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WEIGHTS FOR THE HARDY OPERATOR ON NON-DECREASING FUNCTIONS

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Abstract: We present single weight and two-weight integral inequalities for the Hardy operator on non-decreasing functions. In this context the Hardy operator is the minimal operator, but the inequalities are much nicer than those one expects from the minimal operator theory [3].

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1. Introduction

We will use the notation $f\uparrow, f\downarrow$ to indicate that $f:\mathbb{R}_+\to\mathbb{R}_+$ is either non-decreasing, non-increasing, respectively. For $0<r<\infty$, $\chi_r=\chi_{[0,r]}$, $\chi^r=\chi_{[r,\infty)}$, and $f/\chi_r=f\chi_r+\infty\chi^r$. A non-negative locally integrable function is called a weight.

The Hardy operator on $f:\mathbb{R}_+\to\mathbb{R}_+$ is

$$Hf(x)=\frac{1}{x}\int_0^x f(t)dt.$$

If $f\downarrow$, then $Hf(x)$ is the maximal operator

$$Hf(x)=\sup_{x\in I}\frac{1}{|I|}\int_I f(t)dt,$$

and if $f\uparrow$, then $Hf(x)$ is the minimal operator

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$$Hf(x) = \inf_{x \in I} \frac{1}{|I|} \int_I f(t)dt.$$

The theory of weights for the maximal operator and minimal operator is well developed [6, 3] and can only provide sufficient conditions for weighted integral inequalities of Hf on monotone functions. In particular, the weighted integral inequalities

$$\int_0^\infty Hf(x)^p w(x)dx \le c \int_0^\infty f(x)^p w(x)dx, \quad f \downarrow, 0 < p < \infty$$

are controlled by

$$w \in B_p : \int_r^\infty \left(\frac{r}{x}\right)^p w(x)dx \le c \int_0^r w(x)dx, \quad 0 < r < \infty,$$

and this condition is obtained by using the test function $f(x) = \chi_r(x)$. The converse can be found in [1, 4].

On the other hand, if we restrict ourselves to $f \uparrow$, then the corresponding single weight integral inequality is

$$\int_0^\infty \frac{w(x)}{Hf(x)^p} dx \le c \int_0^\infty \frac{w(x)}{f(x)^p} dx, \quad f \uparrow, 0 < p < \infty. \tag{S_p}$$

The condition on the weight $w \in S_p$ is no longer obtained using the test function $f = 1/\chi_r$, since then $Hf(x) = f(x)$ and S_p gives the trivial $\int_0^r w(x)dx \le c \int_0^r w(x)dx$ satisfied by any w. This is not enough for S_p as the following example shows. Let $w(x) = e^{-1/x}$ and suppose that S_p holds for $p = 1$ and constant c. Let $f(x) = x^n$. Then $Hf(x) = x^n/(n+1)$ and thus

$$\int_0^\infty \frac{w(x)}{Hf(x)} dx = (n+1) \int_0^\infty \frac{w(x)}{f(x)} dx.$$

Since $0 < \int_0^\infty w(x)/x^n dx < \infty$, this would contradict S_1 if $n + 1 > c$. We will prove (Theorem 1) that $w \in S_p$ iff $w \in B_\infty = \cup_{p<\infty} B_p$.

Remark. Theorem 1 is contained in the recent paper [2]. We will include our proof since it has some features of independent interest. The present paper was completed while [2] appeared and rather than referring to the proof of Theorem 1 to [2] we include our version for the sake of completeness. The same method of proof will be used to study the behavior of iterated Hardy operators in Theorem 3.

We will refer to the two weight integral inequality

$$\int_0^\infty \frac{u(x)}{Hf(x)^p} dx \le c \int_0^\infty \frac{v(x)}{f(x)^p} dx, \quad f \uparrow, 0 < p < \infty, \tag{T_p}$$

by saying that $(u, v) \in T_p$ with constant c. In Section 3 we will introduce a two weight B_∞ condition which will imply that $(u, v) \in T_p, 0 < p < \infty$. We will also see that the two weight B_∞ condition is equivalent with the behavior of (u, v) at 0 and ∞, and this will give us that $(u, v) \in T_p$ if supp $u \subset [a, b], a > 0$ (Theorem 5). This allows us to characterize $(u, v) \in T_p$ by the behavior of $(u_n, v) \in T_p, u_n = u\chi_{[1/n,n]}$.

In Section 4 we exploit the dilation of u or v, $u_s(x) = u(sx)$, in connection with $(u, v) \in T_p$. We will see that for every u, $(u, u_s) \in T_p, s > 1$ (Theorem 7). The next question which we will study is the following: if $(u, v) \in T_{p_0}$, when is $(u, v) \in T_p, p \neq p_0$? It is easy to see that this holds if $0 < p < p_0$. We do not know what happens when $p > p_0$, except that the bahavior of the iterated Hardy operator may come into play here (Theorem 11). We will assume throughout that for a pair (u, v), $Hv(r) > 0, r > 0$.

Remark. (1) The characterization of a pair of weights for which T_p holds has been mentioned in [2] as an open problem.

(2) The two-weight condition for

$$\int_0^\infty Hf(x)^p u(x)dx \leq c \int_0^\infty f(x)^p v(x)dx, \quad f \downarrow, 0 < p < \infty$$

is known [5], Theorem 13, p. 106. (3) Heinig and Stepanov [4] characterized the weights u, v for which $||Hf||_{q,u} \leq c||f||_{p,v}$ for $f \uparrow$. By interpreting Hf for $f \uparrow$ as a minimal operator we were led to look at $1/Hf$ rather than Hf (see [3]).

2. Single Weights S_p

We begin with the proof of

Theorem 1. *Let $0 < p < \infty$. Then $w \in S_p$ if and only if $w \in B_\infty = \cup_{p<\infty} B_p$.*

Proof. If $w \in B_\infty$, then $w \in B_{p_0}$ for some p_0. By Hölder's inequality

$$1 = \frac{1}{x}\int_0^x f^\alpha f^{-\alpha} \leq H(f^{\alpha s})(x)^{1/s} H(f^{-\alpha s'})(x)^{1/s'}$$

or

$$\frac{1}{H(f^{\alpha s})(x)} \leq H(f^{-\alpha s'})(x)^{s-1}.$$

Fix $0 < p < \infty$, and let $s = 1 + p_0/p$ and $\alpha s' = p/p_0$. Then $\alpha s = 1$ and we get

$$\frac{1}{Hf(x)^p} \leq H(f^{-p/p_0})(x)^{p_0}.$$

Since $f^{-p/p_0}\downarrow$ and $w\in B_{p_0}$, we get S_p.

Conversely, assume S_p holds. Choose $0<p_0<\infty$ such that $c<(p_0+1)^p$, where c is the constant in S_p. Let $0<r<N<\infty$ and let

$$f_N(x)=\chi_r(x)+(p_o+1)(x/r)^{p_0}\chi^r(x)+\infty\chi^N(x).$$

Then $f_N\uparrow$ and

$$Hf_N(x)=\chi_r(x)+(x/r)^{p_0}\chi^r(x)+\infty\chi^N(x).$$

From S_p,

$$\int_0^r w(x)dx+\int_r^N (r/x)^{pp_0}w(x)dx \\ \le c\int_0^r w(x)dx+c(p_0+1)^{-p}\int_r^N (r/x)^{pp_0}w(x)dx.$$

This gives us

$$(1-c(p_0+1)^{-p})\int_r^N (r/x)^{pp_0}w(x)dx\le c\int_0^r w(x)dx.$$

Let $N\to\infty$ to get $w\in B_{pp_0}$. □

We will now study the behavior of iterated Hardy operators

$$H_jf(x)=\underbrace{H\circ H\circ\cdots\circ Hf(x)}_{j\text{ times}}.$$

The weight class for which

$$\int_0^\infty H_{j+1}f(x)^p w(x)dx\le c\int_0^\infty H_jf(x)^p w(x)dx,\quad f\downarrow,$$

is characterized by $w\in B_p$ or $\int_N^\infty\left(\frac{\log^{j-1}(x)}{x}\right)^p w(x)dx=\infty$ for every $N<\infty$ [7], Theorem 4 with $k=1$.

The analogous result for the case $f\uparrow$ does not need an additional condition, but the proof which follows the lines of Theorem 1 is a good deal more complicated.

Theorem 2. *The integral inequality*

$$\int_0^\infty \frac{w(x)}{H_{j+1}f(x)^p}dx\le c\int_0^\infty \frac{w(x)}{H_jf(x)^p}dx,\quad f\uparrow, 0<p<\infty,$$

holds if and only if $w\in B_\infty$.

Proof. Since $H_jf\uparrow$ if $f\uparrow$ Theorem 1 gives us the inequality if $w\in B_\infty$. For the converse fix $0<r<N<\infty$-later $N\to\infty$- and let

$$f(x)=\chi_r(x)+(x/r)^q\chi^r(x)+\infty\chi^N(x),$$

where $q > 0$ will be determined later. We need the following estimates:

$$H_{j+1}f(x) \le \chi_r(x) + \left(1 + \frac{1}{(q+1)^{j+1}}(x/r)^q\right)\chi^r(x) + \infty\chi^N(x), \tag{1}$$

$$H_j f(x) \ge \chi_{sr}(x) + \frac{1}{(q+1)^j}(x/r)^q\chi^{sr}(x) + \infty\chi^N(x), \quad r < sr < N. \tag{2}$$

The inequality (1) holds for $j = 0$ since

$$\begin{aligned} Hf(x) &= \chi_r(x) + \frac{1}{x}\left(r + \frac{x^{q+1} - r^{q+1}}{(q+1)r^q}\right)\chi^r(x) + \infty\chi^N(x) \\ &\le \chi_r(x)\left(1 + \frac{1}{q+1}(x/r)^q\right)\chi^r(x) + \infty\chi^N(x). \end{aligned}$$

Assume now that (1) holds with $j+1$ replaced by j. Then

$$\begin{aligned} H_{j+1}f(x) &\le \chi_r(x) + \frac{1}{x}\left(r + \frac{x^{q+1} - r^{q+1}}{(q+1)^{j+1}r^q}\right)\chi^r(x) + \infty\chi^N(x) \\ &\le \chi_r(x) + \left(1 + \frac{1}{(q+1)^{j+1}}(x/r)^q\right)\chi^r(x) + \infty\chi^N(x). \end{aligned}$$

For the proof of (2) we need the following representation of $H_j f(x)$ easily verified by induction. For arbitrary $f : \mathbb{R}_+ \to \mathbb{R}_+$,

$$H_j f(x) = \frac{1}{(j-1)!x}\int_0^x f(t)\log^{j-1}(x/t)dt.$$

Then for our special $f(x)$,

$$\begin{aligned} H_j f(x) =& \frac{1}{(j-1)!x}\int_0^x (\chi_r(t) + (t/r)^q\chi^r(t))\log^{j-1}(x/t)dt + \infty\chi^N(x) = \chi_r(x) \\ &+ \left(\frac{1}{(j-1)!x}\int_0^r \log^{j-1}(x/t)dt + \frac{1}{(j-1)!x}\int_r^x (t/r)^q\log^{j-1}(x/t)dt\right) \\ &\times \chi^r(x) + \infty\chi^N(x). \end{aligned}$$

Since $H_j f \uparrow$ we see that

$$H_j f(x) \ge \chi_{sr}(x) + (\cdots)\chi^{sr}(x) + \infty\chi^N(x).$$

The inequality (2) follows by observing that for $x > r$

$$\begin{aligned} &\frac{1}{(j-1)!x}\int_0^r \log^{j-1}(x/t)dt + \frac{1}{(j-1)!x}\int_r^x (t/r)^q\log^{j-1}(x/t)dt \\ &\ge \frac{1}{(j-1)!x}\int_0^x (t/r)^q\log^{j-1}(x/t)dt = \frac{1}{(q+1)^j}(x/r)^q. \end{aligned}$$

We substitute now (1) for $H_{j+1}f(x)$ and (2) for $H_j f(x)$ in our original integral inequality and get

$$\int_{sr}^{N} \frac{(r/x)^{qp} w(x)dx}{\left((r/x)^q + \frac{1}{(q+1))^{j+1}}\right)^p}$$
$$\leq c \int_0^{sr} w(x)dx + c(q+1)^{jp} \int_{sr}^{N} (r/x)^{qp} w(x)dx.$$

Let $D = (r/x)^q + (q+1)^{-(j+1)}$. Since $x \geq sr$, $D \leq s^{-q} + (q+1)^{-(j+1)}$, and since $s > 1$, we can choose $q_0 > 0$ such that $D \leq 2(q+1)^{-(j+1)}, q \geq q_0$. All this gives us

$$\frac{(q+1)^{(j+1)p}}{2^p} \int_{sr}^{N} (r/x)^{qp} w(x)dx \leq c \int_0^{sr} w(x)dx + c(q+1)^{jp} \int_{sr}^{N} (r/x)^{qp} w(x)dx.$$

Finally, choose $q \geq q_0$ so large that

$$\frac{(q+1)^{(j+1)p}}{2^p} - c(q+1)^{jp} = (q+1)^{jp}((q+1)^p/2^p - c) \equiv c_* > 0.$$

Then, letting $N \to \infty$ we get

$$\frac{1}{s^{qp}} \int_{sr}^{\infty} (sr/x)^{qp} w(x)dx \leq \frac{c}{c_*} \int_0^{sr} w(x)dx,$$

and $w \in B_{qp}$. □

3. Two Weights

We begin with some observations that will be used frequently.

α. *Let $u, v : \mathbb{R}_+ \to \mathbb{R}_+$ and assume that there is a constant $0 < c < \infty$ such that $\int_0^r u(x)dx \leq c \int_0^r v(sx)dx, 0 < r < \infty$ for some $s > 0$. Then for $\phi \downarrow$*

$$\int_0^{\infty} \phi(x)u(x)dx \leq c \int_0^{\infty} \phi(x)v(sx)dx.$$

Proof. Let $r(y) \downarrow$. Then

$$L \equiv \int_0^{\infty} \int_0^{r(y)} u(x)dxdy \leq c \int_0^{\infty} \int_0^{r(y)} v(sx)dxdy \equiv R.$$

Interchange the order of integration to get

$$L = \int_0^{\infty} \int_0^{r^{-1}(x)} dyu(x)dx = \int_0^{\infty} r^{-1}(x)u(x)dx.$$

Let $\phi(x) = r^{-1}(x)$. □

$\boldsymbol{\beta}$. If $f\uparrow$ and $s>1$, then
$$Hf(x)\geq\frac{1}{s'}f(x/s),\qquad \frac{1}{s}+\frac{1}{s'}=1.$$

Proof. $Hf(x)\geq\frac{1}{x}\int_{x/s}^{x}f(t)dt\geq f(x/s)\frac{1}{s'}$. □

$\boldsymbol{\gamma}$. *H commutes with dilation: $Hf(tx)=H(f_t)(x), f_t(x)=f(tx), t>0$.*

Proof. Make the obvious change of variables. □

$\boldsymbol{\epsilon}$. *If $(u,v)\in T_p$ with constant c, then $\int_0^r u(x)dx\leq c\int_0^r v(x)dx, 0<r<\infty$.*

Proof. Apply T_p to the function f/χ^r. □

Definition. We say that $(u,v)\in B_\infty$ if for some $s>1$ there is $0<c=c_s<\infty$ such that $Hu(sr)\leq cHv(r), 0<r<\infty$.

Remark. This is an extension to (u,v) of a single weight w to be in B_∞: $w\in B_\infty$ iff $\int_0^{sx}w\leq c\int_0^x w, s>1$ ([9]).

Theorem 3. *$(u,v)\in B_\infty$ if and only if $Hu(sr)=\mathcal{O}(Hv(r))$ as $r\to 0,\infty$ for some $s>1$.*

Proof. The necessity is clear. For the sufficiency, the $\mathcal{O}$ condition implies that there exists a constant $0<c_1<\infty$ such that $Hu(sr)\leq c_1Hv(r)$ for $0<r<r_0, R<r<\infty$. Since we assume that $\int_0^r v>0, r>0$, the quotient $Hu(sr)/Hv(r)$ is continuous on $[r_0,R]$. Hence there exists a constant $0<c_2<\infty$ such that $Hu(sr)\leq c_2Hv(r), r_0\leq r\leq R$. Hence if $c=\max(c_1,c_2)$, then $Hu(sr)\leq cHv(r), 0<r<\infty$. □

Theorem 4. *If $(u,v)\in B_\infty$, then $(u,v)\in T_p, 0<p<\infty$.*

Proof. Since $Hu(sr)\leq cHv(r), 0<r<\infty$ for some $s>1$, we get with $f\uparrow$ from β,α
$$\int_0^\infty\frac{u(x)}{Hf(x)^p}dx=s\int_0^\infty\frac{u(sx)}{Hf(sx)^p}dx$$
$$\leq s(s')^p\int_0^\infty\frac{u(sx)}{f(x)^p}dx\leq cs(s')^p\int_0^\infty\frac{v(x)}{f(x)^p}dx.\quad\square$$

Theorem 5. *Let $u,v:\mathbb{R}_+\to\mathbb{R}_+$ with $\int_0^r v>0, r>0$, and supp $u\subset[a,b], a>0$. Then $(u,v)\in T_p, 0<p<\infty$.*

Proof. The $\mathcal{O}$ condition of Theorem 3 is clearly satisfied and hence $(u,v)\in B_\infty$. Theorem 4 implies our conclusion. □

Theorem 6. *$(u,v)\in T_p$ if and only if $(u_n,v)\in T_p, u_n=u\chi_{[1/n,n]}$, with constant independent of n.*

Proof. If $(u,v)\in T_p$, then

$$\int_0^\infty \frac{u_n(x)}{Hf(x)^p}dx \le \int_0^\infty \frac{u(x)}{Hf(x)^p}dx \le c\int_0^\infty \frac{v(x)}{f(x)^p}dx.$$

The converse follows from the monotone convergence theorem, since $u_n \uparrow u$ and the constant of $(u_n,v)\in T_p$ is independent of n. □

We conclude this section with the following observation.

Let $\mathcal{F}$ be the collection of all $f\uparrow$ and concave. If $Hu(r)\le c_0 Hv(r), 0<r<\infty$, then for $0<p<\infty$, $(u,v)\in T_p$ restricted to $\mathcal{F}$ with constant $4^p c_0$.

Proof. By α,β with $s=2$

$$\int_0^\infty \frac{u(x)}{Hf(x)^p}dx \le c_0\int_0^\infty \frac{v(x)}{Hf(x)^p}dx \le 2^p c_0\int_0^\infty \frac{v(x)}{f(x/2)^p}dx$$
$$\le 4^p c_0\int_0^\infty \frac{v(x)}{f(x)^p}dx,$$

since f is concave and hence $f(x/2)\ge \frac{1}{2}f(x)$. □

4. Dilations of Weights

For $\phi:\mathbb{R}_+\to\mathbb{R}_+$ let $\phi_s(x)=\phi(sx), s>0$. We have the following results.

Theorem 7. *If $u:\mathbb{R}_+\to\mathbb{R}_+$ and $0<p<\infty$, then:*

(i) $(u,u_s)\in T_p, s>1$.

(ii) $(u_\sigma,u)\in T_p, 0<\sigma<1$.

Proof. By (β) for $f\uparrow$

$$\int_0^\infty \frac{u(x)}{Hf(x)^p}dx = s\int_0^\infty \frac{u(sx)}{Hf(sx)^p}dx \le \frac{cs}{(s-1)^p}\int_0^\infty \frac{u(sx)}{f(x)^p}dx.$$

This proves (i), and (ii) since $(u_\sigma,u)=(u_\sigma,(u_\sigma)_{1/\sigma})$. □

Theorem 8. *Assume that $u,v:\mathbb{R}_+\to\mathbb{R}_+$ and $Hu(r)\le cHv(r), 0<r<\infty$ (in particular if $(u,v)\in T_p$). Then for $s>1$ the pair $(u,v_s)\in T_p, 0<p<\infty$ with constant depending upon p,s.*

Proof. Since $Hf(x)^{-p}\downarrow$ for $f\uparrow$ we have by (α)

$$\int_0^\infty \frac{u(x)}{Hf(x)^p}dx \le c\int_0^\infty \frac{v(x)}{Hf(x)^p}dx = cs\int_0^\infty \frac{v(sx)}{Hf(sx)^p}dx$$
$$\le \frac{cs}{(s-1)^p}\int_0^\infty \frac{v(sx)}{f(x)^p}dx,$$

by (β). □

The next Theorem chatacterizes $(u,v) \in T_p$ in terms of dilations.

Theorem 9. *For $0 < p < \infty$, $(u,v) \in T_p$ if and only if $(u_s, v_s) \in T_p$ with constant independent of $s > 0$.*

Proof. From (γ) we get
$$\int_0^\infty \frac{u(sx)}{Hf(x)^p}dx = \frac{1}{s}\int_0^\infty \frac{u(x)}{Hf(x/s)^p}dx \le \frac{c}{s}\int_0^\infty \frac{v(x)}{f(x/s)^p} \le c\int_0^\infty \frac{v(sx)}{f(x)^p}dx.$$
Conversely, again by (γ),
$$\int_0^\infty \frac{u(x)}{Hf(x)^p}dx = s\int_0^\infty \frac{u(sx)}{Hf(sx)^p}dx \le cs\int_0^\infty \frac{v(sx)}{f(sx)^p}dx$$
$$\le c\int_0^\infty \frac{v(x)}{f(x)^p}dx. \quad \square$$

In what follows we will discuss the question: if $(u,v) \in T_{p_0}$, when is $(u,v) \in T_p$ for $p \ne p_0$?

Theorem 10. *If $(u,v) \in T_{p_0}$ with constant c and $0 < p < p_0$, then $(u,v) \in T_p$ with the same constant c.*

Proof. Since by Hölder's inequality $Hf(x)^p = Hf(x)^{\frac{p}{p_0}p_0} \ge H(f^{p/p_0}(x)^{p_0}$, we see that
$$\int_0^\infty \frac{u(x)}{Hf(x)^p}dx \le \int_0^\infty \frac{u(x)}{H(f^{p/p_0})(x)^{p_0}}dx \le c\int_0^\infty \frac{v(x)}{f(x)^p}dx. \qquad \square$$

The next result shows that $(u,v) \in T_p$ is connected with the behavior of the iterated Hardy operator. Let $H_j f$ be the j times iterated Hardy operator. We say that $(u,v) \in T_{j,p}, j \in \mathbb{N}, 0 < p < \infty$, with constant c if
$$\int_0^\infty \frac{u(x)}{H_j f(x)^p}dx \le c\int_0^\infty \frac{v(x)}{f(x)^p}dx, \quad f\uparrow. \tag{$T_{j,p}$}$$

Theorem 11. *For $0 < p_0 < p < \infty$ let j be the least integer $\ge p/p_0$. If $(u,v) \in T_{j,p_0}$ with constant c, then $(u,v) \in T_p$ with the same constant.*

Proof. By Theorem 10 we may assume that $(u,v) \in T_{j,p_1}, jp_1 = p$. We claim that
$$Hf(x)^j \ge H_j(f^j)(x), \quad f\uparrow, j = 1, 2, \cdots.$$
This is obvious for $j = 1$. If the inequality is true up to $j-1$, then using repeatedly the property that $\phi Hf \ge H(\phi f), \phi\uparrow$, we get
$$Hf(x)^j = Hf(x)Hf(x)^{j-1}(x) \ge Hf(x)H_{j-1}(f^{j-1})(x)$$
$$\ge H\{fH_{j-1}(f^{j-1})\}(x) \ge H \circ H_{j-1}(f^j)(x) = H_j(f^j)(x).$$

Since $jp_1 = p$, we have $Hf(x)^p \geq H_j(f^j)(x)^{p_1}$ and hence

$$\int_0^\infty \frac{u(x)}{Hf(x)^p}dx \leq \int_0^\infty \frac{u(x)}{H_j(f^j)(x)^{p_1}}dx \leq c\int_0^\infty \frac{v(x)}{f(x)^p}dx. \qquad \square$$

5. Two Examples

It may be instructive to study the following special case as an application of the above results. Let (u, v) be a pair of weights with $Hu(r) \to L < \infty$ and $Hv(r) \to K > 0$ as $r \to 0$ with a similar situation at ∞. Then for $0 < \sigma < 1, Hu(r)/Hv(\sigma r) \to L/K$ as $r \to 0$ with a similar statement at ∞. Then there is a constant $0 < c_1 < \infty$ such that $Hu(r) \leq c_1 Hv(\sigma r), 0 < r < r_0, R < r < \infty$. Since $Hu(r)/Hv(\sigma r)$ is continuous on $[r_0, R]$, there is a constant $0 < c_2 < \infty$ with $Hu(r) \leq c_2 Hv(\sigma r), r_0 \leq r \leq R$. If $c_* = \max(c_1, c_2)$, then $Hu(r) \leq Hv(\sigma r), 0 < r < \infty$.

(i) By α,

$$\int_0^\infty \frac{u(x)}{Hf(x)^p}dx \leq c_* \int_0^\infty \frac{v(\sigma x)}{Hf(x)^p}dx,$$

since $Hf \uparrow$ if $f \uparrow$, and hence

$$\int_0^\infty \frac{u(x)}{Hf(x)^p}dx \leq (c_*/\sigma)\int_0^\infty \frac{v(x)}{Hf(x/\sigma)^p}dx$$

$$\leq \frac{c_*}{\sigma(1-\sigma)^p}\int_0^\infty \frac{v(x)}{f(x)^p}dx,$$

by β. Let $K_1(\sigma) = \frac{c_*}{\sigma(1-\sigma)^p}$ be the constant in the above integral inequality.

(ii) We can also apply Theorem 11 to get $(u, v) \in T_p$ by showing that $(u, v) \in T_{j,p_1}, jp_1 = p$. Let $\sigma = \sigma_0 < \sigma_1 < \cdots < \sigma_j = 1, \sigma_i = \sigma_0 + (i/j)(1 - \sigma_0), i = 0, \cdots, j$. Then for $f \uparrow$ by α and Theorem 7,

$$\int_0^\infty \frac{u(x)}{H_j f(x)^{p_1}}dx \leq c_* \int_0^\infty \frac{v(\sigma_0 x)}{H_j f(x)^{p_1}}dx$$

$$\leq c_*(\sigma_1/\sigma_0)\left(\frac{\sigma_1}{\sigma_1 - \sigma_0}\right)^{p_1}\int_0^\infty \frac{v(\sigma_1 x)}{H_{j-1}f(x)^{p_1}}dx$$

$$\leq c_*(\sigma_1/\sigma_0)\left(\frac{\sigma_1}{\sigma_1 - \sigma_0}\right)^{p_1}(\sigma_2/\sigma_1)\left(\frac{\sigma_2}{\sigma_2 - \sigma_1}\right)^{p_1}\int_0^\infty \frac{v(\sigma_2 x)}{H_{j-2}f(x)^{p_1}}dx \leq \cdots$$

$$\leq (c_*/\sigma_0)\Pi_{i=1}^j\left(\frac{\sigma_i}{\sigma_i - \sigma_{i-1}}\right)^{p_1}\int_0^\infty \frac{v(x)}{f(x)^{p_1}}dx.$$

Thus $(u, v) \in T_{j,p_1}$ with constant

$$K_2(\sigma) = (c_*/\sigma)\Pi_{i=1}^{j}\left(\frac{j\sigma_i}{1-\sigma}\right)^{p_1}.$$

The relative magnitude of the constants $K_1(\sigma)$ and $K_2(\sigma)$ is the content of

Theorem 12. *If $1/j \le \sigma < 1$, then $K_1(\sigma) < K_2(\sigma)$, and if $0 < \sigma < 1/j^j$, then $K_2(\sigma) < K_1(\sigma)$.*

Proof. First note that

$$K_2(\sigma) = K_1(\sigma) j^p \Pi_1^j \sigma_i^{p_1}.$$

If $1/j \le \sigma < \sigma_1 < \cdots < \sigma_j = 1$, then $j^p \Pi_1^j \sigma_i^{p_1} > j^p/j^p = 1$ and the first inequality follows. Since $\sigma_i < 1, \sigma_j = 1$, we see that $j^p \Pi_1^j \sigma_i^{p_1} < j^p \sigma^{p_1} < 1$ if $0 < \sigma < 1/j^j$, and we have the second inequality. □

Our final example consists of showing that there exist $(u, v) \in T_p$ so that the $\mathcal{O}$ condition of Theorem 3 does not hold. Let $v(x) = e^{-1/x}$ and $u_n(x) = e^{-s_n/x}$ where $s_n \downarrow 1$. By Theorem 7 $(u_n, v) \in T_p$ with constant c_n. Choose now $\epsilon_n > 0$ such that $\sum \epsilon_n c_n < \infty$ and let $u(x) = \sum \epsilon_n u_n(x)$. Then

$$\int_0^\infty \frac{u(x)}{Hf(x)^p} dx = \sum \epsilon_n \int_o^\infty \frac{u_n(x)}{Hf(x)^p} dx \le \sum \epsilon_n c_n \int_0^\infty \frac{v(x)}{f(x)^p} dx.$$

To see that the $\mathcal{O}$ condition at $r = 0$ does not hold let $s > 1$. Then $1 < s_n < s, n \ge N$. For $n \ge N$,

$$\frac{\int_0^{sr} u(x)dx}{\int_0^r v(x)dx} \ge \epsilon_n \frac{\int_0^{sr} u_n(x)dx}{\int_0^r v(x)dx}$$

and the limit as $r \to 0$ is by l'Hospital rule the same as the limit as $r \to 0$ of

$$\epsilon_n \frac{se^{-s_n/sr}}{e^{-1/r}} \to \infty.$$

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