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COMPUTING THE FROBENIUS NUMBER

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Abstract: The Frobenius number $g(A)$ of a finite subset $A \subset \mathbb{N}$ such that $\gcd(A) = 1$ is the largest integer which cannot be expressed as $\sum_{a \in A} a x_a$ with non-negative integers x_a. We present an algorithm for the computation of $g(A)$. Without loss of generality we suppose that there exist $a, b \in A$ such that $\gcd(a, b) = 1$. We give a formula for $g(A)$ in the particular case that for all $c, d \in A$, $c + d$ can be written in the form $c + d = xa + yb$ with $x, y \geq 0$ (e.g. $c + d > ab - a - b$). Using Euler polynomials we give a formula for $g(A)$ in the case that $A = \{a, b, c\}$.

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1. Introduction and Statement of Results

Throughout this paper, small letters denote integers. We will set $\mathbb{N} = \{1, 2, 3 \ldots\}$ and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. We consider a finite subset $A \subset \mathbb{N}$ such that $\gcd(A) = 1$. We define $S(A)$ the additive semigroup of $\mathbb{N}_0$ generated by A, i.e. $S(A) = \{\sum_{a \in A} x_a a \mid x_a \in \mathbb{N}_0\}$. The Frobenius number $g(A)$ is defined as the largest integer which does not belong to $S(A)$. We are interested in computing $g(A)$ and equivalently $f(A) = g(A) + \sum_{a \in A} a$ the largest integer which cannot be expressed as $\sum_{a \in A} a x_a$ with positive integers x_a. It is well known that $g(a, b) = ab - a - b$ if $\gcd(a, b) = 1$, and $g(A) = -1$ if and only if $1 \in A$.

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For card$(A) \geq 3$, no general formula for $g(A)$ is known, except in particular cases, see [6] and [8]. Algorithms are developed in [4] and [7] in the case $A = \{a, b, c\}$.

Without loss of generality, we can consider only sets A containing two coprime integers a, b (see Section 2). We fix two integers $a > 1$ and $b > 1$ such that $\gcd(a, b) = 1$, and $\mathbf{c} = (c_1, \ldots, c_n) \in \mathbb{N}^n$. Let $A = \{a, b, c_1, \ldots, c_n\}$.

For all $\mathbf{t} = (t_1 \ldots, t_n) \in \mathbb{N}_0^n$, we set $\mathbf{t}.\mathbf{c} = \sum_1^n t_i c_i$.

For every $m \in \mathbb{Z}$, there exist unique integers $\hat{m}$ and $\check{m}$ such that $m = \hat{m}a - \check{m}b$ with $0 \leq \hat{m} < b$. We write $\mathbf{t}.\mathbf{c} = b(\mathbf{t})a - a(\mathbf{t})b$ where $b(\mathbf{t}) = \overset{\wedge}{\mathbf{t}.\mathbf{c}}$ and $a(\mathbf{t}) = \overset{\vee}{\mathbf{t}.\mathbf{c}}$.

A finite subset T of $\mathbb{N}_0^n$ is said to be *appropriate* if

$$S(A) = \bigcup_{\mathbf{t} \in T} (S(a, b) + \mathbf{t}.\mathbf{c}). \tag{1}$$

It is said to be *pruned* if $\mathbf{0} \in T$ and for $\mathbf{t} \in T \setminus \{\mathbf{0}\}$, we have $b(\mathbf{t}) > 0$ and $a(\mathbf{t}) > 0$.

Let T be an *appropriate* and *pruned* subset of $\mathbb{N}_0^n$. We can write $T = \{\mathbf{0} = \mathbf{t}_0, \mathbf{t}_1, \ldots, \mathbf{t}_l\}$ with $0 = b(\mathbf{t}_0) < b(\mathbf{t}_1) \leq b(\mathbf{t}_2) \leq \cdots \leq b(\mathbf{t}_l) < b$. We set $b_i = b(\mathbf{t}_i), a_i = a(\mathbf{t}_i)$ for $0 \leq i \leq l$, $b_{l+1} = b$ and $a_{l+1} = a$.

We obtain the sequences

$$0 = b_0 < b_1 \leq b_2 \leq \cdots \leq b_l < b_{l+1} = b$$

and

$$0 = a_0 < a_1, a_2, \ldots, a_l < a_{l+1} = a.$$

For $0 \leq i \leq l$, we set $m_i = \max\{a_j \mid 0 \leq j \leq i\}$ and $g_i = (b_{i+1} - 1)a - (m_i + 1)b$.

Our main result is the following

Theorem 1.

$$g(A) = \max\{g_i \mid 0 \leq i \leq l\}. \tag{2}$$

A subset $M \subset \mathbb{N}_0$ is said to be *trimmed* if for all $m \in M \setminus \{0\}, \check{m} > 0$ and for all $m, d \in M$,

$$\hat{m} < \hat{d} \Leftrightarrow \check{m} < \check{d}. \tag{3}$$

We show that there exists an *appropriate* set T such that $T.\mathbf{c} = \{\mathbf{t}.\mathbf{c} \mid \mathbf{t} \in T\}$ is *trimmed*.

Theorem 2. *Let T be an appropriate set. If $T.\mathbf{c}$ is trimmed then*

$$g(A) = \max\{a(b_{i+1} - b_i) + \mathbf{t}_i.\mathbf{c} \mid 0 \leq i \leq l\} - (a + b), \tag{4}$$

where $\mathbf{t}_0 = \mathbf{0}$.

We give an algorithm for computing $g(A)$. We use Theorem 1 if we start with an arbitrary *appropriate* set T. And we use Theorem 2 if we start with an *appropriate* set T such that $T.\mathbf{c}$ is *trimmed*.

In a particular case we derive a formula for $g(A)$ from Theorem 2:

Theorem 3. *If* $\{c_1, \ldots, c_n\}$ *is trimmed and for all* $1 \le i, j \le n$, *we have* $c_i + c_j \in S(a,b)$ *(e.g.* $c_i + c_j > ab - a - b$*) then*

$$g(A) = \max\{a(b_{i+1} - b_i) + c_i \mid 0 \le i \le n\} - (a+b), \tag{5}$$

where $b_{n+1} = b, b_0 = 0, c_0 = 0$ *and for all* $1 \le i \le n$, $b_i = \hat{c}_i$.

In the case $n = 1$, i.e. $A = \{a, b, c\}$, we take $c \notin S(a,b)$ and we write $c = au_0 - bv_0$ with $0 < u_0 < b$ and $0 < v_0 < a$.

By successive Euclidean divisions we get:

$$\begin{cases} a = u_{-1} = p_0 v_0 + u_1, \ldots, u_{n-1} = p_n v_n + u_{n+1}, \ldots, \\ b = v_{-1} = q_0 u_0 + v_1, \ldots, v_{n-1} = q_n u_n + v_{n+1}, \ldots. \end{cases} \tag{6}$$

The triplet (a,b,c) is said to be *of level* $n = n(a,b,c)$ if

$$p_0 = q_0, p_1 = q_1, \cdots, p_{n-1} = q_{n-1}$$

and $p_n \neq q_n$ or $v_{n+1} = 0$.

Let (a,b,c) a triplet of level n. For all $0 \le i \le n$, we set $w_{i+1} = u_{i-1} - q_i v_i$ so for $i \le n$, $w_i = u_i$ and $w_{n+1} = (p_n - q_n)v_n + u_{n+1}$.

We denote by

$$L(a,b,c) = (L_0, L_1) = \begin{cases} (bv_n, bw_{n+1}) & \text{if } n \text{ is even },\\ (av_n, aw_{n+1}) & \text{if } n \text{ is odd },\end{cases}$$

$$l(a,b,c) = (l_0, l_1) = \begin{cases} (aw_n, av_{n+1}) & \text{if } n \text{ is even },\\ (bw_n, bv_{n+1}) & \text{if } n \text{ is odd }.\end{cases}$$

For an integer n we denote by $k(n) = k = \lfloor \frac{n}{2} \rfloor$ and $h(n) = h = \lfloor \frac{n+1}{2} \rfloor$ where $\lfloor x \rfloor$ is the greatest integer such that $\lfloor x \rfloor \le x$.

Theorem 4.

$$\begin{aligned} f(a,b,c) &= aw_{2h} + bw_{2k+1} - \min\{bv_{2h}, av_{2k+1}\} \\ &= l_0 + L_1 - \min\{l_1, L_0\}. \end{aligned} \tag{7}$$

In particular, when $p_n = q_n$ *and* $v_{n+1} = 0$,

$$f(a,b,c) = \begin{cases} a\gcd(b,c) + \operatorname{lcm}(b,c) & \textit{if } n \textit{ is even,}\\ b\gcd(a,c) + \operatorname{lcm}(a,c) & \textit{if } n \textit{ is odd.}\end{cases} \tag{8}$$

Theorem 5. *Let* $(x_{-1}, x_0, \ldots, x_{l+1}), (y_{-1}, y_0, \ldots, y_{m+1}), (d_0, d_1, \ldots, d_l)$ *and*

$(e_0, e_1, \ldots, e_m)$ *be sequences such that the following conditions*

$$\begin{cases} x_{-1} = d_0 x_0 - x_1, \ldots, x_{l-1} = d_l x_l - x_{l+1}, \\ y_{-1} = e_0 y_0 - y_1, \ldots, y_{m-1} = e_m y_m - y_{m+1}, \end{cases} \tag{9}$$

$$\begin{cases} a = x_{-1} > x_0 > x_1 > \cdots > x_l > x_{l+1} = 0, \\ b = y_{-1} > y_0 > y_1 > \cdots > y_m > y_{m+1} = 0 \end{cases} \tag{10}$$

hold. Let n *be an integer such that*

$$\begin{cases} d_0 = e_0, d_1 = e_1, \cdots, d_{n-1} = e_{n-1} \\ \text{ and } (d_n \neq e_n \text{ or } x_{n+1} = 0). \end{cases} \tag{11}$$

Then,

$$f(a, b, c) = a y_{n+1} + b(x_{n-1} - e_n x_n) + \max\{a(y_n - y_{n+1}), b x_n\}. \tag{12}$$

As a consequence of Theorem 5 we prove the following theorem see [7].

Theorem 6. (Rodseth) *Let* R_i *polynomials defined by induction as follows:* $R_{-1} = 0, R_0 = 1$ *and for all* $i > 0$,

$$R_{i+1} = e_i R_i - R_{i-1}. \tag{13}$$

Let n *be the unique integer such that*

$$\frac{y_{n+1}}{R_{n+1}} \leq \frac{c}{a} < \frac{y_i}{R_i} \text{ for all } 0 \leq i \leq n. \tag{14}$$

Then,

$$f(a, b, c) = c R_{n+1} + a y_n - \min\{a y_{n+1}, c R_n\}. \tag{15}$$

2. Reduction to the Case $\gcd(a, b) = 1$

Let $A = \{b, c_1, \ldots, c_n\}$ be a subset of $\mathbb{N}$ such that $\gcd(A) = 1$.

We recall the following classical result: It is always possible to choose integers $x, x_1 \ldots, x_n$ such that $1 = xb + x_1 c_1 + \cdots + x_n c_n$. Then, for k an integer sufficiently large, we get

$a = 1 + [k(c_1 + \cdots + c_n) - x]b = (x_1 + kb)c_1 + \cdots + (x_n + kb)c_n \in S(c_1, \ldots, c_n)$. Hence, $g(A) = g(A \cup \{a\})$ and $\gcd(a, b) = 1$.

In the particular case that $A = \{a, b, c_1, \ldots, c_n\}$ such that $\gcd(a, b) = \gcd(a, b, c_1, \ldots, c_{n-1}) = d \geq 1$ we observe that $\gcd(\frac{a}{d}, \frac{b}{d}) = 1$. Therefore, to compute $g(A)$ we can use Brauer's formula

$$f(A) = df(\frac{a}{d}, \frac{b}{d}, \frac{c_1}{d}, \dots, \frac{c_{n-1}}{d}, c_n), \quad (16)$$

see [3].

From now on we suppose that $\gcd(a,b) = 1$.

3. Case $n \geq 1$

We need some remarks.

R1. If $m \geq 0$ then $a > \check{m}$.

Proof. Since $m = \hat{m}a - \check{m}b \geq 0$ we have $ba > \hat{m}a \geq \check{m}b$, hence $a > \check{m}$. □

R2. Let $m = xa - yb$. Then,

$$0 \leq x < b \Leftrightarrow \hat{m} = x \Leftrightarrow \check{m} = y$$

and

$$-b \leq x < 0 \Leftrightarrow \hat{m} = x + b \Leftrightarrow \check{m} = y + a.$$

Proof. To prove the second claim we write $m = xa - yb = (x+b)a - (y+a)b$ and we use the uniqueness of $\hat{m}$ and $\check{m}$. □

R3. Let $m = xa - yb$. Then, there exists a unique integer $p \in \mathbb{Z}$ such that $x = pb + \hat{m}$ and $y = pa + \check{m}$.

Proof. We write $m = xa - yb = \hat{m}a - \check{m}b$ thus $(x - \hat{m})a = (y - \check{m})b$. Since $\gcd(a,b) = 1$, $p = \frac{x-\hat{m}}{b} = \frac{y-\check{m}}{a}$ is an integer. □

R4. We have $m \in S(a,b)$ if and only if $\check{m} \leq 0$.

Proof. Clearly $\check{m} \leq 0 \Rightarrow m = \hat{m}a - \check{m}b \in S(a,b)$. Conversely, if $m \in S(a,b)$ then $m = xa + yb$ with $x \geq 0$ and $y \geq 0$. By **R3** there exists $p \in \mathbb{Z}$ such that $x = pb + \hat{m} \geq 0$ thus $p \geq 0$. We also have $-y = pa + \check{m} \leq 0$ and so $\check{m} = -y - pa \leq 0$. □

R5. For $d \geq 0$ we set $G(d) = S(a,b) \bigcup (S(a,b) + d)$. Then, we have

$$m \notin G(d) \Leftrightarrow \check{m} > 0 \text{ and } (\hat{m} < \hat{d} \text{ or } \check{m} > \check{d}).$$

Proof. Let $n = m - d$. Since $d \geq 0$, **R1** shows that $a > \check{d}$. Hence, $\check{m} - \check{d} + a > \check{m}$. Moreover, we have $n = (\hat{m} - \hat{d})a - (\check{m} - \check{d})b$ and $-b < \hat{m} - \hat{d} < b$. It follows from **R2** that $\check{n} = \check{m} - \check{d}$ if $\hat{m} - \hat{d} \geq 0$ and $\check{n} = \check{m} - \check{d} + a > \check{m}$ if $\hat{m} - \hat{d} < 0$. We deduce that

$$\check{m} > 0 \text{ and } (\hat{m} < \hat{d} \text{ or } \check{m} > \check{d}) \Leftrightarrow \check{m} > 0 \text{ and } (\hat{m} - \hat{d} \geq 0 \Rightarrow \check{m} - \check{d} > 0)$$

$$\Leftrightarrow \check{m} > 0 \text{ and } \check{n} > 0 \Leftrightarrow m \notin S(a,b) \text{ and } n = m - d \notin S(a,b)$$
$$\Leftrightarrow m \notin G(d). \qquad \square$$

As a consequence of **R5** we obtain

R6. If T is *appropriate* then

$$m \notin S(A) \Leftrightarrow \check{m} > 0 \text{ and } \forall \mathbf{t} \in T, \hat{m} < b(\mathbf{t}) \text{ or } \check{m} > a(\mathbf{t}).$$

Proof of Theorem 1. For $0 \leq i \leq l$, we set $F(i) = \{m \in \mathbb{Z} \mid \hat{m} < b_{i+1} \text{ and } \check{m} > m_i\}$.

Step 1.

$$\mathbb{Z} \setminus S(A) = \bigcup_{i=0}^{l} F(i). \tag{17}$$

Let $m \in F(i)$. For every $0 \leq j \leq l$, $\hat{m} < b_{i+1} \leq b_j$ if $j \geq i+1$ and $\check{m} > m_i \geq a_j$ if $j \leq i$. Hence, **R6** shows that $m \notin S(A)$.

Conversely, let $m \notin S(A)$ then $\check{m} > 0$ by **R4**. Since $b_0 = 0 \leq \hat{m} < b = b_{l+1}$ and $0 = b_0 < b_1 \leq b_1 \leq \cdots \leq b_l \leq b_{l+1} = b$, there exists $0 \leq j \leq l$ such that $\hat{m} < b_{j+1}$. We put $i = \min\{j \mid \hat{m} < b_{j+1}\}$. We thus get for $0 \leq j \leq i$, $b_j \leq b_i \leq \hat{m} < b_{i+1}$ and $\check{m} > a_j$ by **R6**. Hence, $\check{m} > m_i$. We conclude that $m \in F(i)$.

Step 2. For $0 \leq i \leq l$, $g_i = \max F(i)$. Since $0 \leq b_{i+1} - 1 < b$, we have $\hat{g}_i = b_{i+1} - 1$ and $\check{g}_i = m_i + 1 > m_i$. We thus get $g_i \in F(i)$. Moreover, for all $m \in F(i), m = \hat{m}a - \check{m}b \leq (b_{i+1} - 1)a - (m_i + 1)b$ then $g_i = \max F(i)$. $\square$

In particular, when $T = \{\mathbf{0}\}$, i.e. when all $c_i \in S(a,b)$, we have $b_0 = 0 < b_1 = b, a_0 = 0$ and $m_0 = 0$. Therefore, $g(A) = g_0 = (b_1 - 1)a - (m_0 + 1)b = ab - a - b = g(a,b)$.

Proof of Theorem 2. If $T.\mathbf{c}$ is *trimmed* then $0 = b_0 < b_1 < b_1 < \cdots < b_l < b_{l+1} = b$ and thus $0 = a_0 < a_1 < a_1 < \cdots < a_l < a_{l+1} = a$. In particular, T is *pruned* and for all $0 \leq i \leq l$, $m_i = a_i$. We can write $g_i = a(b_{i+1} - b_i) + ab_i - ba_i - (a+b) = a(b_{i+1} - b_i) + \mathbf{t}_i.\mathbf{c} - (a+b)$. $\square$

Algorithm. 1. For every i, we choose $\lambda_i > 0$ such that $\lambda_i c_i \in S(a,b)$. Numbers λ_i exist. Indeed, it is sufficient to take $\lambda_i > \frac{g(a,b)}{c_i} = \frac{ab-a-b}{c_i}$. The following set

$$U = \{\mathbf{t} = (t_1, \ldots, t_n) \in \mathbb{N}_0^n \mid t_i < \lambda_i, 1 \leq i \leq n\}$$

is thus *appropriate*.

We remove from U all elements $\mathbf{t} \neq \mathbf{0}$ such that $b(\mathbf{t}) = 0$ or $a(\mathbf{t}) \leq 0$. The set T of all remainding elements is *pruned* and still *appropriate*.

We number the elements of $T = \{\mathbf{0} = \mathbf{t}_0, \mathbf{t}_1, \ldots, \mathbf{t}_l\}$ in such a way that $0 = b(\mathbf{t}_0) < b(\mathbf{t}_1) \leq b(\mathbf{t}_2) \leq \cdots \leq b(\mathbf{t}_l)$. We compute $m_i = \max\{a_j \mid 0 \leq j \leq i\}$, $g_i = (b_{i+1} - 1)a - (m_i + 1)b$ and $g(A) = \max\{g_i \mid 0 \leq i \leq l\}$ where $b_i = b(\mathbf{t}_i), a_i = a(\mathbf{t}_i)$ for $0 \leq i \leq l$, $b_{l+1} = b$ and $a_{l+1} = a$.

2. The algorithm can be modified as follows: For every $1 \leq i \leq n$, we start removing from A all elements c_j such that $\hat{c}_i \leq \hat{c}_j$ and $\check{c}_j \leq \check{c}_i$. We choose an *appropriate* and *pruned* set $T = \{\mathbf{0} = \mathbf{t}_0, \mathbf{t}_1, \ldots, \mathbf{t}_l\}$. We can suppose that $0 = b(\mathbf{t}_0) < b(\mathbf{t}_1) \leq b(\mathbf{t}_2) \leq \cdots \leq b(\mathbf{t}_l) < b$. For all $0 \leq i \leq l$, we remove from T all $\mathbf{t}_j$ such that $(b(\mathbf{t}_i) \leq b(\mathbf{t}_j)$ and $a(\mathbf{t}_j) \leq a(\mathbf{t}_i))$. Considering the set of the remainding elements we can suppose that $T.\mathbf{c}$ is trimmed. The Frobenius number can therefore be computed using Theorem 2.

Proof. 1. For all $\mathbf{t} \in \mathbb{N}_0^n$, we have $G(\mathbf{t.c}) = S(a, b) \bigcup (S(a, b) + \mathbf{t.c}) \subset S(A)$ thus $\bigcup_{\mathbf{t} \in U} G(\mathbf{t.c}) \subset S(A)$. Conversely, let $m \in S(A)$, then $m = xa + yb + \sum_{i=1}^n x_i c_i$ with $x \geq 0, y \geq 0$ and $x_i \geq 0$. By Euclidean division we write $x_i = q_i \lambda_i + t_i$. We thus get $\mathbf{t} = (t_1, \ldots, t_n) \in U$ and $m = xa + yb + \sum_{i=1}^n q_i \lambda_i c_i + \mathbf{t.c}$. Since $\lambda_i c_i \in S(a, b)$, we have $m \in G(\mathbf{t.c})$. The equality

$$S(A) = \bigcup_{\mathbf{t} \in U} G(\mathbf{t.c}) \tag{18}$$

follows. Therefore, U is *appropriate*.

By construction T is *pruned* and it is still *appropriate*. Indeed, for $\mathbf{t} \neq \mathbf{0}$, $\mathbf{t.c} \notin S(a, b)\}$ if and only if $a(\mathbf{t}) > 0$ by **R4**. Therefore,

$$T = \{\mathbf{t} \in U \mid \mathbf{t} = \mathbf{0} \text{ or } \mathbf{t.c} \notin S(a, b)\} \tag{19}$$

and

$$S(A) = \bigcup_{\mathbf{t} \in T} G(\mathbf{t.c}). \tag{20}$$

2. If $\hat{c}_i \leq \hat{c}_j$ and $\check{c}_j \leq \check{c}_i$ then $c_j = c_i + (\hat{c}_j - \hat{c}_i)a + (\check{c}_i - \check{c}_j)b \in S(c_i, a, b)$. Therefore, $g(A) = g(A \setminus \{c_j\})$. If $b(\mathbf{t}_i) \leq b(\mathbf{t}_j)$ and $a(\mathbf{t}_j) \leq a(\mathbf{t}_i)$ then $\mathbf{t_j.c} = \mathbf{t_i.c} + (b(\mathbf{t}_j) - b(\mathbf{t}_i))a + (a(\mathbf{t}_i) - a(\mathbf{t}_j))b \in G(\mathbf{t_i.c})$ thus $G(\mathbf{t_j.c}) \subset G(\mathbf{t_i.c})$. We see that (20) is not altered by removing $\mathbf{t_j}$ from T. □

Proof of Theorem 3. Since $2c_i \in S(a, b)$ for all $1 \leq i \leq n$, $U = \{\mathbf{t} = (t_1, \ldots, t_n) \in \mathbb{N}_0^n \mid 0 \leq t_i \leq 1, 1 \leq i \leq n\}$ is *appropriate*. Furthermore, for $\mathbf{t} = (t_1, \ldots, t_n) \in U$, if $\mathbf{t.c} \notin S(a, b)$ then there exists c_i and $y \in S(a, b)$ such that $\mathbf{t.c} = c_i + y \in G(c_i)$. Therefore, the set $T = \{\mathbf{0} = \mathbf{t}_0, \mathbf{t}_1, \ldots, \mathbf{t}_n\}$ where $\mathbf{t}_i = (0, \ldots, t_i, \ldots, 0)$ and $t_i = 1$ is also *appropriate*. Moreover, $T.\mathbf{c} = \{0, c_1, \ldots, c_n\}$ is *trimmed* and $b(\mathbf{t}_i) = \hat{c}_i$ for all $1 \leq i \leq n$. □

4. Case $n = 1$

Let E be a totally ordered set and $x \in E$. We call *successor* of x in E, and we denote x^+, the smallest element of E (if there exists any) such that $x < x^+$.

For an *appropriate* set T, we put $b(T) = \{b(\mathbf{t}) \mid \mathbf{t} \in T\}$ and $\mathcal{B} = b(T) \cup \{b\}$. We equip $\mathcal{B}$ with the natural order $\leq$.

To apply Theorem 2, in the case that $n = 1$, it is convenient to formulate it as follows:

Theorem 2′. *Let T be an appropriate set such that $T.\mathbf{c}$ is trimmed. Then,*

$$g(A) = \max\{a(b(\mathbf{t})^+ - b(\mathbf{t})) + \mathbf{t}.\mathbf{c} \mid \mathbf{t} \in T\} - (a + b), \tag{21}$$

where $b(\mathbf{t})^+$ is the successor of $b(\mathbf{t})$ in $\mathcal{B}$.

Suppose that there exists integers $\alpha_1, \ldots, \alpha_q$ and a partition $E_1, \ldots, E_q$ of T such that $b(\mathbf{t})^+ - b(\mathbf{t}) = \alpha_i$ for all $\mathbf{t} \in E_i$ and for all $1 \leq i \leq q$. Then,

$$g(A) = \max\{a\alpha_i + \beta_i \mid 1 \leq i \leq q\} - (a + b), \tag{22}$$

where we put $\beta_i = \max E_i.\mathbf{c}$.

To compute such a partition, we are led to introduce what we call Euler order on T. We will use Euler polynomials.

Euler Polynomials. Let $(q_0, q_1, \ldots, q_i, \ldots)$ be a sequence of positive integers. We define Euler polynomials Q_i by induction as follows: $Q_{-1} = 0, Q_0 = 1$ and for $i \geq 0$,

$$Q_{i+1}(q_0, \ldots, q_i) = q_i Q_i(q_0, \ldots, q_{i-1}) + Q_{i-1}(q_0, \ldots, q_{i-2}). \tag{23}$$

We set $Q_{i+1} = Q_{i+1}(q_0, \ldots, q_i), Q_i^1 = Q_i(q_1, \ldots, q_i),\ P_{i+1} = Q_i + Q_{i+1}$ and $P_{i+1}^1 = Q_i^1 + Q_{i+1}^1$.

We deduce immediately that

$$Q_{n+1} = q_n Q_n + q_{n-2} Q_{n-2} + \ldots + q_{n-2i} Q_{n-2i} + Q_{n-2i-1} \tag{24}$$

for $0 \leq 2i \leq n$.

Euler order $\leq_e$. Proposition 1. *Every integer $t \in \mathbb{N}_0$ can be written uniquely in the form*

$$t = t_0 Q_0 + t_1 Q_1 + \cdots + t_n Q_n, \tag{25}$$

where

$$\min\{i \mid t_i > 0\} \quad \textit{is even}\ , \tag{26}$$

$$0 \leq t_i \leq q_i \quad \textit{for} \quad 0 \leq i \leq n, \tag{27}$$

and

$$t_i = q_i \Rightarrow t_{i-1} = 0 \quad \textit{for} \quad 1 \le i \le n. \tag{28}$$

Equality (25) is called Euler expansion of t.

Proof. There exists $n \in \mathbb{N}_0$ such that $t < Q_{n+1}$. By successive Euclidean divisions we can write

$$\begin{aligned} t &= t_n Q_n + s_n \quad \text{with} \quad 0 \le s_n < Q_n, \\ s_n &= t_{n-1}Q_{n-1} + s_{n-1} \quad \text{with} \quad 0 \le s_{n-1} < Q_{n-1}, \\ &\vdots \\ s_2 &= t_1 Q_1 + s_1 \quad \text{with} \quad 0 \le s_1 < Q_1, \\ s_1 &= t_0 Q_0 \quad \text{with} \quad t_0 = s_1. \end{aligned}$$

We put $i = \min\{j \mid t_j > 0\}$. If $i = 2e$ then $t = t_{2e}Q_{2e} + t_{2e+1}Q_{2e+1} + \cdots + t_nQ_n$ is Euler expansion of t. If $i = 2e+1$, using (24) we take $t = q_0Q_0 + \cdots + q_{2e}Q_{2e} + (t_{2e+1}-1)Q_{2e+1} + \cdots + t_nQ_n$ as Euler expansion of t. Conditions (27) and (28) follow from $s_{i+1} < Q_{i+1} = Q_{i-1} + q_iQ_i$. The uniqueness follows from the uniqueness of the Euclidean division and the fact that $\sum_{i=0}^{2e} t_iQ_i < Q_{2e+1}$ if and only if there exists $0 \le j \le e$ such that $t_{2j} < q_{2j}$. □

For $t, x \in \mathbb{N}_0$, let $n \in \mathbb{N}_0$ such that $t, x \le Q_{n+1}$. We consider Euler expansions of t and x respectively $t = t_0Q_0 + t_1Q_1 + \cdots + t_nQ_n$ and $x = x_0Q_0 + x_1Q_1 + \cdots + x_nQ_n$. We define Euler order $\le_e$ as follows: $t \le_e x$ if $(t_0, -t_1, \ldots, (-1)^i t_i, \ldots, (-1)^n t_n) \le_l (x_0, -x_1, \ldots, (-1)^i x_i, \ldots, (-1)^n x_n)$ where $\le_l$ is the lexicographic order on $\mathbb{Z}^n$.

Lemma. *Let $n \in \mathbb{N}_0$. We consider $U_n = \{0, 1, \ldots, Q_{n+1} - 1\}$ and $V_n = \{0, 1, \ldots, P_n - 1\}$ equipped with induced Euler order $\le_e$.*

1. *We define a partition of $U_n = F_1 \cup F_2$ as follows:*

$$F_1 = \begin{cases} \{0, 1, \ldots, Q_{2k+1} - Q_{2k} - 1\} & \textit{if } n = 2k\ , \\ \{Q_{2k+1}, \ldots, Q_{2k+2} - 1\} & \textit{if } n = 2k+1\ , \end{cases}$$

$$F_2 = \begin{cases} \{Q_{2k+1} - Q_{2k}, \ldots, Q_{2k+1} - 1\} & \textit{if } n = 2k\ , \\ \{0, 1, \ldots, Q_{2k+1} - 1\} & \textit{if } n = 2k+1\ . \end{cases}$$

Then, the successor, in U_n, of all $t \in F_1$ (resp. $t \in F_2$) is $t^+ = t + (-1)^n Q_n$ (resp. $t^+ = t + (-1)^n[Q_n - Q_{n+1}]$). In particular, if $q_n = 1$ then for all $t \in F_2$, $t^+ = t + (-1)^{n-1}Q_{n-1}$.

2. *We define a partition of $V_n = E_1 \cup E_2$ as follows: $E_1 = \{0, \ldots, Q_{2h-1} - 1\}$,*

$E_2 = \{Q_{2h-1}, \ldots, P_n - 1\}$. *Then, the successor, in* V_n, *of all* $t \in E_1$ *(resp.* $t \in E_2$*) is* $t^+ = t + Q_{2k}$ *(resp.* $t^+ = t - Q_{2h-1}$*).*

Proof. Let $t = t_0Q_0 + t_1Q_1 + \cdots + t_nQ_n$ be Euler expansion of t.

1.(a) Suppose that $t \in F_1$. It is easily seen that $t_n > 0$ if $n = 2k+1$ and $t_n < q_n - 1$ or ($t_n = q_n - 1$ and $t_{n-1} = 0$) if $n = 2k$. Therefore, $t^+ = t + (-1)^n Q_n$.

(b) Suppose that $t \in F_2$, then $t = t_0Q_0 + t_1Q_1 + \cdots + t_{2k}Q_{2k}$. Since $t < Q_{2k+1}$, there exists $j \leq k$ such that $t_{2j} < q_{2j}$. Taking $i = \max\{j \mid t_{2j} < q_{2j}\}$ we can write $t = t_0Q_0 + t_1Q_1 + \cdots + t_{2i}Q_{2i} + [q_{2i+2}Q_{2i+2} + \cdots + q_{2k}Q_{2k}] = t_0Q_0 + t_1Q_1 + \cdots + t_{2i}Q_{2i} - Q_{2i+1} + Q_{2k+1}$. If $t_{2i} < q_{2i} - 1$ or ($t_{2i} = q_{2i} - 1$ and $t_{2i-1} = 0$) then $t^+ = t_0Q_0 + t_1Q_1 + \cdots + (t_{2i}+1)Q_{2i} + (q_{2i+1} - 1)Q_{2i+1} + \cdots + q_{2h-1}Q_{2h-1} = t + Q_{2h} - Q_{2k+1}$. If $t_{2i} = q_{2i} - 1$ and $t_{2i-1} > 0$ then $t^+ = t_0Q_0 + t_1Q_1 + \cdots + (t_{2i-1} - 1)Q_{2i-1} + q_{2i+1}Q_{2i+1} + \cdots + q_{2h-1}Q_{2h-1} = t + Q_{2h} - Q_{2k+1}$.

2. It is a particular case: Taking $q_n = 1$ we get $P_n = Q_{n-1} + Q_n = Q_{n+1}$ and $V_n = U_n$. Moreover, in this case we have $E_1 = F_2$ and $E_2 = F_1$ if $n = 2k$ and $E_1 = F_1$ and $E_2 = F_2$ if $n = 2k+1$. □

Let $(r = r_{-1}, r_0, \ldots, r_n)$ and $(q_0, q_1, \ldots, q_n)$ be sequences of positive integers and $r_{n+1} \geq 0$. We suppose that

$$r_{i-1} = q_i r_i + r_{i+1} \tag{29}$$

for $0 \leq i \leq n$.

We thus have $r > r_0 > r_1 > \cdots > r_n > 0$ and $r_{n-1} > r_{n+1} \geq 0$.

We prove by induction the following identities:

$$r = r_i Q_{i+1} + r_{i+1} Q_i \tag{30}$$

and

$$r_0 Q_i = (-1)^i r_i + r Q^1_{i-1}. \tag{31}$$

It follows from (31) that

$$r_0 P_{i+1} = (-1)^i (r_i - r_{i+1}) + r P^1_i. \tag{32}$$

Let $t \in \mathbb{N}_0$. Given $t = t_0Q_0 + t_1Q_1 + \cdots + t_nQ_n$ its Euler expansion, we associate with t the following numbers: $r(t) = \sum_{i=0}^n (-1)^i t_i r_i$ and $E(t) = \sum_{i=0}^n t_i Q^1_{i-1}$. It follows from (31) that

$$t r_0 = r(t) + r E(t). \tag{33}$$

Moreover, if $0 < t < Q_{n+1}$ then

$$r_n \leq r(t) \leq r - r_n, \tag{34}$$

and if $0 < t < P_n$ then

$$r_{2k} \leq r(t) \leq r - r_{2h-1}. \tag{35}$$

Indeed, for $0 < t < Q_{n+1}$, let $t = t_0 Q_0 + \cdots + t_n Q_n$ be Euler expansion of t. We can write $t = \sum_{i=e}^{k} t_{2i} Q_{2i} + \sum_{i=e}^{h-1} t_{2i+1} Q_{2i+1}$ with $t_{2e} > 0$. Hence, $r(t) = \sum_{i=e}^{k} t_{2i} r_{2i} - \sum_{i=e}^{h-1} t_{2i+1} r_{2i+1}$.

Using $r_{2e} = \sum_{i=e}^{h-1} q_{2i+1} r_{2i+1} + r_{2h}$ we get

$$\begin{aligned} r(t) &= (t_{2e} - 1) r_{2e} + \textstyle\sum_{i=e+1}^{k} t_{2i} r_{2i} \\ &+ \textstyle\sum_{i=e}^{h-1} (q_{2i+1} - t_{2i+1}) r_{2i+1} + r_{2h}. \end{aligned}$$

Now if $n = 2k+1$ we get

$$r_{2h} + r_{2k+1} \leq r(t) \leq \sum_{i=0}^{k} q_{2i} r_{2i} = r - r_{2k+1} = r - r_n. \tag{36}$$

If $n = 2k$ there exists $j \leq k$ such that $t_{2j} < q_{2j}$ since otherwise $t = Q_{n+1}$. We then get

$$r_n \leq r(e) \leq r(t) \leq r - r_{2j} - r_{2k+1} \leq r - r_{2k} - r_{2k+1}. \tag{37}$$

Hence, assumption (34) follows. In the particular case that $q_n = 1$ we have $P_n = Q_{n+1}$ and $r_{n-1} = r_n + r_{n+1}$. In this case if $n = 2k$ we get $r_{n-1} \leq r(t) \leq r - r_n$ by (36) and if $n = 2k+1$ we get $r_n \leq r(t) \leq r - r_{n-1}$ by (37) thus assumption (35) follows.

Proposition 2. *We suppose that $r_{n+1} = 0$. We equip $\{0, 1, \ldots, r-1\}$ with the natural order $\leq$ and $U_n = \{0, 1, \ldots, Q_{n+1} - 1\}$ with Euler order $\leq_e$. Then, the mapping $(U_n, \leq_e) \to (\{0, 1, \ldots, r-1\}, \leq), t \mapsto r(t)$ is strictly increasing.*

Proof. Given $t \leq_e x$ in U_n we have

$$(t_0, -t_1, \ldots, (-1)^j t_j, \ldots, (-1)^n t_n) \ \leq_l \ (x_0, -x_1, \ldots, (-1)^j x_j, \ldots, (-1)^n x_n)$$

and

$$r(x) - r(t) = (-1)^j (x_j - t_j) r_j + \sum_{i=j+1}^{n} (-1)^i x_i r_i - \sum_{i=j+1}^{n} (-1)^i t_i r_i,$$

where j is the smallest integer such that $t_j \neq x_j$.

1. When $j = 2i$ we get $x_{2i} > t_{2i} \geq 0$ and $x_{2i+1} \leq q_{2i+1} - 1$. Then, $r(x) - r(t) = (x_{2i} - t_{2i}) r_{2i} + [-x_{2i+1} r_{2i+1} + \cdots + (-1)^n x_n r_n] - [-t_{2i+1} r_{2i+1} + \cdots + (-1)^n t_n r_n] \geq (x_{2i} - t_{2i}) r_{2i} - [(q_{2i+1} - 1) r_{2i+1} + \cdots + q_{2h-1} r_{2h-1}] - [q_{2i+2} r_{2i+2} + \cdots + q_{2k} r_{2k}] \geq (x_{2i} - t_{2i}) r_{2i} - r_{2i} + r_{2h} + r_{2k+1} > 0$.

2. When $j = 2i-1$ we get $t_{2i-1} > x_{2i-1} \geq 0$ and $t_{2i} \leq q_{2i} - 1$. Then, $r(x) - r(t) = (t_{2i-1} - x_{2i-1}) r_{2i-1} + [x_{2i} r_{2i} + \cdots + (-1)^n x_n r_n] - [t_{2i} r_{2i} + \cdots + (-)^n t_n r_n] \geq (t_{2i-1} - x_{2i-1}) r_{2i-1} - [q_{2i+1} r_{2i+1} + \cdots + q_{2h-1} r_{2h-1}] - [(q_{2i} - 1) r_{2i} +

$\cdots + q_{2k}r_{2k}] \geq (t_{2i-1} - x_{2i-1})r_{2i-1} + r_{2h} - r_{2i-1} + r_{2k+1} > 0.$ □

Now we consider another sequence of positive integers $(s = s_{-1}, s_0, s_1, \ldots, s_n)$ and $s_{n+1} \geq 0$ such that $s_{i-1} = q_i s_i + s_{i+1}$ for $0 \leq i \leq n$.

We also define $s(t) = \sum_{i=0}^{n}(-1)^i t_i s_i$.

Using (31) we prove the following identity

$$(rs_0 - sr_0)Q_i = (-1)^i(rs_i - sr_i). \tag{38}$$

We derive

$$(rs_0 - sr_0)P_{i+1} = (-1)^i[r(s_i - s_{i+1}) + s(r_{i+1} - r_i)] \tag{39}$$

from (32) and

$$t(rs_0 - sr_0) = rs(t) - sr(t) \tag{40}$$

from (33).

Proof of Theorem 4. First Step. Reduction to the case $n = 2k$.

Suppose that $n = n(a, b, c) = 2k + 1$.

Since $g(a, b, c) = g(b, a, c)$, it suffices to show that $n(b, a, c)$ is even, $L(a, b, c) = L(b, a, c)$ and $l(a, b, c) = l(b, a, c)$.

We write $c = b(a - v_0) - a(b - u_0)$. We consider two cases:

If $p_0 = q_0 > 1$ then we can write $a = (a - v_0) + v_0, a - v_0 = (q_0 - 1)v_0 + v_1, b = (b - u_0) + u_0, b - u_0 = (q_0 - 1)u_0 + u_1$. Therefore, $n(b, a, c) = n(a, b, c) + 1$, $L(b, a, c) = L(a, b, c)$ and $l(b, a, c) = l(a, b, c)$.

If $p_0 = q_0 = 1$ we get $a - v_0 = u_1$ and $b - u_0 = v_1$. We therefore get $a = (q_1 + 1)(a - v_0) + v_2$ and $b = (p_1 + 1)(b - u_0) + u_2$. It follows that $n(b, a, c) = n(a, b, c) - 1$. Furthermore, we observe that if $n(a, b, c) > 1$ then we have obviously $L(a, b, c) = L(b, a, c)$ and $l(a, b, c) = l(b, a, c)$. If $n(a, b, c) = 1$ we have $L(b, a, c) = (av_1, a(b - (q_1 + 1)v_1)) = (av_1, a(u_0 - q_1v_1)) = L(a, b, c)$ and $l(b, a, c) = (bu_1, bv_2)) = l(a, b, c)$. In both cases, $n(b, a, c)$ is even. Therefore, the assumption follows.

Moreover, we can write

$$c = w_n w_{n+1} - v_n v_{n+1}. \tag{41}$$

Since $c > 0$, we get $w_{n+1} = (p_n - q_n)v_n + u_{n+1} > 0$ thus $p_n > q_n$ or ($p_n = q_n$ and $v_{n+1} = 0$). When $p_n = q_n$ and $v_{n+1} = 0$ we then have

$$c = w_n w_{n+1} = u_n u_{n+1}. \tag{42}$$

In the following steps we suppose that $n = 2k$.

Second Step. Case $p_n > q_n$.

Taking $a = r, v_0 = r_0, b = s$ and $u_0 = w_0 = s_0$ it follows from (39) that

$cP_{n+1} = a(w_n - v_{n+1}) + b(w_{n+1} - v_n) \in S(a,b)$ because $w_{n+1} > v_n$ and $w_n > v_{n+1}$. Moreover, it follows from (40) and (35) that $s(t) = b(t)$ and $r(t) = a(t)$. Hence, for all $0 < t < P_{n+1}$, $tc = ab(t) - ba(t) \notin S(a,b)$ by **R4**. We conclude that

$$P_{n+1} = \min\{t > 0 \mid tc \in S(a,b)\} \tag{43}$$

and by (19)-(20), that $V_n = \{0, 1, \dots, P_{n+1} - 1\}$ is *appropriate*. Let us show that $V_n.c$ is *trimmed*. Indeed, for $0 < t < x < P_{n+1}$, since $x - t < P_{n+1}$, we have $0 < (x-t)c = (b(x) - b(t))a - (a(x) - a(t))b \notin S(a,b)$. Then, $b(x) < b(t)$ if and only if $a(x) < a(t)$ by **R4**.

Furthermore, it follows from proposition 2 that $b(t)^+ - b(t) = b(t^+) - b(t) = b(t^+ - t)$. Writing $V_n = E_1 \cup E_2$ with the notation of the lemma, Theorem 2 and (22) show that $g(a,b,c) = \max\{a\alpha_1 + \beta_1, a\alpha_2 + \beta_2\} - (a+b)$ where $\alpha_1 = b(t^+) - b(t) = b(Q_n)$ for all $t \in E_1$, $\alpha_2 = b(t^+) - b(t) = -b(Q_{n+1})$ for all $t \in E_2$, and $\beta_i = \max E_i.c$.

It follows from (31) that $\alpha_1 = w_n$, and $\alpha_2 = v_{n+1}$. Moreover, we see that $\beta_2 = c(P_{n+1} - 1)$, $\beta_1 = c(Q_{n+1} - 1) = -cQ_n + cP_{n+1} - c = bv_n - aw_n + cP_{n+1} - c$ by (38). We therefore obtain the formula

$$g(a,b,c) = cP_{n+1} + \max\{av_{n+1}, bv_n\} - (a+b+c) \tag{44}$$

which can be written in the form

$$f(a,b,c) = a(w_n - v_{n+1}) + b(w_{n+1} - v_n) + \max\{av_{n+1}, bv_n\} \tag{45}$$

by (39). Finally,

$$f(a,b,c) = aw_n + bw_{n+1} - \min\{av_{n+1}, bv_n\}. \tag{46}$$

Third Step. Case $p_n = q_n$ and $v_{n+1} = 0$.

We have $cQ_{n+1} = bw_{n+1} \in S(a,b)$ by (38) and for all $0 < t < Q_{n+1}$, $tc = ab(t) - ba(t) \notin S(a,b))$ by (34). We deduce that

$$Q_{n+1} = \min\{t > 0 \mid tc \in S(a,b)\}. \tag{47}$$

Using (19)-(20) we show by a similar argument that $U_n = \{0, 1, \dots, Q_{n+1} - 1\}$ is *appropriate* and $U_n.c$ is *trimmed*. Furthermore, since $v_{n+1} = 0$ the lemma shows that $b(t^+) - b(t) = w_n$ for all $t \in U_n$. Therefore, by (21) $f(a,b,c) = aw_n + cQ_{n+1} = aw_n + bw_{n+1}$.

Moreover, since $v_{n+1} = 0$, we get $w_n = \gcd(b, w_0) = \gcd(b,c)$ and $b = w_n Q_{n+1}$ by (30). Using (42) we deduce that $\operatorname{lcm}(b,c) = bw_{n+1}$. $\square$

Remark. The case that ($p_n = q_n$ and $v_{n+1} = 0$) can be deduced from Brauer's formula (16): We put $d = \gcd(b,c) = \gcd(b, w_0) = w_n$, $c' = \frac{c}{d} = w_{n+1}$ and $b' = \frac{b}{d} = Q_{n+1}$. Using (30) we get $a = v_n Q_{n+1} + w_{n+1} Q_n > w_{n+1} Q_{n+1} =$

$b'c'$. We thus have $f(a, b', c') = g(a, b', c') + a + b' + c' = b'c' + a$. Hence, (16) show that $f(a, b, c) = df(a, b', c') = bc' + da = \mathrm{lcm}(b, c) + a\gcd(b, c)$. $\square$

Proof of Theorem 5. We first prove by induction that polynomials R_i satisfy the following properties: For $i \leq n$,

$$cR_i = ay_i - bx_i, \tag{48}$$

$$cR_{n+1} = ay_{n+1} - b(e_n x_n - x_{n-1}), \tag{49}$$

$$y_{-1} = y_i R_{i+1}(e_0, \dots, e_i) - y_{i+1} R_i(e_0, \dots, e_{i-1}) \tag{50}$$

and for $j \leq i$,

$$y_{j-1} = y_i R_{i-j+1}(e_j, \dots, e_i) - y_{i+1} R_{i-j}(e_j, \dots, e_{i-1}). \tag{51}$$

In the particular case that $e_j = e_{j+1} = \cdots = e_i = 2$, we get

$$R_{i-j+1}(e_j, \dots, e_i) = (i - j + 2) \tag{52}$$

and

$$y_{j-1} - y_j = y_j - y_{j+1} = \cdots = (y_i - y_{i+1}), \tag{53}$$

so in this case (51) can be written in the form

$$y_{j-1} = (i - j + 2)(y_i - y_{i+1}) + y_{i+1}. \tag{54}$$

Now we consider the set $K = \{e_i \mid i = 0 \text{ or } (0 < i < m \text{ and } e_i > 2)\} = \{e_0 = e_{k_0}, e_{k_1}, \dots, e_{k_{s-1}}\}$ and we set $k_s = m$. We can suppose that $0 = k_0 < k_1 < \cdots < k_s = m$. We have

$$y_{-1} = (e_0 - 1)y_0 + (y_0 - y_1) \tag{55}$$

and for $0 < i \leq s$,

$$y_{k_i - 1} - y_{k_i} = (e_{k_i} - 2)y_{k_i} + (y_{k_i} - y_{k_i + 1}). \tag{56}$$

Furthermore, (54) shows that

$$y_{k_{(i-1)}} = (k_i - k_{(i-1)})(y_{k_i - 1} - y_{k_i}) + y_{k_i} \tag{57}$$

and (53) that for $0 < i < s$,

$$y_{k_{(i-1)}} - y_{k_{(i-1)}+1} = \cdots = y_{k_i - 1} - y_{k_i} > y_{k_i}. \tag{58}$$

To apply Theorem 4 we set

$$v_{-1} = y_{-1}, \quad u_0 = y_0, \quad v_1 = (y_0 - y_1), \quad q_0 = (e_0 - 1),$$

and for $0 < i < s$,

$$v_{2i-1} = y_{k_i - 1} - y_{k_i}, \quad u_{2i} = y_{k_i}, \quad q_{2i} = (e_{k_i} - 2), \quad p_{2i-1} = (k_i - k_{(i-1)}).$$

Furthermore, if $e_m > 2$ we set

$$v_{2s-1} = (y_{m-1} - y_m), \quad v_{2s+1} = y_{m+1} = 0,$$

$$p_{2s-1} = (m - k_{(s-1)}), \quad u_{2s} = y_m, \quad q_{2s} = (e_m - 1),$$

and if $e_m = 2$ we set

$$v_{2s-1} = y_m, p_{2s-1} = (m+1-k_{(s-1)}) \text{ and } u_{2s} = y_{m+1} = 0.$$

We thus get

$$v_{-1} > u_0 > \cdots > v_{2i-1} > u_{2i} > v_{2i+1} > \cdots$$

and

$$v_{2i-1} = q_{2i}u_{2i} + v_{2i+1}; u_{2i} = p_{2i+1}v_{2i+1} + u_{2i+2}.$$

Using (11) we get $c = ay_0 - bx_0 = x_n y_n(d_n - e_n) + x_n y_{n+1} - y_n x_{n+1} > 0$. Then, $d_n > e_n$ or ($d_n = e_n$ and $x_{n+1} = 0$).

To prove (12) we consider two cases:

Case $n = 0$. We can write $a = (d_0 - 1)x_0 + (x_0 - x_1)$ and $b = (e_0 - 1)y_0 + (y_0 - y_1)$ with $d_0 - 1 > e_0 - 1$ or $d_0 - 1 = e_0 - 1$ and $x_1 = 0$. Then, $L(a,b,c) = (bx_0, b(a-(e_0-1)x_0)), l(a,b,c) = (ay_0, a(y_0 - y_1))$. We conclude that $f(a,b,c) = ay_0 + b(a-(e_0-1)x_0) - \min\{a(y_0-y_1), bx_0\} = ay_1 + b(a - e_0x_0) + \max\{a(y_0-y_1), bx_0\}$ by (7).

Case $n > 0$. Let $r = \max\{i \mid k_i < n\}$.

1. Suppose that $d_n \geq e_n > 2$. We can write $x_{n-1} - x_n = (d_n - 2)x_n + (x_n - x_{n+1})$ and $y_{n-1} - y_n = (e_n - 2)y_n + (y_n - y_{n+1})$ with $d_n - 2 > e_n - 2$ or $d_n - 2 = e_n - 2$ and $x_{n+1} = 0$. It follows that the level of (a,b,c) is even, $L(a,b,c) = (bx_n, b(x_{n-1} - (e_n - 1)x_n)$ and $l(a,b,c) = (ay_n, a(y_n - y_{n+1}))$. Therefore, $f(a,b,c) = ay_{n+1} + b(x_{n-1} - e_n x_n) + \max\{a(y_n - y_{n+1}), bx_n\}$ by (7).

2. Suppose that $d_n > e_n = 2$. We thus have $k_{(r+1)} > n$. We can write $y_{k_r} = (k_{(r+1)} - k_r)(y_{k_{(r+1)}-1} - y_{k_{(r+1)}}) + y_{k_{(r+1)}}$ and $x_{k_r} = (n-k_r)(x_{n-1} - x_n) + x_n$ by (57). Moreover, we have $y_{k_{(r+1)}-1} - y_{k_{(r+1)}} = y_{n-1} - y_n = y_n - y_{n+1}$ by (53) and $y_{k_r} - (n - k_r)(y_{n-1} - y_n) = y_n$ by (54). Therefore, the level of (a,b,c) is odd, $L = (a(y_n - y_{n+1}), ay_n)$ and $l = (b(x_{n-1} - x_n), bx_n)$. We deduce that $f(a,b,c) = ay_{n+1} + b(x_{n-1} - 2x_n) + \max\{a(y_n - y_{n+1}), bx_n\}$ by (7).

3. Suppose that $d_n = e_n = 2$ and $x_{n+1} = 0$ then $k_{(r+1)} \geq n+1$. Using (57) we can write $y_{k_r} = (k_{(r+1)} - k_r)(y_{k_{(r+1)}-1} - y_{k_{(r+1)}}) + y_{k_{(r+1)}} = (n+1-k_r)(y_n - y_{n+1}) + y_{n+1}$ and $x_{k_r} = (n+1-k_r)x_n$. Hence, the level of (a,b,c) is odd. Since $y_{k_r} - (n+1-k_r)(y_n - y_{n+1}) = y_{n+1}$, we get $L = (a(y_n - y_{n+1}), ay_{n+1})$ and $l = (bx_n, 0)$. Therefore, $f(a,b,c) = ay_{n+1} + bx_n = ay_{n+1} + \max\{a(y_n - y_{n+1}), bx_n\}$ because, by (48), we have $bx_n - ay_n + ay_{n+1} = -cR_n + cR_{n+1}$ and it is easily seen that $-cR_n + cR_{n+1} < 0$.

Finally we have proved that

$$f(a,b,c) = ay_{n+1} + b[x_{n-1} - e_n x_n] + \max\{a(y_n - y_{n+1}), bx_n\}. \qquad \square \qquad (59)$$

To prove Theorem 6 we observe, using (48)-(49), that (11) and the following condition

$$\frac{y_{n+1}}{R_{n+1}} \le \frac{c}{a} < \frac{y_i}{R_i} \text{ for all } 0 \le i \le n. \tag{60}$$

are equivalent. Taking account of (48)-(49) we obtain

$$f(a,b,c) = cR_{n+1} + ay_n - \min\{ay_{n+1}, cR_n\}. \qquad \square \tag{61}$$

5. Examples

1. $A = \{31, 44, 462, 674, 402, 932, 1214\}$.

We take $a = 31, b = 44$. We obtain $(\overset{\wedge}{462}, \overset{\wedge}{674}, \overset{\wedge}{402}, \overset{\wedge}{932}, \overset{\wedge}{1214}) = (22, 26, 30, 40, 42)$ and $(\overset{\vee}{462}, \overset{\vee}{674}, \overset{\vee}{402}, \overset{\vee}{932}, \overset{\vee}{1214}) = (5, 3, 12, 7, 2)$.

We remove 674, 932, 1214 from A without altering $g(A)$. We consider $(a, b, c_1, c_2) = (31, 44, 462, 402)$. Applying Theorem 3 we obtain $g(A) = 761$.

2. $A = \{57, 83, 367, 543, 605\}$.

We take $a = 57, b = 83$. We have $(\overset{\wedge}{367}, \overset{\wedge}{543}, \overset{\wedge}{605}) = (21, 27, 31)$, $(\overset{\vee}{367}, \overset{\vee}{543}, \overset{\vee}{605}) = (10, 12, 14)$, $\mathcal{B} = b(T) \cup \{b\} = \{21, 27, 31, 42, 48, 52, 58, 62, 63, 69, 73, 79, 83\}$, $\mathcal{A} = a(T) \cup \{a\} = \{10, 12, 14, 20, 22, 24, 26, 28, 30, 32, 34, 36, 57\}$. We obtain $g(A) = 1603$.

3. $A = \{a, b, c\} = \{137, 250, 337\}$. We have $(\hat{c}, \check{c}) = (101, 54), l = 4$, $\mathcal{B} = \{53, 101, 154, 202, 250\}, \mathcal{A} = \{25, 54, 79, 108, 137\}$. Using Theorem 2 we obtain $g(137, 250, 337) = \max\{g_1, g_2, g_3\} = 7537$.

Let us compute $g(A)$ by Theorem 5. We get $n(a, b, c) = 1$, $L(a, b, c) = (6576, 7261), l(a, b, c) = (7250, 6250)$. We obtain $g(a, b, c) = 7250 + 7261 - 6250 - 137 - 250 - 337 = 7537$.

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