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CUSPIDAL PROJECTIONS OF SPACE CURVES

E. Ballico
Department of Mathematics
University of Trento
38 123 Povo (Trento) - Via Sommarive, 14, ITALY
e-mail: ballico@science.unitn.it

Abstract: Let $X \subset \mathbb{P}^3$ be an integral and non-degenerate curve. A point $O \in \mathbb{P}^3 \backslash X$ is said to give a cuspidal projection of X if the linear projection from O induces an injective map $X \to \mathbb{P}^2$. Here we produce two classes of pairs (X, O) with O inducing a cuspidal projection of X. In the second class (only in positive characteristic) X is contained in a smooth quadric surface Σ and we may take as O any point of $\Sigma \backslash X$.

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For any $O \in \mathbb{P}^3$ let $\ell_O : \mathbb{P}^3 \setminus \{O\} \to \mathbb{P}^2$ denote the linear projection from O. Fix an integral and non-degenerate curve $X \subset \mathbb{P}^3$ such that $O \notin X$. Since $O \notin X$, the map $\ell_O|X : X \to \mathbb{P}^2$ is a morphism. As in [5] (in which X is assumed smooth) we say that O gives a cuspidal projection of X if $\ell_O|X$ is injective. This is related to the notion of X-rank (see [4], [3], [2]) and references therein): O gives a cuspidal projection of the space curve X if and only if it has X-rank at least 3. In characteristic zero any point of $\mathbb{P}^3$ has X-rank at most 3 (see [4], Proposition 5.1). Hence in characteristic zero O gives a cuspidal projection of X has X-rank 3; in positive characteristic we need to exclude the stange point of X (if any). R. Piene proved the existence of some smooth space curves with cuspidal projections and that many smooth space curves have no

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cuspidal projection (see [5]). She also listed all the examples of curves with at least one cuspidal constructed up to that time by other authors (see [5]). G. Sacchiero gave many examples of non-linearly normal space curves without cuspidal projections (see [6]). Here we produce two classes of pairs (X, O) with O inducing a cuspidal projection of X (see Examples 1 and 2 for their description). Both classes are contained in a smooth quadric surface Σ. In the second class (only in positive characteristic) X is contained in a smooth quadric surface Σ and we may take as O any point of $\Sigma \backslash X$.

Example 1. Fix bihomogeneous coordinates x_0, x_1, y_0, y_1 on the quadric surface $\Sigma := \mathbb{P}^1 \times \mathbb{P}^1 \subset \mathbb{P}^3$. For all integers $a \geq 0$ and $b \geq 0$ a curve of type (a, b) on Q is the zero-locus of a polynomial $f = f(x_0, x_1, y_0, y_1)$ of bidegree (a, b), i.e. $e_1 + e_2 = a$ and $e_3 + e_4 = b$ for every monomial $x_0^{e_1} x_1^{e_2} y_0^{e_3} y_1^{e_4}$ appearing with non-zero coefficient in f. The associated space curve $X := \{f = 0\}$ has degree $a + b$. Assume $a \geq 1$ and $b \geq 3$. Assume X integral, i.e. assume that f is irreducible. Assume the existence of $Q \in X$, say $Q = (Q_1, Q_2)$, such that the line $D_1(Q) := \{Q_1\} \times \mathbb{P}^1 \subset \Sigma$ has intersection multiplicity b with X at Q. Since $D_1(Q) \cdot X = b$ (intersection product in the smooth surface Σ), we obtain $(D_1(Q) \cap X)_{red} = \{Q\}$. Fix any $P \in D_1(Q) \backslash \{Q\}$ and any line $D \subset \mathbb{P}^3$ such that $P \in D$. If $D = D_1(P)$, then $Q := (D \cap X)_{red} = \{Q\}$. Now assume D not contained in Σ. Since $P \in \Sigma \setminus X$ and $\deg(\Sigma) = 2$, Bezout's theorem gives $\sharp((D \cap X)_{red}) \leq 1$. Now assume $D \neq D_1(Q)$ and $D \subset \Sigma$. Thus $D = D_2(P) = \mathbb{P}^1 \times \{P_2\}$, where $P = (P_1, P_2)$. We have $D_2(P) \cdot X = a$. Thus if either $a = 1$ or there is a point $(A_1, P_2) \in X$ such that $D_2(P)$ has order of contact a with X at (A_1, P_2), then $\sharp((X \cap D_2(P))_{red}) = 1$. Hence $r_X(P) \geq 3$, i.e. $P \notin X$ and $\ell_P | X$ is injective, if these conditions are satisfied.

Example 2. Assume $p := \mathrm{char}(\mathbb{K}) > 0$ and fix an integer $e \geq 1$. Take $a = 1$ and $b = p^e$ in the set-up of Example 1. Since $a = 1$, X is smooth and rational (assuming that it is integral, i.e. assuming that its equation f is irreducible). Take f irreducible such that either $e_3 = 0$ or $e_3 = p^e$ for every monomial $x_0^{e_1} x_1^{e_2} y_0^{e_3} y_1^{e_4}$ appearing with non-zero coefficient in f. Notice that $e_4 = p^e - e_3 \in \{0, p^e\}$. Thus the projection $X \to \mathbb{P}^1$ onto the first factor of Σ is purely inseparable of degree p^e and hence it is injective. Since $a = 1$, every line of type $(0, 1)$ on Σ meets X at a unique point. Hence every point of $\Sigma \backslash X$ gives a cuspidal projection of X.

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