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ON THE STRATIFICATION BY X-RANKS FOR A PROJECTIVE CURVE $X \subset \mathbb{P}^n$

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Abstract: Let $X \subset \mathbb{P}^n$, $n \geq 3$, be an integral and non-degenerate curve For any $P \in \mathbb{P}^n$ let $r_X(P)$ be the minimal cardinality of a set $S \subset X$ such that $P \in \langle S \rangle$. Set $F_X(k) := \{P \in \mathbb{P}^n : r_X(P) \geq k\}$. Here we prove that $F_X(k)$ contains no linear $(n-k+1)$-dimensional subspace and no $(n-k+2)$-dimensional complete subvariety.

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Let $X \subset \mathbb{P}^n$ be an integral and non-degenerate m-dimensional variety defined over an algebraically closed field $\mathbb{K}$ such that $\mathrm{char}(\mathbb{K}) = 0$. For any $P \in \mathbb{P}^n$ the X-rank $r_X(P)$ of P is the minimal integer k such that there is $S \subset X$ with $\sharp(S) = k$ and $P \in \langle S \rangle$. Since X is non-degenerate, $r_X(P) \leq n+1$ for all $P \in \mathbb{P}^n$. Landsberg and Titler proved that $r_X(P) \leq n-m+1$ for all $P \in \mathbb{P}^n$ (see [2], 5.1). For any integer $k \geq 1$ set $E_X(k) := \{P \in \mathbb{P}^n : r_X(P) = k\}$ and $F_X(k) := \cup_{x \geq k} E_X(k)$. Obviously $E_X(1) = X$ and $F_X(1) = \mathbb{P}^n$. Hence $E_X(1)$ contains a complete curve and $F_X(1)$ contains an n-dimensional complete subvariety. Here we show that for $k \geq 2$ the picture is quite different. We look at the case $m = 1$ and prove the following result.

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Theorem 1. *Let $X \subset \mathbb{P}^n$, $n \geq 3$, be an integral and non-degenerate curve. Fix an integer k such that $2 \leq k \leq n$. Then $F_X(k)$ contains no $(n-k+1)$-dimensional linear subspace and no $(n-k+2)$-dimensional complete subvariety.*

In the set-up of Theorem 1 assume that X is a rational normal curve. Since $E_X(n) = F_X(n) = TX\backslash X$ (see [1] or [2], Theorem 4.1), $E_X(n)$ may be 2-dimensional. The case $k = n$ of Theorem 1 says that it contains no line and no complete surface. In this case $E_X(n) = \cup_{Q\in X}(T_QX\backslash\{Q\})$ is an infinite union of affine lines. Similarly, for all $k \geq \lfloor (n+3)/2 \rfloor$ the set $E_X(k)$ is described as a union of non-empty open subsets of certain $(n-k+1)$-dimensional linear subspaces (see [1] or [2], Theorem 4.1).

Proof of Theorem 1. In order to obtain a contradiction we assume the existence of an integral and complete subvariety B of $F_X(k)$ such that $\dim(B) = n-k+2$. Fix a general $U \subset X$ such that $\sharp(U) = k-1$. Since X is non-degenerate and U is general, $\dim(\langle U\rangle) = k-2$. Since $\dim(B) + \dim(\langle U\rangle) \geq n$, there is $P \in B \cap \langle U\rangle)$. Thus $r_X(P) \leq \sharp(U)$, contradiction. Now assume the existence of an $(n-k+1)$-dimensional linear subspace of $F_X(k)$. Fix a general $S \subset X$ such that $\sharp(S) = k-2$. Set $V := \langle S\rangle$. Since X is non-degenerate and S is general, we have $\dim(V) = k-3$. Let $\ell_V : \mathbb{P}^n \to \mathbb{P}^{n-k+2}$ denote the linear projection from V. Since each point of T has X-rank $> \sharp(S)$, then $V \cap T = \emptyset$. Set $Y := \ell_V(T)$. Since $\mathrm{char}(\mathbb{K}) = 0$, a general hyperplane section of X is in linearly general position. Since $k-2 \leq n-1$ and S is general, we get $V \cap X = S$. Since S is general in X, each point of S is a smooth point of X. Hence the rational map $\ell_V|X\backslash V\cap X \to \mathbb{P}^{n-k+2}$ extends to a morphism $u : X \to \mathbb{P}^{n-k+2}$. Set $A := u(X)$. Since $\dim(A) + \dim(Y) = n-k+2$, there is $Q \in A \cap Y$. Fix $P \in T$ such that $\ell_V(P) = T$ and $O \in X$ such that $u(O) = P$. First assume $O \notin S$. In this case we get $P \in \langle\{O, S\rangle$. Thus $r_X(P) \leq k-1$, contradiction. Now assume $O \in S$. Set $S' := S\backslash\{O\}$ and $W := \langle S'\rangle$. The definitions of ℓ_V and of P give $P \in \langle T_OX \cup S'\rangle$. Let $\ell_W : \mathbb{P}^n\backslash W \to \mathbb{P}^{n-k+3}$ denote the linear projection from W (with the convention that ℓ_W is the identity if $W = \emptyset$, i.e. if $k = 3$). Since $S' \subset X_{reg}$ the morphism $\ell_W|X\backslash S' \to \mathbb{P}^{n-k+3}$ extends to a morphism $w : X \to \mathbb{P}^{n-k+3}$. Since $P \in \langle T_OX \cup S'\rangle$, we get that $T_{v(O)}v(X)$ intersects the linear space $\ell_W(T)$. We fix S' and then move O. We get that a general tangent line of $v(X)$ intersects the linear subspace $\ell_W(T)$. Hence the restriction to $v(X)$ of the linear projection from $\ell_W(T)$ has generically vanishing differential. Since $v(X)$ is non-degenerate and $\mathrm{char}(\mathbb{K}) = 0$, this is absurd. □

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References

[1] G. Comas, M. Seiguer, On the rank of a binary form, *ArXiv:* math.AG/0112311.

[2] J.M. Landsberg, Z. Teiler, On the ranks and border ranks of symmetric tensors, *ArXiv:* 0901.0487v3.