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HYBRID PROXIMAL POINT ALGORITHMS ON $(A, \eta)-$MAXIMAL MONOTONICITY FOR NONLINEAR VARIATIONAL INCLUSIONS

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Abstract: A general framework for a hybrid proximal point algorithm using the notion of (A, η)-monotonicity is developed. Convergence analysis for this algorithm in the context of solving a class of nonlinear inclusion problems is explored along with some results on the resolvent operator corresponding to (A, η)-monotonicity.

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1. Introduction

Let X be a real Hilbert space with the norm $\|\cdot\|$ and the inner product $\langle \cdot, \cdot \rangle$. We consider a class inclusion problems of the form: find a solution to

$$0 \in M(x), \tag{1}$$

where $M : X \to 2^X$ is a set-valued mapping on X.

Pennanen [5] over-relaxed the proximal point algorithm considered by Eckstein and Bertsekas [2], and using it has shown that the sequence converges linearly to a solution of (1). Based on this new version of proximal point algorithm, Pennanen [5] studied a localized version of the maximal monotonicity, and has shown that it ensures the local convergence of the over-relaxed proximal point algorithm. Furthermore, the local convergence of multiplier methods for a

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general class of problems is established. This, in a way, presents specializations as new convergence results for multiplier methods for nonmonotone variational inequalities and nonconvex nonlinear programming.

Motivated by the recent algorithmic advances [5], [2], we develop a hybrid version for the relaxed proximal point algorithm considered by Eckstein and Bertsekas [2] based on the notions of A-maximal monotonicity [10] and (A, η)-maximal monotonicities [11] for solving general inclusion problems. These notions generalize the general class of maximal monotone set-valued mappings, including the notion of H-maximal monotonicity introduced by Fang and Huang [3] in a Hilbert space setting. This clearly reflects that a general class of problems of variational character, including minimization or maximization (whether constraint or not) of functions, variational problems, and minimax problems can be unified into the form (1). Recently, the author [11] introduced and studied the notion of (A, η)-maximal monotonicity in the context of approximating the solution of an inclusion problem based on the generalized resolvent operator technique. The generalized resolvent operator technique can also be applied to other problems, such as equilibria problems in economics, management sciences, optimization and control theory, operations research, and mathematical programming. For more details on the resolvent operator technique and its applications, we refer the reader [1-17].

2. (A, η)-Maximal Monotonicity

In this section we discuss some results based on basic properties of (A, η)-maximal monotonicity and its variant forms. Let $M : X \to 2^X$ be a multivalued mapping on X. We shall denote both the map M and its graph by M, that is, the set $\{(x, y) : y \in M(x)\}$. This is equivalent to stating that a mapping is any subset M of $X \times X$, and $M(x) = \{y : (x, y) \in M\}$. If M is single-valued, we shall still use $M(x)$ to represent the unique y such that $(x, y) \in M$ rather than the singleton set $\{y\}$. This interpretation shall much depend on the context. The domain of a map M is defined (as its projection onto the first argument) by

$$D(M) = \{x \in X : \exists\, y \in X : (x, y) \in M\} = \{x \in X : M(x) \neq \emptyset\}.$$

$D(M) = X$, shall denote the full domain of M, and the range of M is defined by

$$R(M) = \{y \in X : \exists\, x \in X : (x, y) \in M\}.$$

The inverse M^{-1} of M is $\{(y,x) : (x,y) \in M\}$. For a real number ρ and a mapping M, let $\rho M = \{x, \rho y) : (x,y) \in M\}$. If L and M are any mappings, we define

$$L + M = \{(x, y+z) : (x,y) \in L, (x,z) \in M\}.$$

Definition 2.1. Let $M : X \to 2^X$ be a multivalued mapping on X. The map M is said to be:

(i) (r)-strongly monotone if there exists a positive constant r such that

$$\langle u^* - v^*, u - v\rangle \geq r\|u-v\|^2 \quad \forall\,(u,u^*),(v,v^*) \in \text{graph}(M).$$

(ii) (m)-relaxed monotone if there exists a positive constant m such that

$$\langle u^* - v^*, u - v\rangle \geq (-m)\|u-v\|^2 \quad \forall\,(u,u^*),(v,v^*) \in \text{graph}(M).$$

Definition 2.2. Let $M : X \to 2^X$ be a multivalued mapping on X, and let $\eta : X \times X \to X$ be another mapping. The map M is said to be:

(i) (r,η)-strongly monotone if there exists a positive constant r such that

$$\langle u^* - v^*, \eta(u,v)\rangle \geq r\|u-v\|^2 \quad \forall\,(u,u^*),(v,v^*) \in \text{graph}(M).$$

(ii) $(1,\eta)$-strongly monotone if

$$\langle u^* - v^*, \eta(u,v)\rangle \geq \|u-v\|^2 \quad \forall\,(u,u^*),(v,v^*) \in \text{graph}(M).$$

(iii) η is said to be (τ)-Lipschitz continuous if there is a positive constant τ such that

$$\|\eta(u,v)\| \leq \tau\|u-v\|.$$

Definition 2.3. (see [10]) Let $A : X \to X$ be (r)-strongly monotone. The map $M : X \to 2^X$ is said to be A-maximal monotone if:

(i) M is (m)-relaxed monotone;

(ii) $R(A + \rho M) = X$ for $\rho > 0$.

Definition 2.4. Let $A : X \to X$ be (r,η)-strongly monotone. The map $M : X \to 2^X$ is said to be (A,η)-maximal monotone if:

(i) M is (m,η)-relaxed monotone;

(ii) $R(A + \rho M) = X$ for $\rho > 0$.

Definition 2.5. Let $A : X \to X$ be an (r,η)-strongly monotone mapping and let $M : X \to 2^X$ be an (A,η)-maximal monotone mapping. Then the generalized resolvent operator $J^{M,\eta}_{\rho,A} : X \to X$ is defined by

$$J^{M,\eta}_{\rho,A}(u) = (A + \rho M)^{-1}(u).$$

Proposition 2.1. *Let $A : X \to X$ be an (r,η)-strongly monotone mapping* *and let* $M : X \to 2^X$ *be an* (A,η)*-monotone mapping. Then the operator* $(A+\rho M)^{-1}$ *is single-valued.*

Proof. Suppose that for some $z \in X$, there is $x, y \in (A+\rho M)^{-1}(z)$. Then we have

$$-A(x) + z \in \rho M(x) \quad \text{and} \quad -A(y) + z \in \rho M(y).$$

Since M is (A,η)-monotone and A is (r,η)-strongly monotone, it follows that

$$\begin{aligned} &-\langle A(x) - A(y), \eta(x,y)\rangle \geq -\rho m\|x-y\|^2 \\ &\quad\Rightarrow \quad -\rho m\|x-y\|^2 \leq -\langle A(x) - A(y), \eta(x,y)\rangle \leq -r\|x-y\|^2 \\ &\quad\Rightarrow \quad (r-\rho m)\|x-y\|^2 \leq 0 \quad \Rightarrow \quad x = y \quad \text{for} \quad (r-\rho m) > 0. \end{aligned}$$

3. Hybrid Proximal Point Algorithms

This section deals with a hybrid proximal point algorithm to the relaxed version of the proximal point algorithm [2] and its application to approximation solvability of the inclusion problem (1) based on the (A,η)-maximal monotonicity.

Lemma 3.1. (see [11]) *Let* X *be a real Hilbert space, let* $A : X \to X$ *be* (r,η)*-strongly monotone, let* $M : X \to 2^X$ *be* (A,η)*-maximal monotone, and let* $\eta : X \times X \to X$ *be* (τ)*-Lipschitz continuous. Then the generalized resolvent operator associated with* M *and defined by*

$$J_{\rho,A}^{M,\eta}(u) = (A+\rho M)^{-1}(u) \quad \forall\, u \in X,$$

is $(\frac{\tau}{r-\rho m})$*-Lipschitz continuous.*

Furthermore, we have

$$\langle J_{\rho,A}^{M,\eta}(u) - J_{\rho,A}^{M,\eta}(v), u - v\rangle \leq \frac{\tau}{r-\rho m}\|u-v\|^2.$$

Theorem 3.1. *Let* X *be a real Hilbert space, let* $A : X \to X$ *be* (r,η)*-strongly monotone, and let* $M : X \to 2^X$ *be* (A,η)*-maximal monotone. Then the following statements are mutually equivalent:*

(i) An element $u \in X$ *is a solution to* (1).

(ii) For an $u \in X$*, we have*

$$u = J_{\rho,A}^{M,\eta}(A(u)),$$

where

$$J_{\rho,A}^{M,\eta}(u) = (A+\rho M)^{-1}(u).$$

Theorem 3.2. (see [3]) *Let X be a real Hilbert space, let $H : X \to X$ be (r, η)-strongly monotone, and let $M : X \to 2^X$ be (H, η)-maximal monotone. Then the following statements are mutually equivalent:*

(i) An element $u \in X$ is a solution to (1).

(ii) For an $u \in X$, we have

$$u = J_{\rho,H}^{M,\eta}(H(u)),$$

where

$$J_{\rho,H}^{M,\eta}(u) = (H + \rho M)^{-1}(u).$$

Lemma 3.2. *Let X be a real Hilbert space, let $A : X \to X$ be (r, η)-strongly monotone and (s)-Lipschitz continuous, and let $M : X \to 2^X$ be (A, η)-maximal monotone. Furthermore, let $\eta : X \times X \to X$ be (τ)-Lipschitz continuous. Then*

$$\|(J_{\rho,A}^{M,\eta} oA)(u) - (J_{\rho,A}^{M,\eta} oA)(v)\| \le \frac{s\tau}{r - \rho m}\|u - v\|,$$

and hence,

$$\langle (J_{\rho,A}^{M,\eta} oA)(u) - (J_{\rho,A}^{M,\eta} oA)(v), u - v\rangle \le \frac{s\tau}{r - \rho m}\|u - v\|^2 \quad \forall\, u, v \in X,$$

where $r - \rho m > 0$.

Lemma 3.3. *Let X be a real Hilbert space, let $H : X \to X$ be (r, η)-strongly monotone and (s)-Lipschitz continuous, and let $M : X \to 2^X$ be (H, η)-maximal monotone. Furthermore, let $\eta : X \times X \to X$ be (τ)-Lipschitz continuous. Then*

$$\|(J_{\rho,H}^{M,\eta} oH)(u) - (J_{\rho,H}^{M,\eta} oH)(v)\| \le \frac{s\tau}{r}\|u - v\|,$$

and hence,

$$\langle (J_{\rho,H}^{M,\eta} oH)(u) - (J_{\rho,H}^{M,\eta} oH)(v), u - v\rangle \le \frac{s\tau}{r}\|u - v\|^2 \quad \forall\, u, v \in X.$$

In the following theorem, we apply the hybrid proximal point algorithm to approximate the solution of (1), and as a result, we end up showing linear convergence.

Theorem 3.3. *Let X be a real Hilbert space, let $A : X \to X$ be (r, η)-strongly monotone and (s)-Lipschitz continuous, and let $M : X \to 2^X$ be (A, η)-maximal monotone. Let $\eta : X \times X \to X$ be (τ)-Lipschitz continuous. For an arbitrarily chosen initial point x^0, let the sequence $\{x^k\}$ be generated* *by an iterative algorithm of the form*

$$x^{k+1} = (1-\alpha_k)x^k + \alpha_k y^k \quad \forall\, k \geq 0, \tag{2}$$

and y^k *satisfies*

$$\|y^k - J^{M,\eta}_{\rho_k,A}(A(x^k))\| \leq \delta_k \|y^k - x^k\|,$$

where $J^{M,\eta}_{\rho_k,A} = (A+\rho_k M)^{-1}$, *and*

$$\{\delta_k\}, \{\alpha_k\}, \{\rho_k\} \subseteq [0,\infty)$$

are scalar sequences such that $\sum_{k=0}^{\infty} \delta_k < \infty$, $\delta_k \to 0$, $\alpha = \limsup_{k\to\infty} \alpha_k$, $\rho_k \uparrow \rho \leq \infty$, $\alpha_k < 1$ *and* $\tau < 1$.

Then the sequence $\{x^k\}$ *converges linearly to a solution of* (1) *with convergence rate*

$$\sqrt{[1 - 2\alpha\{1-(1-\alpha)\frac{s\tau}{r-\rho m} - \frac{1}{2}\alpha[\frac{s\tau}{r-\rho m}]^2 - \frac{1}{2}\alpha\}]} < 1, \tag{3}$$

for $c = r - \rho m > 0$,

$$c > \frac{(1-\alpha)s\tau + \sqrt{((1-\alpha)s\tau)^2 + (2-\alpha)\alpha s^2\tau^2}}{2-\alpha}.$$

Proof. Suppose that x^* is a zero of M. From Theorem 3.1, it follows that any solution to (1) is a fixed point of $J^{M,\eta}_{\rho_k,A} oA$. For all $k \geq 0$, we express

$$z^{k+1} = (1-\alpha_k)x^k + \alpha_k J^{M,\eta}_{\rho_k,A}(A(x^k)).$$

Next, we find the estimate using Lemma 3.2 that

$$\begin{aligned}
&\|z^{k+1} - x^*\|^2 = \|(1-\alpha_k)x^k + \alpha_k J^{M,\eta}_{\rho_k,A}(A(x^k)) \\
- \quad & [(1-\alpha_k)x^* + \alpha_k J^{M,\eta}_{\rho_k,A}(A(x^*))]\|^2 \\
= \quad & \|(1-\alpha_k)(x^k - x^*) + \alpha_k(J^{M,\eta}_{\rho_k,A}(A(x^k)) - J^{M,\eta}_{\rho_k,A}(A(x^*)))\|^2 \\
= \quad & (1-\alpha_k)^2\|x^k - x^*\|^2 + 2\alpha_k(1-\alpha_k)\langle x^k - x^*, J^{M,\eta}_{\rho_k,A}(A(x^k)) - J^{M,\eta}_{\rho_k,A}(A(x^*))\rangle \\
+ \quad & \alpha_k^2\|J^{M,\eta}_{\rho_k,A}(A(x^k)) - J^{M,\eta}_{\rho_k,A}(A(x^*))\|^2 \\
\leq \quad & (1-\alpha_k)^2\|x^k - x^*\|^2 + 2\alpha_k(1-\alpha_k)\frac{s\tau}{r-\rho_k m}\|x^k - x^*\|^2 \\
+ \quad & \alpha_k^2\|J^{M,\eta}_{\rho_k,A}(A(x^k)) - J^{M,\eta}_{\rho_k,A}(A(x^*))\|^2 \\
\leq \quad & (1-\alpha_k)^2\|x^k - x^*\|^2 + 2\alpha_k(1-\alpha_k)\frac{s\tau}{r-\rho_k m}\|x^k - x^*\|^2 \\
+ \quad & \alpha_k^2(\frac{s\tau}{r-\rho_k m})^2\|x^k - x^*\|^2
\end{aligned}$$

$$= \quad [1-2\alpha_k\{1-(1-\alpha_k)\frac{s\tau}{r-\rho_k m}-\frac{1}{2}\alpha_k(\frac{s\tau}{r-\rho_k m})^2-\frac{1}{2}\alpha_k\}]\|x^k-x^*\|^2.$$

It follows that

$$\|z^{k+1}-x^*\| \le \theta_k\|x^k-x^*\|,$$

where

$$\theta_k=\sqrt{[1-2\alpha_k\{1-(1-\alpha_k)\frac{s\tau}{r-\rho_k m}-\frac{1}{2}\alpha_k(\frac{s\tau}{r-\rho_k m})^2-\frac{1}{2}\alpha_k\}]}.$$

Since $x^{k+1}=(1-\alpha_k)x^k+\alpha_k y^k$, we have $x^{k+1}-x^k=\alpha_k(y^k-x^k)$. It follows that

$$\begin{aligned}\|x^{k+1}-z^{k+1}\| &= \|(1-\alpha_k)x^k+\alpha_k y^k-[(1-\alpha_k)x^k+\alpha_k J^{M,\eta}_{\rho_k,A}(A(x^k))]\|\\ &= \|\alpha_k(y^k-J^{M,\eta}_{\rho,A}(A(x^k)))\|\\ &\le \alpha_k\delta_k\|y^k-x^k\|.\end{aligned}$$

Next, we find the estimate

$$\begin{aligned}\|x^{k+1}-x^*\| &\le \|z^{k+1}-x^*\|+\|x^{k+1}-z^{k+1}\|\\ &\le \|z^{k+1}-x^*\|+\alpha_k\delta_k\|y^k-x^k\|\\ &\le \|z^{k+1}-x^*\|+\delta_k\|x^{k+1}-x^k\|\\ &\le \|z^{k+1}-x^*\|+\delta_k\|x^{k+1}-x^*\|+\delta_k\|x^k-x^*\|\\ &\le \theta_k\|x^k-x^*\|+\delta_k\|x^{k+1}-x^*\|+\delta_k\|x^k-x^*\|. \qquad (4)\end{aligned}$$

This implies that

$$\|x^{k+1}-x^*\| \le \frac{\theta_k+\delta_k}{1-\delta_k}\|x^k-x^*\|,$$

where

$$\begin{aligned}\limsup\frac{\theta_k+\delta_k}{1-\delta_k} &= \limsup\,\theta_k\\ &= \sqrt{[1-2\alpha\{1-(1-\alpha)\frac{s\tau}{r-\rho m}-\frac{1}{2}\alpha(\frac{s\tau}{r-\rho m})^2-\frac{1}{2}\alpha\}]}<1. \quad \square\end{aligned}$$

Theorem 3.4. *Let X be a real Hilbert space, let $H:X\to X$ be (r,η)-strongly monotone and (s)-Lipschitz continuous, and let $M:X\to 2^X$ be (H,η)-maximal monotone. Let $\eta: X\times X\to X$ be (τ)-Lipschitz continuous. For an arbitrarily chosen initial point x^0, let the sequence $\{x^k\}$ be generated by another iterative algorithm*

$$x^{k+1}=(1-\alpha_k)x^k+\alpha_k y^k \quad \forall\, k\ge 0, \qquad (5)$$

and y^k satisfies

$$\|y^k-J^{M,\eta}_{\rho_k,H}(H(x^k))\| \le \delta_k\|y^k-x^k\|,$$

where $J^{M,\eta}_{\rho_k,H} = (H + \rho_k M)^{-1}$, *and*

$$\{\delta_k\}, \{\alpha_k\}, \{\rho_k\} \subseteq [0, \infty)$$

are scalar sequences such that $\sum_{k=0}^{\infty} \delta_k < \infty$, $\delta_k \to 0$, $\alpha = \limsup_{k\to\infty} \alpha_k$, $\rho_k \uparrow \rho \le \infty$, $\alpha_k < 1$ *and* $\tau < 1$.

Then the sequence $\{x^k\}$ *converges linearly to a solution of* (1) *for*

$$r > \frac{(1-\alpha)s\tau + \sqrt{((1-\alpha)s\tau)^2 + (2-\alpha)\alpha s^2\tau^2}}{2-\alpha}.$$

Proof. The proof is similar to that of Theorem 3.3. Suppose that x^* is a zero of M. From Theorem 3.2, it follows that any solution to (1) is a fixed point of $J^{M,\eta}_{\rho_k,H} oH$. For all $k \ge 0$, we express

$$z^{k+1} = (1-\alpha_k)x^k + \alpha_k J^{M,\eta}_{\rho_k,H}(H(x^k)).$$

Next, we find the estimate using Lemma 3.3 that

$$\begin{aligned} &\|z^{k+1} - x^*\|^2 = \|(1-\alpha_k)x^k + \alpha_k J^{M,\eta}_{\rho_k,H}(H(x^k)) \\ -\ & [(1-\alpha_k)x^* + \alpha_k J^{M,\eta}_{\rho_k,H}(H(x^*))]\|^2 \\ =\ & \|(1-\alpha_k)(x^k - x^*) + \alpha_k(J^{M,\eta}_{\rho_k,H}(H(x^k)) - J^{M,\eta}_{\rho_k,H}(H(x^*)))\|^2 \\ =\ & (1-\alpha_k)^2\|x^k - x^*\|^2 + 2\alpha_k(1-\alpha_k)\langle x^k - x^*, J^{M,\eta}_{\rho_k,H}(H(x^k)) - J^{M,\eta}_{\rho_k,H}(A(x^*))\rangle \\ +\ & \alpha_k^2\|J^{M,\eta}_{\rho_k,H}(H(x^k)) - J^{M,\eta}_{\rho_k,H}(H(x^*))\|^2 \\ \le\ & (1-\alpha_k)^2\|x^k - x^*\|^2 + 2\alpha_k(1-\alpha_k)\frac{s\tau}{r}\|x^k - x^*\|^2 \\ +\ & \alpha_k^2\|J^{M,\eta}_{\rho_k,H}(H(x^k)) - J^{M,\eta}_{\rho_k,H}(H(x^*))\|^2 \\ \le\ & (1-\alpha_k)^2\|x^k - x^*\|^2 + 2\alpha_k(1-\alpha_k)\frac{s\tau}{r}\|x^k - x^*\|^2 + \alpha_k^2(\frac{s\tau}{r})^2\|x^k - x^*\|^2 \\ =\ & [1 - 2\alpha_k\{1 - (1-\alpha_k)\frac{s\tau}{r} - \frac{1}{2}\alpha_k(\frac{s\tau}{r})^2 - \frac{1}{2}\alpha_k\}]\|x^k - x^*\|^2. \end{aligned}$$

It follows that

$$\|z^{k+1} - x^*\| \le \theta_k\|x^k - x^*\|,$$

where

$$\theta_k = \sqrt{[1 - 2\alpha_k\{1 - (1-\alpha_k)\frac{s\tau}{r} - \frac{1}{2}\alpha_k(\frac{s\tau}{r})^2 - \frac{1}{2}\alpha_k\}]}.$$

Since $x^{k+1} = (1-\alpha_k)x^k + \alpha_k y^k$, we have $x^{k+1} - x^k = \alpha_k(y^k - x^k)$. It follows that

$$\|x^{k+1} - z^{k+1}\| = \|(1-\alpha_k)x^k + \alpha_k y^k - [(1-\alpha_k)x^k + \alpha_k J^{M,\eta}_{\rho_k,H}(H(x^k))]\|$$

$$= \|\alpha_k(y^k - J_{\rho,H}^{M,\eta}(H(x^k)))\| \leq \alpha_k\delta_k\|y^k - x^k\|.$$

Next, we find the estimate

$$\begin{aligned}
\|x^{k+1} - x^*\| &\leq \|z^{k+1} - x^*\| + \|x^{k+1} - z^{k+1}\| \\
&\leq \|z^{k+1} - x^*\| + \alpha_k\delta_k\|y^k - x^k\| \\
&\leq \|z^{k+1} - x^*\| + \delta_k\|x^{k+1} - x^k\| \\
&\leq \|z^{k+1} - x^*\| + \delta_k\|x^{k+1} - x^*\| + \delta_k\|x^k - x^*\| \\
&\leq \theta_k\|x^k - x^*\| + \delta_k\|x^{k+1} - x^*\| + \delta_k\|x^k - x^*\|. \qquad (6)
\end{aligned}$$

This implies that

$$\|x^{k+1} - x^*\| \leq \frac{\theta_k + \delta_k}{1 - \delta_k}\|x^k - x^*\|,$$

where

$$\begin{aligned}
\limsup \frac{\theta_k + \delta_k}{1 - \delta_k} &= \limsup \theta_k \\
&= \sqrt{[1 - 2\alpha\{1 - (1-\alpha)\frac{s\tau}{r} - \frac{1}{2}\alpha(\frac{s\tau}{r})^2 - \frac{1}{2}\alpha\}]} < 1. \quad \Box
\end{aligned}$$

4. Concluding Remarks

The general framework of Theorem 3.3 as well as of Theorem 3.4 toward the linear convergence much depends on the Lipschitz continuity constant $\tau < 1$ of mapping η, while the strong monotonicity constant r and Lipschitz continuity constant s for mapping A are always governed by $r \leq s$. In absence of the condition $\tau < 1$, the linear convergence analysis may not be achieved at all. Moreover, even a weak convergence, like in the case Eckstein and Bertsekas [2], may not be established simply because of the relationship $r \leq s$ between the strong monotonicity constant r and the Lipschitz continuity constant s of mapping A.

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