In case (i), we must have $f(\tau) < a$. Thus $\tau \subseteq K(a)$. Since α is a face of τ, we have $\alpha \subseteq K(a)$ so that $K(a) = K(b)$ and again there is nothing to prove.

In case (ii), we have $f(\nu) > b$. If $\tilde{\nu} \neq \nu$ is any other $(p-1)$-face of α, we have $f(\tilde{\nu}) < f(\alpha)$ so that $f(\tilde{\nu}) < a$. Thus, $\tilde{\nu}$ and all its faces are contained in $K(a)$.

Let $\tilde{\alpha}^{(p)} \neq \alpha$ be any other p-face of K with $\tilde{\alpha} > \nu$, this implies $f(\tilde{\alpha}) > f(\nu) > b$. Thus, $\nu \cap K(a) = \emptyset$. It follows that $K(b)$ can be expressed as a disjoint union $K(b) = K(a) \cup \alpha \cup \nu$, where ν is a free face of α. Therefore, $K(b)$ is homotopy equivalent to $K(a)$. ☐

Lemma 6. *If there is a single critical simplex $\alpha^{(p)}$ with $f(\alpha) \in [a, b]$ then $K(b)$ is homotopy equivalent to $K(a) \cup_{\dot{e}^p} e^p$, where e^p denotes a p-dimensional*

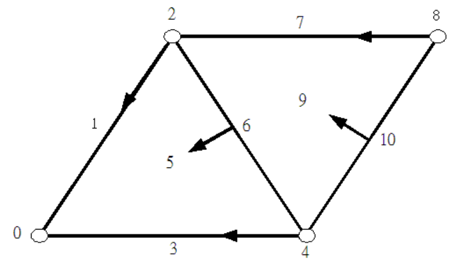

Figure 2: A gradient vector field

cell with boundary $\dot{e}^p$.

Proof. We assume $a < a' < b' < b$ with $\alpha = f^{-1}([a', b']$. By Lemma 5, $K(a)$ is homotopy equivalent to $K(a')$ and $K(b)$ is homotopy equivalent to $K(b')$. Thus, $K(b')$ is homotopy equivalent to $K(a') \cup_{\dot{e}^p} e^p$.

Since α is critical, if $\tau^{(p+1)} \succ \alpha$ we have $f(\tau) > f(\alpha)$ so that $f(\tau) > b'$. If τ is any face of K with $\tau > \alpha$ then $f(\tau) > b'$. Thus $\alpha \cap K(a') = \emptyset$.

It also follows from the criticality of α that for every $\nu^{(p-1)} \prec \alpha$ we have $f(\nu) < f(\alpha)$ so that $f(\nu) < a'$ which implies $\nu \subseteq K(a')$. Therefore, $\dot{\alpha} \subseteq K(a')$ so that $K(b') = K(a') \cup_{\dot{\alpha}} \alpha$. Since α is homeomorphic to e^p, the lemma follows. □

A discrete Morse function gives us a way to build the simplicial complex by attaching the simplices in the order prescribed by the function, i.e. adding first the simplices which are assigned by the the smallest values. For a simplicial complices K with a discrete morse function f , we will draw the arrows as follows. If $\alpha^{(p)}$ is a non-critical simplex with $\beta^{(p+1)} \succ \alpha$ satisfying $f(\beta) \leq f(\alpha)$, we draw an arrow from α to β. By Lemma 2.3, every simplex α satisfies exactly one of the following:

(1) α is the tail of exactly one arrow.

(2) α is the head of exactly one arrow.

(3) α is neither the head nor the tail of an arrow.

Note that a simplex is critical if and only if it is neither the head nor the tail of an arrow. A discrete gradient vector field V is a collection of pairs $\left\{\alpha^{(p)} \prec \beta^{(p+1)}\right\}$ of simplices of K, where $\left\{\alpha^{(p)} \prec \beta^{(p+1)}\right\}$ is in V if and only if $f(\beta) \leq f(\alpha)$. In other words, $\left\{\alpha^{(p)} \prec \beta^{(p+1)}\right\}$ is in V if and only if we draw an arrow that has α as its tail, and β as its head.

In Figure 2, we draw an arrow that has $f^{-1}(2)$ as its tail, $f^{-1}(1)$ as its head, and draw an arrow that has $f^{-1}(10)$ as its tail, $f^{-1}(9)$ as its head. The critical point $f^{-1}(0)$ is neither the head nor the tail of an arrow.

Definition 7. A discrete vector field V on K is a collection of pairs $\{\alpha^{(p)} < \beta^{(p+1)}\}$ of simplices of K such that each simplex is in at most one pair of V.

Definition 8. Let V be a discrete vector field on K, a V-path is a sequence of simplices

$$\alpha_0^{(p)}, \beta_0^{(p+1)}, \alpha_1^{(p)}, \beta_1^{(p+1)}, \cdots\cdots, \beta_r^{(p+1)}, \alpha_{r+1}^{(p)}$$

such that $i = 0, \cdots, r$, $\{\alpha \prec \beta\} \in V$ and $\beta_i \succ \alpha_{i+1} \neq \alpha_i$. Such a path is called a non-trivial closed path if $r \geq 0$ and $\alpha_0 = \alpha_{r+1}$. If V is the gradient vector field of a discrete morse function f, then we refer to a V-path as a gradient path of f.

In Figure 2, we can find

$$f^{-1}(10), f^{-1}(9), f^{-1}(6), f^{-1}(5)$$

is a V-path, and

$$f^{-1}(8), f^{-1}(7), f^{-1}(2), f^{-1}(1), f^{-1}(0)$$

is also a V-path.

Theorem 9. *Let V be a gradient vector field of a discrete Morse function f. Then a sequence of simplices is a V-path if and only if $\alpha_i \prec \beta_i \succ \alpha_{i+1}$ for each $i = 0, 1, \cdots, r$, and*

$$f(\alpha_0) \geq f(\beta_0) > f(\alpha_1) \geq f(\beta_1) > \cdots \geq f(\beta_r) > f(\alpha_{r+1}).$$

In particular, if V is a gradient vector field, then there are no non-trivial closed V-path. Therefore, the converse is true as well.

Using the V-path in Figure 2, we have the following relations

$$10 \geq 9 \geq 6 \geq 5$$

and

$$8 \geq 7 \geq 2 \geq 1 \geq 0.$$

Theorem 10. *A discrete vector field V is the gradient vector field of a discrete Morse function if and only if there is no non-trivial closed V-path.*

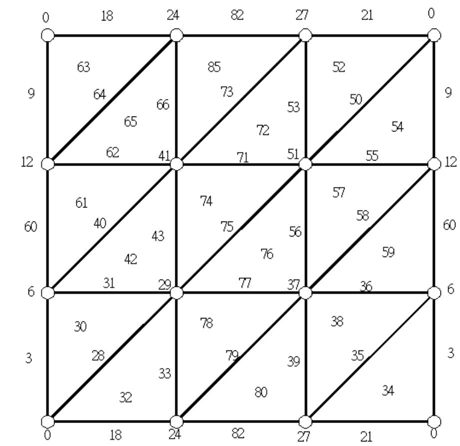

Figure 3: A discrete Morse function on the tours

3. An Application and Discussion

Next we give an example of a discrete Morse function and a gradient vector field on the tours.

In Figure 3, the critical simplices are the vertex $f^{-1}(0)$, the two edges $f^{-1}(56)$, $f^{-1}(71)$, and the 2-cell $f^{-1}(85)$.

A discrete vector field V can be thought as a collection of pairs $\{\alpha^{(p)}, \beta^{(p+1)}\}$ of simplices, where a pair α, β is in V if $\alpha \prec \beta$ and $f(\alpha) \geq f(\beta)$. In Figure 4, we indicate these pairings by drawing an arrow from the vertex $f^{-1}(24)$ to the edge $f^{-1}(18)$, and an arrow from the edge $f^{-1}(43)$ to the 2-cell $f^{-1}(42)$.

Moreover, using the value of the function on a simplex to denote the simplex, we have the following V-path:

$$82 \to 80 \to 79 \to 78 \to 77 \to 76 \to 75 \to 74 \to 43 \to 42 \to 31 \to 30,$$

and

$$36 \to 6 \to 3 \to 0.$$

Remark. As an application in combinatorial geometry, Crowley investigated in [1] the collapsibility and geometry of nonpositively curved simplicial complexes by discrete Morse theory. Moreover, Forman's discrete Morse theory has been adopted by building, graph theory, and visualization. The paper [4] proves that the complexity class of the strategy optimization of Grow and Fold is MAX-SNP hard. A MAX-SNP hard problem is a NP-Hard problem for

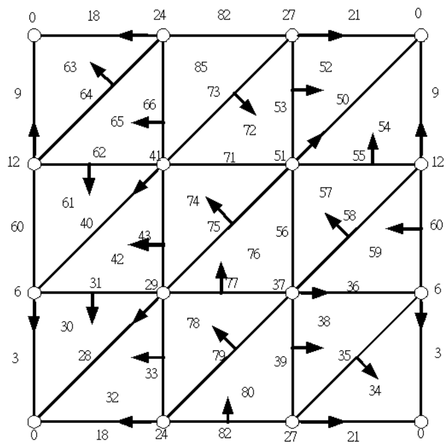

Figure 4: A gradient vector field on the tours

which any polynomial approximation can be arbitrarily far from optimal.

Acknowledgments

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