International Journal of Pure and Applied Mathematics
Volume 60 No. 1 2010, 45-48

A SYMMETRIC TENSOR RANK CHARACTERIZATION OF MINIMAL DEGREE SUBVARIETIES

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Abstract: Let $X \subset \mathbb{P}^n$ be an integral m-dimensional reflexive subvariety. Either X has minimal degree $n+1-m$ or a general point of a general tangent space of X has X-rank at most $n-m$.

AMS Subject Classification: 14N05, 14H99
Key Words: tensor rank, symmetric tensor rank, strange curve, reflexive variety, rational normal curves

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Here we improve [3] and give a modified proof of its main result. We work over an algebraically closed field $\mathbb{K}$. We recall the following definition. Let $X \subset \mathbb{P}^n$ be an integral and non-degenerate subvariety. For any $P \in \mathbb{P}^n$ the X-rank $r_X(P)$ of P is the minimal integer k such that there is $S \subset X$ with $\sharp(S) = k$ and $P \in \langle S \rangle$. Since X is non-degenerate, $r_X(P) \le n+1$ for all $P \in \mathbb{P}^n$. If $\mathrm{char}(\mathbb{K}) = 0$, then $r_X(P) \le n - \dim(X) + 1$ for all P (see [9], 5.1). For the notion of reflexive variety, see [6], [7], [5]. Here we prove the following result.

Theorem 1. *Let $X \subset \mathbb{P}^n$ be an integral and non-degenerate m-dimensional variety. If $m \ge 2$ assume that X is reflexive and* $\mathrm{char}(\mathbb{K}) \ne 2$. *If $n = m+2$ assume either* $\mathrm{char}(\mathbb{K}) = 0$ *or* $\mathrm{char}(\mathbb{K}) > d$. *Then either X is a minimal degree variety, i.e.* $\deg(X) = n - m + 1$ *or $r_X(P) \le n-1$ for a general point P of a general tangent space $T_Q X$ of X.*

Received: January 14, 2010

For the X-ranks of all points of $\mathbb{P}^n$ when X is a rational normal curve and $\mathrm{char}(\mathbb{K}) = 0$, see [4] or [9], 4.1.

We recall that if $\mathrm{chark}(\mathbb{K}) = 0$ every integral subvariety is reflexive. For more on reflexive varieties, see [7]. For results and further references on tensor ranks and symmetric tensor ranks (all in characteristic zero), see [9]).

Proof of Theorem 1. Take a general hyperplane section of X and use that in characteristic $\neq 2$ a general hyperplane section of a reflexive variety is reflexive (use the Hessian criterion, see [6], 3.2, or [7], Theorem 12). Hence it is sufficient to do the case $m = 1$. Assume $m = 1$. For any linear subspace $V \subset \mathbb{P}^n$ let $\ell_V : \mathbb{P}^n \backslash V \to \mathbb{P}^{n-\dim(V)-1}$ denote the linear projection. If $V = \{O\}$ is a point, then we write ℓ_O instead of $\ell_{\{O\}}$. Set $d := \deg(X)$. Thus $d > n$ if and only if X is not a rational normal curve. Fix a general $Q \in X$ and a general $P \in T_QX$. Since P is general, $P \notin X$. Hence $\ell_P|X : X \to \mathbb{P}^{n-1}$ is a morphism. If $\sharp((T_QX \cap X))_{red} \geq 2$, then $r_P(X) \leq 2$. Hence we may assume $(T_Q \cap X)_{red} = \{Q\}$. Set $z := \mathrm{length}(T_QX \cap X)$ (as schemes). Since Q is general, $z = 2$ if either $\mathrm{char}(\mathbb{K}) = 0$ or X is reflexive (see [5]). If $p := \mathrm{char}(\mathbb{K}) > 0$, then either $z = 2$ or $z = p^e$ for some integer $e > 0$ (see [5], p. 14). Let $D \subseteq \mathbb{P}^{n-2}$ be the image of X by the rational map ϕ from X induced by the linear projection ℓ_{T_QX} from T_QX. We have $d - z = \deg(\phi) \cdot \deg(D)$ with the convention $\deg(D) = 1$ if $n = 3$. If $\deg(\phi) \geq 2$, then every $A \in T_QX$ is in the linear span of a set $S \subset X$ such that $\sharp(S) \leq 3$ and $Q \in S$. Hence if $n \geq 4$ we may assume $\deg(\phi) = 1$. Hence $\deg(D) = d - z$.

First assume $n \geq 4$. Assume $d - z > n - 2$. Hence there is $A \subset D$ such that $A \subset D_{reg}$, $\phi|\phi^{-1}(A) : \phi^{-1}(A) \to A$ is a bijection, $\sharp(A) = n-1$, $Q \notin \phi^{-1}(A)$ and $\dim(\langle A \rangle) = n-3$. Hence $\langle \phi^{-1}(A) \rangle \cap (T_QX \backslash \{Q\}) \neq \emptyset$. For a general hyperplane $\langle A \rangle$ of $\mathbb{P}^{n-2}$ the point $\langle \phi^{-1}(A) \rangle \cap (T_QX \backslash \{Q\})$ may be considered as a general points P' of T_QX. We have $r_X(P') \leq \sharp(A) = n - 1$. Now assume $d - z = n - 2$. Since $d > n$, we get $z \geq 3$, $p := \mathrm{char}(\mathbb{K}) > 0$ and $z \geq p$. Any $\lfloor n/2 \rfloor$ lines of $\mathbb{P}^n$ are contained in a hyperplane of $\mathbb{P}^n$. Hence there is a hyperplane H containing $\lfloor n/2 \rfloor$ general tangent lines of X. Hence $d = \mathrm{length}(X \cap H) \geq z\lfloor n/2 \rfloor$. Since $d = z+n-2$, $z \geq 3$ and $n \geq 4$, we got a contradiction, unless $(n, d, z) = (5, 6, 3)$. In this case we take a hyperplane H' containing two tangent lines of X and another point of X, obtaining $6 = d \geq \mathrm{length}(X \cap H') \geq 6 + 1$, contradiction.

Now assume $n = 3$. Since either $\mathrm{char}(\mathbb{K}) = 0$ or $\mathrm{char}(\mathbb{K}) > d$ and $d > z$, we have $z = 2$ (see [5]). Hence X is reflexive. Under the reflexivity assumption we may give a very short proof which works for arbitrary $n \geq 3$. Since X is reflexive, $z = 2$. Fix a general $Q \in X$ and let $H \subset \mathbb{P}^n$ be a general hyperplane containing T_QX. If $(X \cap T_QX)_{red} \neq \{Q\}$, then $r_X(P) \leq 2$ for every $P \in T_QX$.

Hence we may assume $(X \cap T_Q X)_{red} \neq \{Q\}$. Hence the generality of X gives $\mathrm{Sing}(X) \cap H = \emptyset$. Since $z = 2$, the connected component of the scheme $X \cap H$ containing Q has length 2. Since X is reflexive, it is not strange. Hence at most finitely many tangent lines of X meet $T_Q X$. Hence we may assume that $X \cap H$ is reduced, outside Q. Thus $\sharp((X \cap H)_{red}) = d - 1 > n$. Take a codimension 1 linear subspace of H spanned by some $A \subset (X \cap H)_{red} \backslash \{Q\}$ with $\sharp(A) = n - 1$; it exists by the reflexivity of X. Hence $T_Q X \cap M$ is a unique point, P'. We have $r_X(P') \leq \sharp(A) = n - 1$. Varying H the points P' varies. Hence $r_X(P) \leq n - 1$ for a general $P \in TX$. □

Acknowledgments

The author was partially supported by MIUR and GNSAGA of INdAM (Italy).

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