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SUBCLASSES OF HARMONIC UNIVALENT FUNCTIONS ASSOCIATED WITH HYPERGEOMETRIC FUNCTIONS

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Abstract: In the present paper two subclasses of S_H are defined using convolution of harmonic univalent functions with hypergeometric functions and several interesting properties like coefficient bound, distortion theorem and extreme points are obtained.

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1. Introduction

A continuous function $f = u + iv$ in any simply connected domain $D \subset C$ is a complex-valued, harmonic function if both u and v are real harmonic in D. In [4], Clunie and Sheil-Small developed the basic theory of harmonic functions which are univalent in $U = \{z : |z| < 1\}$ having normalization $f(0) = 0$ and $f_z(0) = 1$. Such functions admit representation

$$f = h + \bar{g}, \tag{1.1}$$

where h and g are analytic. In this case f is sense-preserving if $|h'(z)| > |g'(z)|$ in U or equivalently, if dilation function $\omega = \frac{g'}{h'}$ satisfies $|\omega| < 1$ for $z \in U$. To this end, without loss of generality we may write

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$$h(z) = z + \sum_{n=2}^{\infty} A_n z^n, \quad g(z) = \sum_{n=1}^{\infty} B_n z^n, \quad z \in U, \quad |B_1| < 1. \tag{1.2}$$

Let S_H denote the family of functions of the form (1.1) which are harmonic, univalent and sense-preserving in U. For harmonic functions $f = h + \bar{g}$ we call h as the analytic part of f and g as the co-analytic part of f. Note that the familiar class S of analytic functions is contained in S_H. S_H reduces to S, if the co-analytic part of f is identically zero.

Let S_H^* and K_H denote the subclasses of S_H consisting of harmonic univalent functions which map U onto starlike and convex domains respectively. In [7] and [8], Jahangiri studied the subclasses $S_H^*(\alpha)$ and $K_H(\alpha)(0 \leq \alpha < 1)$, which are harmonic starlike functions of order α and harmonic convex functions of order α in U respectively. Since then, there have been several related papers on S_H and its subclasses. For more information see [3, 5, 6, 11].

Recently Ahuja [2], studied the connection between various subclasses of S_H and hypergeometric functions.

In the present paper we define two new subclasses of S_H which are generated by taking convolution of harmonic univalent functions with hypergeometric functions. Consider the Gaussian hypergeometric function defined by

$$F(a_i, b_i; c_i; z) = \sum_{n=0}^{\infty} \frac{(a_i)_n (b_i)_n}{(c_i)_n (1)_n} z^n, \quad a_i, b_i, c_i \in C \text{ and } c_i \neq 0, -1, -2, \dots.$$

It is well-known that hypergeometric functions play an important role in the theory of univalent functions. The extensive use of such functions in geometric function theory has been shown by a number of authors which include Srivastava and Manocha [12], Owa and Srivastava [9], and Ruscheweyh and Singh [10].

We shall consider the functions $\Phi_1(z)$ and $\Phi_2(z)$ defined by

$$\Phi_1(z) = zF(a_1, b_1; c_1; z) = z + \sum_{n=2}^{\infty} \frac{(a_1)_{n-1} (b_1)_{n-1}}{(c_1)_{n-1} (1)_{n-1}} z^n \tag{1.3}$$

and

$$\Phi_2(z) = zF(a_2, b_2; c_2; z) = \sum_{n=1}^{\infty} \frac{(a_2)_{n-1} (b_2)_{n-1}}{(c_2)_{n-1} (1)_{n-1}} z^n. \tag{1.4}$$

Definition 1. Let $S_H^*(\Phi_1, \Phi_2, \alpha)$ denote the subclass of S_H consisting of the functions $f = h + \bar{g}$ that satisfies the condition

$$\operatorname{Re}\left\{ \frac{z(h(z) * \Phi_1(z))' - \overline{z(g(z) * \Phi_2(z))'}}{(h(z) * \Phi_1(z)) + \overline{(g(z) * \Phi_2(z))}} \right\} \geq \alpha, \tag{1.5}$$

where $0 \leq \alpha < 1, \Phi_1(z)$ and $\Phi_2(z)$ are given by (1.3) and (1.4) respectively.

Definition 2. Let $K_H(\Phi_1, \Phi_2, \alpha)$ denote subclass of S_H consisting of the functions $f = h + \bar{g}$ that satisfies the condition

$$\mathrm{Re}\left\{ \frac{\begin{array}{c} z^2(h(z) * \Phi_1(z))'' + z(h(z) * \Phi_1(z))' \\ +\overline{z^2(g(z) * \Phi_2(z))'' + z(g(z) * \Phi_2(z))'} \end{array}}{z(h(z) * \Phi_1(z))' - \overline{z(g(z) * \Phi_2(z))'}} \right\} \geq \alpha, \qquad (1.6)$$

where $0 \leq \alpha < 1, \Phi_1(z)$ and $\Phi_2(z)$ are given by (1.3) and (1.4) respectively.

We denote by T_H the subclass of S_H consisting functions of the form $f = h + \bar{g}$ where h and g are given by

$$h(z) = z - \sum_{n=2}^{\infty} |A_n| z^n, \quad g(z) = \sum_{n=1}^{\infty} |B_n| z^n, \quad z \in U, \quad |B_1| < 1. \qquad (1.7)$$

Also let

$TS_H^*(\Phi_1, \Phi_2, \alpha) = S_H^*(\Phi_1, \Phi_2, \alpha) \cap T_H$ and $TK_H(\Phi_1, \Phi_2, \alpha) = K_H(\Phi_1, \Phi_2, \alpha) \cap T_H$. Note that the subclasses $TS_H^*(\Phi_1, \Phi_2, \alpha)$ and $TK_H(\Phi_1, \Phi_2, \alpha)$ are non-empty as for the particular values of Φ_1 and Φ_2 that is by taking $\Phi_1(z) = zF(1, 1; 1; z)$ and $\Phi_2(z) = zF(1, 1; 1; z)$ we observe that $TS_H^*(\Phi_1, \Phi_2, \alpha)$ and $TK_H(\Phi_1, \Phi_2, \alpha)$ reduces to the classes studied by Jahangiri in [7, 8].

2. Results and Proofs

Theorem 2.1. *Let $f = h + \bar{g}$, be such that h and g are given by (1.2). Furthermore let*

$$\sum_{n=1}^{\infty} \left[\left(\frac{n - \alpha}{1 - \alpha} \right) \frac{(a_1)_{n-1}(b_1)_{n-1}}{(c_1)_{n-1}(1)_{n-1}} |A_n| + \left(\frac{n + \alpha}{1 - \alpha} \right) \frac{(a_2)_{n-1}(b_2)_{n-1}}{(c_2)_{n-1}(1)_{n-1}} |B_n| \right] \leq 2 \qquad (2.1)$$

with $A_1 = 1, 0 \leq \alpha < 1$ and $n \leq \left(\frac{n-\alpha}{1-\alpha}\right) \frac{(a_1)_{n-1}(b_1)_{n-1}}{(c_1)_{n-1}(1)_{n-1}}$ and $n \leq \left(\frac{n+\alpha}{1-\alpha}\right) \frac{(a_2)_{n-1}(b_2)_{n-1}}{(c_2)_{n-1}(1)_{n-1}}$. Then f is harmonic, univalent and sense-preserving in U and $f \in S_H^*(\Phi_1, \Phi_2, \alpha)$.

Proof. First we have to show that $f = h + \bar{g}$ is locally univalent and sense-preserving in U, for which it is sufficient to show that $|h'(z)| > |g'(z)|$.

Consider

$$|h'(z)| \geq 1 - \sum_{n=2}^{\infty} n|A_n| r^{n-1} > 1 - \sum_{n=2}^{\infty} n|A_n|$$

$$\geq 1-\sum_{n=2}^{\infty}\left(\frac{n-\alpha}{1-\alpha}\right)\frac{(a_1)_{n-1}(b_1)_{n-1}}{(c_1)_{n-1}(1)_{n-1}}|A_n| \geq \sum_{n=1}^{\infty}\left(\frac{n+\alpha}{1-\alpha}\right)\frac{(a_2)_{n-1}(b_2)_{n-1}}{(c_2)_{n-1}(1)_{n-1}}|B_n|$$
$$\geq \sum_{n=1}^{\infty} n|B_n| \geq \sum_{n=1}^{\infty} n|B_n|r^{n-1} > |g'(z)|.$$

Now to show f is univalent in U, suppose $z_1, z_2 \in U$ such that $z_1 \neq z_2$ then

$$\left|\frac{f(z_1)-f(z_2)}{h(z_1)-h(z_2)}\right| \geq 1-\left|\frac{g(z_1)-g(z_2)}{h(z_1)-h(z_2)}\right|$$
$$= 1-\left|\frac{\sum_{n=1}^{\infty} B_n(z_1^n-z_2^n)}{(z_1-z_2)+\sum_{n=2}^{\infty} A_n(z_1^n-z_2^n)}\right| > 1-\left|\frac{\sum_{n=1}^{\infty} nB_n}{1-\sum_{n=2}^{\infty} nA_n}\right|$$
$$\geq 1-\frac{\sum_{n=1}^{\infty}\left(\frac{n+\alpha}{1-\alpha}\right)\frac{(a_2)_{n1}(b_2)_{n-1}}{(c_2)_{n1}(1)_{n1}}|B_n|}{1-\sum_{n=2}^{\infty}\left(\frac{n-\alpha}{1-\alpha}\right)\frac{(a_1)_{n-1}(b_1)_{n-1}}{(c_1)_{n-1}(1)_{n-1}}|A_n|} \geq 0.$$

Now, we show that $f \in S_H^*(\Phi_1, \Phi_2, \alpha)$.

Using the fact that $\mathrm{Re}\,\omega \geq \alpha$ if and only if $|1-\alpha+\omega| \geq |1+\alpha-\omega|$, it suffices to show that

$$|A(z)+(1-\alpha)B(z)| - |A(z)-(1+\alpha)B(z)| \geq 0, \tag{2.2}$$

where $A(z) = z(h(z)*\Phi_1(z))' - \overline{z(g(z)*\Phi_2(z))'}$ and $B(z) = (h(z)*\Phi_1(z)) + (g(z)*\Phi_2(z))$ substituting values of $A(z)$ and $B(z)$ in equation (2.2) and making use of (2.1) we obtain

$$|A(z)+(1-\alpha)B(z)|-|A(z)-(1+\alpha)B(z)| = |z(h(z)*\Phi_1(z))'-\overline{z(g(z)*\Phi_2(z))'}$$
$$+(1-\alpha)(h(z)*\Phi_1(z))+\overline{(g(z)*\Phi_2(z))}| - |z(h(z)*\Phi_1(z))'-\overline{z(g(z)*\Phi_2(z))'}$$
$$-(1+\alpha)(h(z)*\Phi_1(z))+\overline{(g(z)*\Phi_2(z))}|$$
$$= \left|(2-\alpha)z+\sum_{n=2}^{\infty}(n+1-\alpha)\frac{(a_1)_{n-1}(b_1)_{n-1}}{(c_1)_{n-1}(1)_{n-1}}A_n z^n\right.$$
$$\left.-\sum_{n=1}^{\infty}\overline{(n-1+\alpha)\frac{(a_2)_{n-1}(b_2)_{n-1}}{(c_2)_{n-1}(1)_{n-1}}B_n z^n}\right|$$
$$-\left|-\alpha z+\sum_{n=2}^{\infty}(n-1-\alpha)\frac{(a_1)_{n-1}(b_1)_{n-1}}{(c_1)_{n-1}(1)_{n-1}}A_n z^n\right.$$
$$\left.-\sum_{n=1}^{\infty}\overline{(n+1+\alpha)\frac{(a_2)_{n-1}(b_2)_{n-1}}{(c_2)_{n-1}(1)_{n-1}}B_n z^n}\right|$$

$$\geq 2(1-\alpha)|z| - \sum_{n=2}^{\infty}(n+1-\alpha)\frac{(a_1)_{n-1}(b_1)_{n-1}}{(c_1)_{n-1}(1)_{n-1}}|A_n||z|^n$$
$$- \sum_{n=1}^{\infty}(n-1+\alpha)\frac{(a_2)_{n-1}(b_2)_{n-1}}{(c_2)_{n-1}(1)_{n-1}}|B_n||z|^n$$
$$- \alpha|z| - \sum_{n=2}^{\infty}(n-1-\alpha)\frac{(a_1)_{n-1}(b_1)_{n-1}}{(c_1)_{n-1}(1)_{n-1}}|A_n||z|^n$$
$$- \sum_{n=1}^{\infty}(n+1+\alpha)\frac{(a_2)_{n-1}(b_2)_{n-1}}{(c_2)_{n-1}(1)_{n-1}}|B_n||z|^n$$
$$= 2(1-\alpha)|z|\left\{1 - \sum_{n=2}^{\infty}\left[\frac{n-\alpha}{1-\alpha}\right]\frac{(a_1)_{n-1}(b_1)_{n-1}}{(c_1)_{n-1}(1)_{n-1}}|A_n||z|^{n-1}\right.$$
$$\left. - \sum_{n=1}^{\infty}\left(\frac{n+\alpha}{1-\alpha}\right)\frac{(a_2)_{n-1}(b_2)_{n-1}}{(c_2)_{n-1}(1)_{n-1}}|B_n||z|^{n-1}\right\}$$
$$\geq 2(1-\alpha)\left\{1 - \sum_{n=2}^{\infty}\left[\frac{n-\alpha}{1-\alpha}\right]\frac{(a_1)_{n-1}(b_1)_{n-1}}{(c_1)_{n-1}(1)_{n-1}}|A_n|\right.$$
$$\left. - \sum_{n=1}^{\infty}\left(\frac{n+\alpha}{1-\alpha}\right)\frac{(a_2)_{n-1}(b_2)_{n-1}}{(c_2)_{n-1}(1)_{n-1}}|B_n|\right\} \geq 0,$$

by the condition (2.1). Hence $f \in S_H^*(\Phi_1, \Phi_2, \alpha)$. □

The coefficient bound is sharp for the function

$$f(z) = z + \sum_{n=2}^{\infty}\left[\frac{1-\alpha}{n-\alpha}\right]\frac{(c_1)_{n-1}(1)_{n-1}}{(a_1)_{n-1}(b_1)_{n-1}}x_n z^n$$
$$+ \sum_{n=1}^{\infty}\left(\frac{1-\alpha}{n+\alpha}\right)\frac{(c_2)_{n-1}(1)_{n-1}}{(a_2)_{n-1}(b_2)_{n-1}}\bar{y}_n \bar{z}^n, \tag{2.3}$$

where $\sum_{n=2}^{\infty}|x_n| + \sum_{n=1}^{\infty}|y_n| = 1$.

Theorem 2.2. *Let the function $f = h + \bar{g}$ be so that h and g are given by (1.7). Then $f \in TS_H^*(\Phi_1, \Phi_2, \alpha)$ if and only if*

$$\sum_{n=1}^{\infty}\left[\left(\frac{n-\alpha}{1-\alpha}\right)\frac{(a_1)_{n-1}(b_1)_{n-1}}{(c_1)_{n-1}(1)_{n-1}}|A_n| + \left(\frac{n+\alpha}{1-\alpha}\right)\frac{(a_2)_{n-1}(b_2)_{n-1}}{(c_2)_{n-1}(1)_{n-1}}|B_n|\right] \leq 2, \tag{2.4}$$

where $A_1 = 1, 0 \leq \alpha < 1$ and $n \leq \left(\frac{n-\alpha}{1-\alpha}\right)\frac{(a_1)_{n-1}(b_1)_{n-1}}{(c_1)_{n-1}(1)_{n-1}}$ and $n \leq \left(\frac{n+\alpha}{1-\alpha}\right)\frac{(a_2)_{n-1}(b_2)_{n-1}}{(c_2)_{n-1}(1)_{n-1}}$.

Proof. In view of Theorem 2.1 we need only to prove that $f \notin TS_H^*$ (Φ_1, Φ_2, α) if the coefficient inequality (2.4) does not hold. To this end we have, if f is in $TS_H^*(\Phi_1, \Phi_2, \alpha)$ then

$$\operatorname{Re}\left\{\frac{z(h(z)*\Phi_1(z))' - \overline{z(g(z)*\Phi_2(z))'}}{(h(z)*\Phi_1(z)) + (g(z)*\Phi_2(z))} - \alpha\right\} \geq 0$$

which is equivalent to

$$\operatorname{Re}\left\{\frac{\begin{array}{c}(1-\alpha)|z| - \sum_{n=2}^{\infty}(n-\alpha)\frac{(a_1)_{n1}(b_1)_{n1}}{(c_1)_{n-1}(1)_{n1}}|A_n||z|^n \\ - \sum_{n=1}^{\infty}(n+\alpha)\frac{(a_2)_{n-1}(b_2)_{n-1}}{(c_2)_{n-1}(1)_{n-1}}|B_n||\bar{z}|^n\end{array}}{|z| - \sum_{n=2}^{\infty}\frac{(a_1)_{n-1}(b_1)_{n-1}}{(c_1)_{n-1}(1)_{n-1}}|A_n||z|^n - \sum_{n=1}^{\infty}\frac{(a_2)_{n-1}(b_2)_{n-1}}{(c_2)_{n-1}(1)_{n-1}}|B_n||\bar{z}|^n}\right\} \geq 0.$$

Upon choosing the values of z on the positive real axis where $0 \leq z = r < 1$, the above inequality reduces to

$$\frac{\begin{array}{c}(1-\alpha) - \sum_{n=2}^{\infty}(n-\alpha)\frac{(a_1)_{n-1}(b_1)_{n-1}}{(c_1)_{n-1}(1)_{n-1}}|A_n|r^{n-1} \\ - \sum_{n=1}^{\infty}(n+\alpha)\frac{(a_2)_{n-1}(b_2)_{n-1}}{(c_2)_{n-1}(1)_{n-1}}|B_n|r^{n-1}\end{array}}{1 - \sum_{n=2}^{\infty}\frac{(a_1)_{n-1}(b_1)_{n-1}}{(c_1)_{n-1}(1)_{n-1}}|A_n|r^{n-1} - \sum_{n=1}^{\infty}\frac{(a_2)_{n-1}(b_2)_{n-1}}{(c_2)_{n-1}(1)_{n-1}}|B_n|r^{n-1}} \geq 0. \tag{2.5}$$

If condition (2.4) does not hold then the numerator in (2.5) is negative for r sufficiently close to 1. Thus there exists $z_0 = r_0$ in $(0,1)$ for which the quotient of (2.5) is negative. This contradicts the required condition for $f \in TS_H^*(\Phi_1, \Phi_2, \alpha)$ and so the proof is completed. □

Theorem 2.3. *If $f = h + \bar{g}$ be such that h and g are given by (1.2). Also let*

$$\sum_{n=1}^{\infty}\left[n\left(\frac{n-\alpha}{1-\alpha}\right)\frac{(a_1)_{n-1}(b_1)_{n-1}}{(c_1)_{n-1}(1)_{n-1}}|A_n| + n\left(\frac{n+\alpha}{1-\alpha}\right)\frac{(a_2)_{n-1}(b_2)_{n-1}}{(c_2)_{n-1}(1)_{n-1}}|B_n|\right] \leq 2 \tag{2.6}$$

with $A_1 = 1, 0 \leq \alpha < 1$ and $n \leq \left(\frac{n-\alpha}{1-\alpha}\right)\frac{(a_1)_{n-1}(b_1)_{n-1}}{(c_1)_{n-1}(1)_{n-1}}$ and $n \leq \left(\frac{n+\alpha}{1-\alpha}\right)\frac{(a_2)_{n-1}(b_2)_{n-1}}{(c_2)_{n-1}(1)_{n-1}}$. Then f is harmonic, univalent and sense-preserving in U and $f \in K_H(\Phi_1, \Phi_2, \alpha)$.

Theorem 2.4. *If $f = h + \bar{g}$ be such that h and g are given by (1.7). Then $f \in TK_H(\Phi_1, \Phi_2, \alpha)$ if and only if (2.6) is satisfied.*

We omit the proofs of Theorems 2.3 and 2.4 since they are similar to that of Theorem 2.2.

Theorem 2.5. *Let $f \in TS_H^*(\Phi_1, \Phi_2, \alpha)$ and*

$$\frac{(2-\alpha)a_1b_1}{(1-\alpha)c_1} \leq \frac{(n-\alpha)}{(1-\alpha)}\frac{(a_1)_{n-1}(b_1)_{n-1}}{(c_1)_{n-1}(1)_{n-1}}$$

and

$$\frac{(2-\alpha)a_1b_1}{(1-\alpha)c_1} \leq \frac{(n+\alpha)}{(1-\alpha)}\frac{(a_2)_{n-1}(b_2)_{n-1}}{(c_2)_{n-1}(1)_{n-1}}$$

for $n \geq 2$. Then we have

$$|f(z)| \leq (1+|B_1|)r + \frac{c_1}{a_1b_1}\left(\frac{(1-\alpha)}{(2-\alpha)} - \frac{(1+\alpha)}{(2-\alpha)}|B_1|\right)r^2, \quad |z| = r < 1 \quad (2.7)$$

and

$$|f(z)| \geq (1-|B_1|)r - \frac{c_1}{a_1b_1}\left(\frac{(1-\alpha)}{(2-\alpha)} - \frac{(1+\alpha)}{(2-\alpha)}|B_1|\right)r^2, \quad |z| = r < 1. \quad (2.8)$$

Proof. Let $f \in TS_H^*(\Phi_1, \Phi_2, \alpha)$, then we have,

$$\begin{aligned}
|f(z)| &\leq (1+|B_1|)r + \sum_{n=2}^{\infty}(|A_n| + |B_n|)r^n \\
&\leq (1+|B_1|)r + \sum_{n=2}^{\infty}(|A_n| + |B_n|)r^2 \\
&\leq (1+|B_1|)r + \left(\frac{1-\alpha}{2-\alpha}\right)\frac{c_1}{a_1b_1}\sum_{n=2}^{\infty} \\
&\quad \times \left(\left(\frac{2-\alpha}{1-\alpha}\right)\frac{a_1b_1}{c_1}|A_n| + \left(\frac{2-\alpha}{1-\alpha}\right)\frac{a_1b_1}{c_1}|B_n|\right)r^2 \\
&\leq (1+|B_1|)r + \left(\frac{1-\alpha}{2-\alpha}\right)\frac{c_1}{a_1b_1}\sum_{n=2}^{\infty}\left(\left(\frac{n-\alpha}{1-\alpha}\right)\frac{(a_1)_{n-1}(b_1)_{n-1}}{(c_1)_{n-1}(1)_{n-1}}|A_n|\right. \\
&\quad \left. + \left(\frac{n+\alpha}{1-\alpha}\right)\frac{(a_2)_{n-1}(b_2)_{n-1}}{(c_2)_{n-1}(1)_{n-1}}|B_n|\right)r^2 \\
&\leq (1+|B_1|)r + \left(\frac{1-\alpha}{2-\alpha}\right)\frac{c_1}{a_1b_1}\left(1 - \frac{1+\alpha}{1-\alpha}|B_1|\right)r^2, \quad \text{by (2.4)} \\
&\leq (1+|B_1|)r + \frac{c_1}{a_1b_1}\left(\frac{1-\alpha}{2-\alpha} - \frac{1+\alpha}{2-\alpha}|B_1|\right)r^2
\end{aligned}$$

and similarly we can prove the other inequality (2.8). □

Theorem 2.6. *Let $f \in TK_H(\Phi_1, \Phi_2, \alpha)$ with*

$$\frac{2(2-\alpha)a_1b_1}{(1-\alpha)c_1} \leq \frac{(n-\alpha)}{(1-\alpha)}\frac{(a_1)_{n-1}(b_1)_{n-1}}{(c_1)_{n-1}(1)_{n-1}}$$

and

$$\frac{2(2-\alpha)a_1b_1}{(1-\alpha)c_1} \le \frac{(n+\alpha)}{(1-\alpha)}\frac{(a_2)_{n-1}(b_2)_{n-1}}{(c_2)_{n-1}(1)_{n-1}}.$$

Then

$$|f(z)| \le (1+|B_1|)r + \frac{c_1}{2a_1b_1}\left(\frac{(1-\alpha)}{(2-\alpha)} - \frac{(1+\alpha)}{(2-\alpha)}|B_1|\right)r^2 \quad |z| = r < 1$$

and

$$|f(z)| \ge (1-|B_1|)r - \frac{c_1}{2a_1b_1}\left(\frac{(1-\alpha)}{(2-\alpha)} - \frac{(1+\alpha)}{(2-\alpha)}|B_1|\right)r^2 \quad |z| = r < 1.$$

The details of the proof of Theorem 2.6 are omitted.

Next, we will derive extreme points for classes $TS^*_H(\Phi_1, \Phi_2, \alpha)$ and $TK_H(\Phi_1, \Phi_2, \alpha)$.

Theorem 2.7. *Let* $h_1(z) = z, h_n(z) = z - \frac{1-\alpha}{n-\alpha}\frac{(c_1)_{n-1}(1)_{n-1}}{(a_1)_{n-1}(b_1)_{n-1}}z^n, (n \ge 2)$ *and* $g_n(z) = z + \frac{1-\alpha}{n+\alpha}\frac{(c_2)_{n-1}(1)_{n-1}}{(a_2)_{n-1}(b_2)_{n-1}}\bar{z}^n, (n \ge 1)$*. Then* $f \in TS^*_H(\Phi_1, \Phi_2, \alpha)$ *if and only if it can be expressed as* $f(z) = \sum_{n=1}^{\infty}(x_nh_n + y_ng_n)$*, where* $x_n \ge 0, y_n \ge 0, \sum_{n=1}^{\infty}(x_n + y_n) = 1$.

In particular the extreme points of $TS^*_H(\Phi_1, \Phi_2, \alpha)$ *are given by* $\{h_n\}$ *and* $\{g_n\}$.

Proof. Let

$$\begin{aligned} f(z) &= \sum_{n=1}^{\infty}(x_nh_n + y_ng_n) \\ &= \sum_{n=1}^{\infty}(x_n + y_n)z - \sum_{n=2}^{\infty}\frac{1-\alpha}{n-\alpha}\frac{(c_1)_{n-1}(1)_{n-1}}{(a_1)_{n-1}(b_1)_{n-1}}x_nz^n \\ &\quad + \sum_{n=1}^{\infty}\frac{1-\alpha}{n+\alpha}\frac{(c_2)_{n-1}(1)_{n-1}}{(a_2)_{n-1}(b_2)_{n-1}}y_n\bar{z}^n. \end{aligned}$$

Then

$$\begin{aligned} &\sum_{n=2}^{\infty}\frac{n-\alpha}{1-\alpha}\frac{(a_1)_{n-1}(b_1)_{n-1}}{(c_1)_{n-1}(1)_{n-1}}\left(\frac{1-\alpha}{n-\alpha}\frac{(c_1)_{n-1}(1)_{n-1}}{(a_1)_{n-1}(b_1)_{n-1}}x_n\right) \\ &+ \sum_{n=1}^{\infty}\frac{n+\alpha(a_2)_{n-1}(b_2)_{n-1}}{1-\alpha(c_2)_{n-1}(1)_{n-1}}\left(\frac{1-\alpha}{n+\alpha}\frac{(c_2)_{n-1}(1)_{n-1}}{(a_2)_{n-1}(b_2)_{n-1}}y_n\right) \\ &= \sum_{n=2}^{\infty}x_n + \sum_{n=2}^{\infty}y_n = 1 - x_1 \le 1, \end{aligned}$$

and hence $f \in TS_H^*(\Phi_1, \Phi_2, \alpha)$. Conversely, if $f \in TS_H^*(\Phi_1, \Phi_2, \alpha)$, then

$$|A_n| \leq \frac{1-\alpha}{n-\alpha} \frac{(c_1)_{n-1}(1)_{n-1}}{(a_1)_{n-1}(b_1)_{n-1}} \quad \text{and} \quad |B_n| \leq \frac{1-\alpha}{n+\alpha} \frac{(c_2)_{n-1}(1)_{n-1}}{(a_2)_{n-1}(b_2)_{n-1}}.$$

Set

$$x_n = \frac{n-\alpha}{1-\alpha} \frac{(a_1)_{n-1}(b_1)_{n-1}}{(c_1)_{n-1}(1)_{n-1}} \quad \text{for} \quad n \geq 2$$

and

$$y_n = \frac{n+\alpha}{1-\alpha} \frac{(a_2)_{n-1}(b_2)_{n-1}}{(c_2)_{n-1}(1)_{n-1}} \quad \text{for} \quad n \geq 1.$$

Then note that by Theorem 2.2, $0 \leq x_n \leq 1 (n \geq 2)$ and $0 \leq y_n \leq 1 (n \geq 1)$ and in view of this we define $x_1 = 1 - \sum_{n=2}^{\infty} x_n - \sum_{n=1}^{\infty} y_n \geq 0$. Consequently, we can see that $f(z)$ can be expressed in the form $f(z) = \sum_{n=1}^{\infty}(x_n h_n + y_n g_n)$ as required. □

Similarly, we have

Theorem 2.8. *Let* $h_1(z) = z, h_n(z) = z - \frac{1-\alpha}{n(n-\alpha)} \frac{(c_1)_{n-1}(1)_{n-1}}{(a_1)_{n-1}(b_1)_{n-1}} z^n$, $(n \geq 2)$ *and* $g_n(z) = z + \frac{1-\alpha}{n(n+\alpha)} \frac{(c_2)_{n-1}(1)_{n-1}}{(a_2)_{n-1}(b_2)_{n-1}} \bar{z}^n, (n \geq 1)$. *Then* $f \in TK_H(\Phi_1, \Phi_2, \alpha)$ *if and only if it can be expressed as* $f(z) = \sum_{n=1}^{\infty}(x_n h_n + y_n g_n)$, *where* $x_n \geq 0, y_n \geq 0, \sum_{n=1}^{\infty}(x_n + y_n) = 1$.

In particular the extreme points of $TK_H(\Phi_1, \Phi_2, \alpha)$ *are given by* $\{h_n\}$ *and* $\{g_n\}$.

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