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APPROXIMATE SOLUTION TO THE OVERDETERMINED TWO-DIMENSIONAL RESECTION PROBLEM

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Abstract: In surveying a resection is the observation of directions or lines of sight from a station to three or more targets with known (fixed or previously computed) coordinates and the computation of the station coordinates from these observations alone. In this article we shall minimize the sum of squares of the distances from the targets to the corresponding lines of sight. This will result in a set of equations linear in the station coordinates (and quadratic in a trigonometric function). Apart from exceptional cases, one of the solutions to the equations gives good approximate values for the station coordinates.

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1. Formulation of the Problem

We use a right-handed Cartesian coordinate system with x-axis and y-axis in the plane. The unknown station coordinates are $(x,\ y)$. The set of direction readings from the station are $\alpha_1, \alpha_2, \dots, \alpha_n$ to the targets $(x_1,\ y_1), (x_2,\ y_2), \dots, (x_n,\ y_n)$. Together with the coordinates of the station we introduce a third unknown variable, the so-called orientation constant α, so that the line of sight to target $(x_i,\ y_i)$ makes angle $\alpha + \alpha_i$ clockwise with the y-axis.

The scalar product of the unit vector $(\sin(\alpha + \alpha_i),\ \cos(\alpha + \alpha_i))$ in the direction of target $(x_i,\ y_i)$ and the vector $(x_i - x,\ y_i - y)$ must satisfy the relation

$$(x_i - x)\sin(\alpha + \alpha_i) + (y_i - y)\cos(\alpha + \alpha_i) > 0 \quad (i = 1, 2, \dots, n). \qquad (1)$$

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The unit vector $(\cos(\alpha+\alpha_i),\ -\sin(\alpha+\alpha_i))$ is perpendicular to the line of sight and its scalar product with vector $(x_i - x,\ y_i - y)$ is numerically equal to the distance p_i from target $(x_i,\ y_i)$ to its corresponding line of sight:

$$p_i = (x_i - x)\cos(\alpha+\alpha_i) - (y_i - y)\sin(\alpha+\alpha_i). \tag{2}$$

The sum of squares to be minimized is therefore

$$\sum_1^n p_i^2 = \sum_1^n ((x_i - x)\cos(\alpha+\alpha_i) - (y_i - y)\sin(\alpha+\alpha_i))^2. \tag{3}$$

2. Reformulation

We introduce the following notation:

$$(c_i,\ s_i) = ((\cos(\alpha_i),\ \sin(\alpha_i)), \qquad (c,\ s) = ((\cos(\alpha),\ \sin(\alpha)),$$

$$\begin{bmatrix} u_i \\ v_i \end{bmatrix} = \begin{bmatrix} c_i & -s_i \\ s_i & c_i \end{bmatrix} \begin{bmatrix} x_i \\ y_i \end{bmatrix}, \qquad \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} c & -s \\ s & c \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}.$$

Then

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} c & s \\ -s & c \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} \tag{4}$$

and from (2) we deduce $p_i = (x_i - x)(c\,c_i - s\,s_i) - (y_i - y)(s\,c_i + c\,s_i) = (c_i\,x_i - s_i\,y_i)c - (s_i\,x_i + c_i\,y_i)s - (c\,x - s\,y)c_i + (s\,x + c\,y)s_i = u_i\,c - v_i\,s - c_i\,u + s_i\,v$ or $p_i = (c_i,\ s_i,\ u_i,\ v_i)\,\boldsymbol{z}$, where

$$\boldsymbol{z} = [-u,\ v,\ c,\ -s]^T. \tag{5}$$

Writing

$$\boldsymbol{p} = [p_1,\ p_2, \ldots, p_n]^T, \tag{6}$$

$$\mathsf{A} = \begin{bmatrix} c_1 & s_1 & u_1 & v_1 \\ c_2 & s_2 & u_2 & v_2 \\ \cdots & \cdots & \cdots & \cdots \\ c_n & s_n & u_n & v_n \end{bmatrix}, \tag{7}$$

$$\boldsymbol{p} = \mathsf{A}\,\boldsymbol{z}, \tag{8}$$

and

$$\mathsf{Q} = \mathsf{A}^T\mathsf{A}, \tag{9}$$

we get the positive-semidefinite quadratic form

$$\boldsymbol{p}^T\boldsymbol{p} = \boldsymbol{z}^T\mathsf{Q}\,\boldsymbol{z}. \tag{10}$$

In order to find the extreme points we take the derivatives of the quadratic form with respect to u, v, and α (observing that c and s are the cosine and sine of α)

$$\begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -s & -c \end{bmatrix} \mathsf{Q}\, \boldsymbol{z} = \mathbf{0}. \tag{11}$$

3. Factorization

Definition 1. The vectors given by $\boldsymbol{c} = (c_1, c_2, \ldots, c_n)$ and $\boldsymbol{s} = (s_1, s_2, \ldots, s_n)$ are the vectors of cosines and sines associated with the resection problem. We shall speak of the case of linear dependence when these vectors are linearly dependent.

Theorem 2. *In the case of linear dependence, the targets and the station lie on the same line, and conversely.*

Proof. In case of linear dependence, either $\boldsymbol{c} = \mathbf{0}$ or $\boldsymbol{s} = t\,\boldsymbol{c}$ for some t. In the first case the direction readings are all 90° or 270°, in the second case the readings are either $(\arctan(\mathrm{t})/\pi)\,180°$ or $(\arctan(\mathrm{t})/\pi + 1)\,180°$. In both cases the direction readings take one of two values differing by 180° and the targets and the station lie on the same line. If the reverse is true, the direction readings will all take one of two values differing by 180°, and the proof can be reduced to the same two cases as before. □

Theorem 3. *Except in the case of linear dependence, we have* $q_{11} > 0$, $q_{22} > 0$, *and* $|\mathsf{Q}_{11}| > 0$, *where* $|\mathsf{Q}_{11}|$ *is the determinant of the* 2×2 *submatrix in the upper left corner of* Q.

Proof. By Cauchy–Schwarz inequality (see [1] and [4]), we have for vectors $\boldsymbol{c}$ and $\boldsymbol{s}$ $\left(\sum_1^n c_i\, s_i\right)^2 \leq \sum_1^n c_i^2 \cdot \sum_1^n s_i^2$ and the equality holds only if the vectors are linearly dependent. In terms of elements of Q the inequality becomes $q_{12}\, q_{21} < q_{11}\, q_{22}$ or $|\mathsf{Q}_{11}| = q_{11}\, q_{22} - q_{12}\, q_{21} > 0$. Clearly q_{11} and q_{22} are then not zero, and since each is a sum of squares, each must be positive. □

Theorem 4. *Except in the case of linear dependence, the matrix* Q *of order* 4×4 *given by (9) can be uniquely written as the product of a lower triangular matrix* L *with units on the diagonal and element* $l_{43} = 0$, *and matrix* M *with zeros below the diagonal in the first two columns. The submatrix of order* 2×2 *in the lower right-hand corner of* M *is symmetric.*

Proof. Let us try to partition the unknown matrices L and M and the given matrix Q into submatrices of order 2×2:

$$\begin{bmatrix} \mathsf{L}_{11} & 0 \\ \mathsf{L}_{21} & \mathsf{I} \end{bmatrix} \begin{bmatrix} \mathsf{M}_{11} & \mathsf{M}_{12} \\ 0 & \mathsf{M}_{22} \end{bmatrix} = \begin{bmatrix} \mathsf{Q}_{11} & \mathsf{Q}_{12} \\ \mathsf{Q}_{21} & \mathsf{Q}_{22} \end{bmatrix}.$$

The relation $\mathsf{L}_1 \mathsf{M}_{11} = \mathsf{Q}_{11}$ gives both the value q_{21}/q_{11} for the subdiagonal element of L_1 and the value of M_{11}, so that

$$\mathsf{L}_{11} = \begin{bmatrix} 1 & 0 \\ q_{21}/q_{11} & 1 \end{bmatrix} \quad \text{and} \quad \mathsf{M}_{11} = \mathsf{L}_{11}^{-1} \mathsf{Q}_{11}.$$

By Theorem 3 the inverse of M_{11} exists and the other products of the submatrices give results as follows:

$$\begin{aligned} \mathsf{L}_{11} \mathsf{M}_{12} &= \mathsf{Q}_{12} \qquad \text{gives} \qquad & \mathsf{M}_{12} &= \mathsf{L}_{11}^{-1} \mathsf{Q}_{12} \\ \mathsf{L}_{21} \mathsf{M}_{11} &= \mathsf{Q}_{21} & \mathsf{L}_{21} &= \mathsf{Q}_{21} \mathsf{Q}_{11}^{-1} \mathsf{L}_{11} \\ \mathsf{L}_{21} \mathsf{M}_{12} + \mathsf{M}_{22} &= \mathsf{Q}_{22} & \mathsf{M}_{22} &= \mathsf{Q}_{22} - \mathsf{Q}_{21} \mathsf{Q}_{11}^{-1} \mathsf{Q}_{12} \end{aligned}$$

This proves the existence and uniqueness of the matrices L and M. The expression for M_{22} shows that it is symmetric. □

Corollary 5. *The form and uniqueness of matrices* L *and* M *show that they may be obtained by the process of LU factorization (see [2]) as applied to a positive-definite matrix, except that the last stage in the process, introduction of element* l_{43} *in matrix* L *and its effect on elements* m_{43} *and* m_{44} *in matrix* M*, is omitted.*

4. Matrix Manipulation

Theorem 6. *If a unique solution to the resection problem exists, it is among the solutions to the equations*

$$\begin{bmatrix} m_{11} & -m_{12} & -c\,m_{13} + s\,m_{14} \\ 0 & m_{22} & c\,m_{23} - s\,m_{24} \\ 0 & 0 & \begin{array}{c} -c\,s\,m_{33} - c^2 m_{43} \\ +s^2 m_{34} + c\,s\,m_{44} \end{array} \end{bmatrix} \begin{bmatrix} u \\ v \\ 1 \end{bmatrix} = \mathbf{0}, \tag{12}$$

where $m_{43} = m_{34}$.

Proof. According to Theorem 4 we can now write

$$\mathsf{Q} = \mathsf{L}\,\mathsf{M} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ l_{21} & 1 & 0 & 0 \\ l_{31} & l_{32} & 1 & 0 \\ l_{41} & l_{42} & 0 & 1 \end{bmatrix} \begin{bmatrix} m_{11} & m_{12} & m_{13} & m_{14} \\ 0 & m_{22} & m_{23} & m_{24} \\ 0 & 0 & m_{33} & m_{34} \\ 0 & 0 & m_{43} & m_{44} \end{bmatrix}, \tag{13}$$

where $m_{43} = m_{34}$. We substitute (13) in (11) to get

$$\begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -s & -c \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ l_{21} & 1 & 0 & 0 \\ l_{31} & l_{32} & 1 & 0 \\ l_{41} & l_{42} & 0 & 1 \end{bmatrix} \mathsf{M}\, \boldsymbol{z}$$
$$= \begin{bmatrix} -1 & 0 & 0 & 0 \\ l_{21} & 1 & 0 & 0 \\ -s\, l_{31} - c\, l_{41} & -s\, l_{32} - c\, l_{42} & -s & -c \end{bmatrix} \mathsf{M}\, \boldsymbol{z}$$
$$= \begin{bmatrix} 1 & 0 & 0 \\ -l_{21} & 1 & 0 \\ s\, l_{31} + c\, l_{41} & -s\, l_{32} - c\, l_{42} & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -s & -c \end{bmatrix} \mathsf{M}\, \boldsymbol{z} = \mathbf{0}.$$

Premultiplying this equation by the inverse of the first factor and substituting for M and $\boldsymbol{z}$ we get for the left-hand side

$$\begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -s & -c \end{bmatrix} \begin{bmatrix} m_{11} & m_{12} & m_{13} & m_{14} \\ 0 & m_{22} & m_{23} & m_{24} \\ 0 & 0 & m_{33} & m_{34} \\ 0 & 0 & m_{43} & m_{44} \end{bmatrix} \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & c \\ 0 & 0 & -s \end{bmatrix} \begin{bmatrix} u \\ v \\ 1 \end{bmatrix}.$$

Multiplication of the matrices gives the formula in the theorem. □

5. Results

1. Except in the case of linear dependence or vanishing of the discriminant $(m_{44} - m_{33})^2 + 4\, m_{34}^2$, proceed as follows:

$$\alpha^{(1)} = \begin{cases} \dfrac{\pi}{4}, & \text{if } m_{33} = m_{44}, \\ \dfrac{1}{2} \arctan(\dfrac{2\, m_{34}}{m_{44} - m_{33}}), & \text{if } m_{33} \neq m_{44}, \end{cases}$$

$$\alpha^{(k)} = \alpha^{(1)} + \frac{(k-1)\,\pi}{2}, \quad \text{for} \quad k = 1,\, 2,\, 3,\, 4.$$

With $\alpha = \alpha^{(1)}$ compute $u^{(1)}$ and $v^{(1)}$ using the first two equations in (12), $(\boldsymbol{p}^T \boldsymbol{p})^{(1)}$ using (5 – 8), $x^{(1)}$ and $y^{(1)}$ using (4). ($\alpha = \alpha^{(3)}$ gives the same values for these quantities).

Exchange $\alpha^{(3)}$ for $\alpha^{(1)}$ if $\alpha^{(1)}$ does not satisfy (1).

With $\alpha = \alpha^{(2)}$ compute $u^{(2)}$ and $v^{(2)}$, $(\boldsymbol{p}^T \boldsymbol{p})^{(2)}$, $x^{(2)}$, and $y^{(2)}$ using the same formulas as before. ($\alpha = \alpha^{(4)}$ gives the same values for these quantities).

Exchange $\alpha^{(4)}$ for $\alpha^{(2)}$ if $\alpha^{(2)}$ does not satisfy (1).

$$(x,\,y,\,\alpha)=\begin{cases}(x^{(1)},\,y^{(1)},\,\alpha^{(1)}), & \text{if } (\boldsymbol{p}^T\boldsymbol{p})^{(1)}<(\boldsymbol{p}^T\boldsymbol{p})^{(2)},\\ (x^{(2)},\,y^{(2)},\,\alpha^{(2)}), & \text{if } (\boldsymbol{p}^T\boldsymbol{p})^{(2)}<(\boldsymbol{p}^T\boldsymbol{p})^{(1)}.\end{cases}$$

2. In case of linear dependence the targets and the station lie on a common line. A unique solution is not available.

3. In case of a vanishing discriminant, $(m_{44}-m_{33})^2+4\,m_{34}^2=0$, the targets and the station lie on a common circle. A unique solution is not available.

6. Conclusion

The method of computation described in this article gives the same result as any classical resection (see [3]) when the number of targets is three. When the number of targets is four or more, all the targets are used in the computation, reducing the likelihood of failure – when all targets and the station lie on or close to a common line or circle.

The author used the method as one of two methods for computing initial values of coordinates in a program for adjustment of triangulation he wrote in 1965 (see [5]).

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