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ON THE SOLVABILITY CONDITIONS FOR SOME NON FREDHOLM OPERATORS

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Abstract: We obtain solvability conditions for some elliptic equations involving non Fredholm operators, which are sums of second order differential operators with the methods of spectral theory and scattering theory for Schrödinger type operators.

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1. Introduction

Fredholm type solvability conditions, which affirm that an operator equation is solvable if and only if its right-hand side is orthogonal to solutions of the homogeneous adjoint equation, are used directly or indirectly in the most methods of linear and nonlinear analysis. If the operator does not satisfy the Fredholm property, applicability of these solvability conditions is not established. In this work we study solvability conditions for some class of non Fredholm operators. We consider the equation

$$Hu \equiv -\Delta u + W(x)u - au = f \tag{1.1}$$

with a non-negative constant a and with some conditions on the potential $W(x)$

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which will be specified below. If $W(x) \to 0$ as $|x| \to \infty$, the case studied in our previous work [23], then the essential spectrum of the operator H consists of the half-axis $[-a, \infty)$ [15], [16], [21], [9]. Since it contains the origin, the operator does not satisfy the Fredholm property, and the usual solvability conditions for equation (1.1) are not applicable.

There are two distinct cases, $a = 0$ and $a > 0$. Let us first discuss the case $a = 0$. It is known that homogeneous elliptic operators with constant coefficients satisfy the Fredholm property if considered in some specially chosen polynomial weighted spaces. For the operator H given by (1.1) this is the case if $a = 0$ and $W(x) \equiv 0$, that is for the Laplace operator. Lower order terms with the coefficients decaying at infinity represent compact operators if the decay rate is sufficiently high. Therefore the operator H in the weighted spaces remains Fredholm under some conditions on the potential. This allows one to make some conclusions about its index and solvability conditions. This approach is based on a priori estimates of solutions obtained in [17]. The Fredholm property of this class of operators in Sobolev spaces is studied in [13], [14]. Similar problems for elliptic operators in Hölder spaces are investigated in [3], [4]. Exterior problems for the Laplace operator in weighted Sobolev spaces are considered in [1], [2], and for more general operators in Hölder spaces in [5]. The dimension of the kernel and the Fredholm property of elliptic operators of the first order are studied in [24], [25].

The case of positive a is qualitatively different. The method described above is not applicable. In the 1D case we can introduce exponential weighted spaces where the operator will satisfy the Fredholm property [21]. However, in $\mathbb{R}^n$ with $n \geq 2$ this method is not applicable neither. The reason for this can be already seen for the case $W(x) \equiv 0$. If the equation is solvable, then the right-hand side is orthogonal to all functions $e^{ipx_1}e^{iqx_2}$, where $p^2 + q^2 = a$, $x = (x_1, x_2) \in \mathbb{R}^2$. Hence there is a continuous family of solvability conditions while the Fredholm property implies only a finite number of them.

The method developed in our previous work is based on the spectral theory of self-adjoint operators [23]. Similar to the case $W(x) \equiv 0$ where we can use the Fourier transform and explicitly find the solution, in the case of nonzero potential we use spectral decomposition with respect to the functions of the continuous spectrum of the operator $H_0 u = -\Delta u + W u$. This allows us to obtain solvability conditions as orthogonality to the functions of the continuous spectrum. To the best of our knowledge, this is the first result on solvability conditions for this class of operators. This method is applicable both for $a = 0$ and $a > 0$. Though these solvability conditions are similar to the usual ones, we should not forget that the operator does not satisfy the Fredholm property. Its range is not closed, the dimension of the kernel and the codimension of the image may not be finite. Hence, the similarity with Fredholm solvability conditions is only formal.

In this work we continue the investigation of equation (1.1) under different assumptions on the potential. We will assume that it can be represented as $W(x) = W_1(x') + W_2(x'')$, where $x = (x_1, ..., x_n)$, $x' = (x_1, ..., x_k)$, $x'' = (x_{k+1}, ..., x_n)$. Though the potential does not converge to zero as $|x| \to \infty$, we can apply the method of [23] using separation of variables. A particular case of such equations with $W_1(x') \equiv 0$ arises in reaction-diffusion problems [22].

2. Formulation of the Results

For the sake of convenience we will denote independent variables by x and y and put $W(x, y) = V(x) + U(y)$. We begin with the operator H_a on $L^2(\mathbb{R}^6)$, such that

$$H_a u = -\Delta_x u + V(x)u - \Delta_y u + U(y)u - au$$

with the Laplace operators Δ_x and Δ_y in $x = (x_1, x_2, x_3)$ and $y = (y_1, y_2, y_3)$, $a \geq 0$ is a parameter, the potentials $V(x)$ and $U(y)$ decay to zero as $x, y \to \infty$. We investigate the conditions on the function $f(x, y) \in L^2(\mathbb{R}^6)$ under which the equations

$$H_a u = f \tag{2.2}$$

and

$$H_0 u = f, \tag{2.3}$$

the second one is the limiting case of the first one as $a \to 0$, have the unique solution in $L^2(\mathbb{R}^6)$. Thus the case of a single Schrödinger operator studied in [23] is being generalized to the case of the sum of two such operators. We will use the spectral decomposition of self-adjoint operators.

For a function $\psi(x)$ belonging to a $L^p(\mathbb{R}^d)$ space with $1 \leq p \leq \infty$, $d \in \mathbb{N}$ its norm is being denoted as $\|\psi\|_{L^p(\mathbb{R}^d)}$. As technical tools for estimating the appropriate norms of functions we will be using, in particular Young's inequality

$$\|f_1 * f_2\|_{L^\infty(\mathbb{R}^3)} \leq \|f_1\|_{L^4(\mathbb{R}^3)} \|f_2\|_{L^{\frac{4}{3}}(\mathbb{R}^3)}, \quad f_1 \in L^4(\mathbb{R}^3), \quad f_2 \in L^{\frac{4}{3}}(\mathbb{R}^3),$$

where * stands for the convolution. The inner product of functions on $L^2(\mathbb{R}^d)$, $d \in$

$\mathbb{N}$ is being denoted as

$$(f_1(x), f_2(x))_{L^2(\mathbb{R}^d)} := \int_{\mathbb{R}^d} f_1(x)\bar{f}_2(x)dx;$$

for a vector function $A(x) = (A_1(x), A_2(x), A_3(x)),\ x \in \mathbb{R}^3$ the inner product $(f_1(x), A(x))_{L^2(\mathbb{R}^3)}$ is the vector with the coordinates $\int_{\mathbb{R}^3} f_1(x)\bar{A}_i(x)dx,\ i = 1,2,3$. Note that with a slight abuse we use the same notation even when functions may not be square integrable, for instance the functions $\varphi_k(x)$ and $\eta_q(y)$ of the continuous spectrum of the operators $-\Delta_x + V(x)$ and $-\Delta_y + U(y)$ respectively are normalized to Dirac delta-functions (see (3.8) and (3.9) in Section 3).

We make the following technical assumptions on the potential functions involved in the equations (2.2) and (2.3) and on the right sides of these equations.

Assumption 1. The potential functions $V(x), U(y) : \mathbb{R}^3 \to \mathbb{R}$ satisfy the bounds $|V(x)| \leq \frac{C}{1+|x|^{3.5+\varepsilon}}$ and $|U(y)| \leq \frac{C}{1+|y|^{3.5+\varepsilon}}$ with some $\varepsilon > 0$ and $x, y \in \mathbb{R}^3$ a.e. such that

$$4^{\frac{1}{9}}\frac{9}{8}(4\pi)^{\frac{-2}{3}}\|V\|^{\frac{1}{9}}_{L^\infty(\mathbb{R}^3)}\|V\|^{\frac{8}{9}}_{L^{\frac{4}{3}}(\mathbb{R}^3)} < 1,\ 4^{\frac{1}{9}}\frac{9}{8}(4\pi)^{\frac{-2}{3}}\|U\|^{\frac{1}{9}}_{L^\infty(\mathbb{R}^3)}\|U\|^{\frac{8}{9}}_{L^{\frac{4}{3}}(\mathbb{R}^3)} < 1,$$

$$\sqrt{c_{HLS}}\|V\|_{L^{\frac{3}{2}}(\mathbb{R}^3)} < 4\pi,\ \sqrt{c_{HLS}}\|U\|_{L^{\frac{3}{2}}(\mathbb{R}^3)} < 4\pi$$

Here c_{HLS} is the constant in the Hardy-Littlewood-Sobolev inequality

$$\Big|\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{f_1(x)f_1(y)}{|x-y|^2}dxdy\Big| \leq c_{HLS}\|f_1\|^2_{L^{\frac{3}{2}}(\mathbb{R}^3)},\ f_1 \in L^{\frac{3}{2}}(\mathbb{R}^3) \tag{2.4}$$

and given on p. 98 of [12].

Assumption 2. The function

$$f(x,y) \in L^2(\mathbb{R}^6) \quad \text{and} \quad |x|f(x,y) \in L^1(\mathbb{R}^6),\ |y|f(x,y) \in L^1(\mathbb{R}^6).$$

Here and further down C denotes a finite positive constant. Due to our assumptions on the potentials the essential spectrum $\sigma_{ess}(H_a)$ of the Schrödinger type operator $H_a = H_0 - a$ fills the interval $[-a,\ \infty)$ (see, e.g. [9]), the Fredholm alternative theorem does not work. The problem can be trivially handled by the method of the Fourier transform when the potential terms $V(x)$ and $U(y)$ vanish. We prove that the method can be generalized in the presence of shallow, short-range $V(x)$ and $U(y)$ replacing the standard Fourier harmonics by the functions $\varphi_k(x)\eta_q(y),\ k, q \in \mathbb{R}^3$ of the continuous spectrum of the operator H_0. These functions satisfy the Lippmann-Schwinger equations (see (3.8), (3.9)) in Section 3 and the explicit formulas (3.10). Note that the conditions $|x|f(x,y) \in L^1(\mathbb{R}^6)$ and $|y|f(x,y) \in L^1(\mathbb{R}^6)$ of Assumption 2 are needed here to show the regularity of the gradient of the generalized Fourier transform with respect to $\varphi_k(x)\eta_q(y),\ k, q \in \mathbb{R}^3$ (see Lemma 11).

While the wave vectors k, q attain all the possible values in $\mathbb{R}^3$, the functions $\varphi_0(x)$ and $\eta_0(y)$ correspond to $k, q = 0$ in the formulas (3.8), (3.9) and (3.10). The sphere of radius r in $\mathbb{R}^d,\ d \in \mathbb{N}$ centered at the origin is being designated as S_r^d, the unit one as S^d and $|S^d|$ denotes its Lebesgue measure. Our first main theorem is as follows.

Theorem 3. *Let Assumptions 1 and 2 hold. Then:*

a) Problem (2.2) admits a unique solution $u(x,y) \in L^2(\mathbb{R}^6)$ if and only if

$$(f(x,y), \varphi_k(x)\eta_q(y))_{L^2(\mathbb{R}^6)} = 0 \quad \textit{for} \quad (k,q) \in S^6_{\sqrt{a}} \quad \textit{a.e.}$$

b) Problem (2.3) has a unique solution $u(x,y) \in L^2(\mathbb{R}^6)$.

Remark 4. Note that as distinct from the single Schrödinger operator case studied in [23], part b) of Theorem 3 does not require any orthogonality conditions at all.

In the second part of the article we study the operator L_a on $L^2(\mathbb{R}^{n+3}),\ n \in \mathbb{N}$, such that

$$L_a u = -\Delta_x u - \Delta_y u + U(y)u - au$$

with $x = (x_1, x_2, ..., x_n) \in \mathbb{R}^n,\ y = (y_1, y_2, y_3) \in \mathbb{R}^3,\ a \geq 0$ and the potential $U(y)$ satisfying the same assumptions as before. We establish the solvability conditions in $L^2(\mathbb{R}^{n+3})$ for the inhomogeneous elliptic problems

$$L_a u = \phi \tag{2.5}$$

and

$$L_0 u = \phi\ , \tag{2.6}$$

where L_0 is the limiting case of the operator L_a when $a \to 0$ and $\phi(x,y) \in L^2(\mathbb{R}^{n+3})$. We impose the following technical conditions on the right side of the equations (2.5) and (2.6).

Assumption 5. We have

$$\phi(x,y) \in L^2(\mathbb{R}^{n+3}),\ |x|\phi(x,y) \in L^1(\mathbb{R}^{n+3}) \quad \text{and} \quad |y|\phi(x,y) \in L^1(\mathbb{R}^{n+3}).$$

The conditions

$$|x|\phi(x,y) \in L^1(\mathbb{R}^{n+3}) \quad \text{and} \quad |y|\phi(x,y) \in L^1(\mathbb{R}^{n+3})$$

of the assumption above will be used to prove Lemma 12 about the regularity of the gradient of the generalized Fourier transform with respect to

$$\frac{e^{ikx}}{(2\pi)^{\frac{n}{2}}}\eta_q(y), \quad k \in \mathbb{R}^n,\ q \in \mathbb{R}^3.$$

As for the potential term $U(y)$ involved in the left sides of the equations (2.5) and (2.6), we assume the same technical conditions to hold as in the first Theorem. Thus for the operator L_a the essential spectrum coincides with the semi-axis $[-a,\ \infty)$ and the Fredholm alternative theorem fails to work.

Theorem 6. *Let Assumption 5 hold and the function $U(y)$ satisfy Assumption 1. Then:*

a) Equation (2.5) possesses a unique solution $u(x,y) \in L^2(\mathbb{R}^{n+3}),\ n \in \mathbb{N}$ if and only if

$$(\phi(x,y), \frac{e^{ikx}}{(2\pi)^{\frac{n}{2}}}\eta_q(y))_{L^2(\mathbb{R}^{n+3})} = 0 \quad \text{for} \quad (k,q) \in S^{n+3}_{\sqrt{a}} \quad \text{a.e.}$$

b) When $n=1$ equation (2.6) admits a unique solution $u(x,y) \in L^2(\mathbb{R}^4)$ if and only if

$$(\phi(x,y), \eta_0(y))_{L^2(\mathbb{R}^4)} = 0.$$

c) When $n \geq 2$ equation (2.6) admits a unique solution $u(x,y) \in L^2(\mathbb{R}^{n+3})$.

Remark 7. Note that the solvability conditions for equation (2.6) depend on the dimensions of the problem, such that we have the explicit one in case b) and no orthogonality conditions are required at all in higher dimensions as stated in case c).

In the third part of the article we consider the operator $\mathcal{L} = -\Delta_x + V(x) - \Delta_y + \mathcal{V}(y)$ on $L^2(\mathbb{R}^{3+m})$ with the Laplacian operators Δ_x and Δ_y such that $x = (x_1, x_2, x_3) \in \mathbb{R}^3,\ y = (y_1, y_2, ..., y_m) \in \mathbb{R}^m,\ m \in \mathbb{N}$ and prove the necessary and sufficient conditions for the solvability in $L^2(\mathbb{R}^{3+m})$ of the following inhomogeneous problem

$$\mathcal{L}u = g(x,y), \tag{2.7}$$

where $g(x,y) \in L^2(\mathbb{R}^{3+m})$. Note that problem (2.7) is the extension of the elliptic equation studied in the second part of [23] to the case of the potential function $V(x)$ explicitly present in it, which is assumed to satisfy the same assumptions as in the first part of our article and $x \in \mathbb{R}^3$. The assumptions on the second potential function $\mathcal{V}(y)$ are analogous to those in [23].

Assumption 8. The function $\mathcal{V}(y) : \mathbb{R}^m \to \mathbb{R}$ is continuous and

$$\lim_{y\to\infty}\mathcal{V}(y) = \mathcal{V}_+ > 0.$$

Thus for the operator $h := -\Delta_y + \mathcal{V}(y)$ on $L^2(\mathbb{R}^m)$ the essential spectrum $\sigma_{ess}(h) = [\mathcal{V}_+, \infty)$. Let us denote the eigenvalues of the operator h located below $\mathcal{V}_+$ as $e_j,\ e_j < e_{j+1},\ j \geq 1$ and the corresponding elements of the orthonormal set of eigenfunctions as ψ_j^l, such that $h\psi_j^l = e_j\psi_j^l,\ 1 \leq l \leq m_j,\ (\psi_i^l, \psi_j^s)_{L^2(\mathbb{R}^m)} = \delta_{i,j}\delta_{l,s}$, where m_j stands for the eigenvalue multiplicity, which is finite since the essential spectrum starts only at $\mathcal{V}_+$ and $\delta_{i,j}$ denotes the Kronecker symbol. We make the following key assumption on the discrete spectrum of the operator h relevant to the problem (2.7).

Assumption 9. The eigenvalues $e_j < 0$ for all $1 \leq j \leq N-1,\ N \geq 1$ and $e_N = 0$.

Hence under our assumptions the operator $\mathcal{L}$ is not Fredholm. The bottom of the essential spectrum of the operator $-\Delta_x + V(x)$, which is unitarily equivalent to the free Laplacian in our problem (see Section 3) is zero and h has the square integrable zero modes. Furthermore, the operator h has the negative eigenvalues $e_j,\ j = 1, ..., N-1$ and $-\Delta_x + V(x)$ has the functions of the continuous spectrum $\varphi_k(x)$, such that $k \in S^3_{\sqrt{-e_j}}$. However, equation (2.7) can be solved on the proper subspace with the orthogonality conditions stated precisely below. Our third main result is as follows.

Theorem 10. *Let the potential function $V(x)$ satisfy Assumption 1, Assumptions 8 and 9 hold, $g(x,y) \in L^2(\mathbb{R}^{3+m})$ and $|x|^{\frac{\alpha}{2}}g(x,y) \in L^2(\mathbb{R}^{3+m})$ with some $\alpha > 5$. Then equation (2.7) admits a unique solution $u(x,y) \in L^2(\mathbb{R}^{3+m})$ if and only if:*

$$(g(x,y), \varphi_0(x)\psi_N^l(y))_{L^2(\mathbb{R}^{3+m})} = 0, \quad 1 \leq l \leq m_N \quad \textit{and}$$

$$(g(x,y), \varphi_k(x)\psi_j^l(y))_{L^2(\mathbb{R}^{3+m})} = 0, \quad 1 \leq l \leq m_j,\ k \in S^3_{\sqrt{-e_j}} \text{ a.e.},\ 1 \leq j \leq N-1$$

Establishing solvability conditions for linear elliptic problems involving non Fredholm operators plays the significant role in various applications including the existence of travelling wave solutions of reaction-diffusion systems (see, e.g. [22]).

3. Properties of the Operator H_0 and Proof of Theorem 3

The functions of the continuous spectrum of the operator $-\Delta_x + V(x)$ are solutions to the Lippmann-Schwinger equation (see, e.g. [18] p. 98)

$$\varphi_k(x) = \frac{e^{ikx}}{(2\pi)^{\frac{3}{2}}} - \frac{1}{4\pi}\int_{\mathbb{R}^3} \frac{e^{i|k||x-y|}}{|x-y|}(V\varphi_k)(y)dy \tag{3.8}$$

with the orthogonality conditions $(\varphi_k(x), \varphi_{k_1}(x))_{L^2(\mathbb{R}^3)} = \delta(k-k_1), \quad k, k_1 \in \mathbb{R}^3$. Let us define the integral operator

$$(Q\varphi)(x) := -\frac{1}{4\pi}\int_{\mathbb{R}^3} \frac{e^{i|k||x-y|}}{|x-y|}(V\varphi)(y)dy, \quad \varphi \in L^\infty(\mathbb{R}^3).$$

Similarly the functions of the continuous spectrum of the operator $-\Delta_y + U(y)$ satisfy

$$\eta_q(y) = \frac{e^{iqy}}{(2\pi)^{\frac{3}{2}}} - \frac{1}{4\pi}\int_{\mathbb{R}^3} \frac{e^{i|q||y-z|}}{|y-z|}(U\eta_q)(z)dz \tag{3.9}$$

and the orthogonality relations $(\eta_q(y), \eta_{q_1}(y))_{L^2(\mathbb{R}^3)} = \delta(q-q_1), \quad q, q_1 \in \mathbb{R}^3$. Therefore, it makes sense to define the integral operator

$$(P\eta)(y) := -\frac{1}{4\pi}\int_{\mathbb{R}^3} \frac{e^{i|q||y-z|}}{|y-z|}(U\eta)(z)dz, \quad \eta \in L^\infty(\mathbb{R}^3).$$

For the potentials $V(x)$ and $U(y)$ satisfying Assumption 1 we have the estimates for the operator norms of $Q : L^\infty(\mathbb{R}^3) \to L^\infty(\mathbb{R}^3)$ and $P : L^\infty(\mathbb{R}^3) \to L^\infty(\mathbb{R}^3)$ denoted as $\|Q\|_\infty$ and $\|P\|_\infty$ respectively. These k and q independent bounds are proven in Lemma 2.1 of [23], namely

$$\|Q\|_\infty \le 4^{\frac{1}{9}}\frac{9}{8}(4\pi)^{-\frac{2}{3}}\|V\|^{\frac{1}{9}}_{L^\infty(\mathbb{R}^3)}\|V\|^{\frac{8}{9}}_{L^{\frac{4}{3}}(\mathbb{R}^3)} < 1$$

and

$$\|P\|_\infty \le 4^{\frac{1}{9}}\frac{9}{8}(4\pi)^{-\frac{2}{3}}\|U\|^{\frac{1}{9}}_{L^\infty(\mathbb{R}^3)}\|U\|^{\frac{8}{9}}_{L^{\frac{4}{3}}(\mathbb{R}^3)} < 1\ ,$$

via Assumption 1 which enables us to express the functions of the continuous spectrum as

$$\varphi_k(x) = (I-Q)^{-1}\frac{e^{ikx}}{(2\pi)^{\frac{3}{2}}}\ , \quad \eta_q(y) = (I-P)^{-1}\frac{e^{iqy}}{(2\pi)^{\frac{3}{2}}},\ k, q \in \mathbb{R}^3, \tag{3.10}$$

and obtain the trivial estimates on their norms (see Corollary 2.2 of [23])

$$\|\varphi_k(x)\|_{L^\infty(\mathbb{R}^3)} \le \frac{1}{1-\|Q\|_\infty}\frac{1}{(2\pi)^{\frac{3}{2}}}, \quad \|\eta_q(y)\|_{L^\infty(\mathbb{R}^3)} \le \frac{1}{1-\|P\|_\infty}\frac{1}{(2\pi)^{\frac{3}{2}}}.$$

We revise the argument of Lemma 2.3 of [23]. By means of inequality (2.4) and Assumption 1 the Rollnik norms for both potentials $V(x)$ and $U(y)$ (see, e.g. [20]) given by

$$\|V\|_R^2 := \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{|V(x)||V(y)|}{|x-y|^2}dxdy, \quad \|U\|_R^2 := \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{|U(x)||U(y)|}{|x-y|^2}dxdy,$$

are bounded above, such that

$$\|V\|_R < 4\pi, \quad \|U\|_R < 4\pi,$$

which yields the self-adjointness of the operators $-\Delta_x + V(x)$ and $-\Delta_y + U(y)$ on $L^2(\mathbb{R}^3)$ and their unitary equivalence to $-\Delta_x$ and $-\Delta_y$ respectively (see, [10], also [19]) via the wave operators

$$\Omega_V^{\pm} := s - \lim_{t\to\mp\infty} e^{it(-\Delta+V)}e^{it\Delta}, \quad \Omega_U^{\pm} := s - \lim_{t\to\mp\infty} e^{it(-\Delta+U)}e^{it\Delta}$$

with the limits understood in the strong L^2 sense (see, e.g. [18], p. 34, [6], p. 90). The spectral theorem implies that any function in $L^2(\mathbb{R}^6)$ can be expanded through the products of the functions of the continuous spectrum $\varphi_k(x)\eta_q(y)$, $k, q \in \mathbb{R}^3$ which form the complete system in $L^2(\mathbb{R}^6)$. Let $\tilde{\tilde{f}}(k,q)$ denote the generalized Fourier transform of the function $f(x,y)$ with respect to these functions, namely

$$\tilde{\tilde{f}}(k,q) := (f(x,y), \varphi_k(x)\eta_q(y))_{L^2(\mathbb{R}^6)}, \quad k, q \in \mathbb{R}^3.$$

We have the following auxiliary statement.

Lemma 11. *Let Assumptions 1 and 2 hold. Then*

$$(\nabla_k + \nabla_q)\tilde{\tilde{f}}(k,q) \in L^\infty(\mathbb{R}^6).$$

Proof. Clearly we need to estimate the sum

$$(f(x,y), \nabla_k\varphi_k(x)\eta_q(y))_{L^2(\mathbb{R}^6)} + (f(x,y), \varphi_k(x)\nabla_q\eta_q(y))_{L^2(\mathbb{R}^6)}. \tag{3.11}$$

From the Lippmann-Schwinger equations (3.8) and (3.9) we easily obtain

$$\nabla_k\varphi_k(x) = \frac{e^{ikx}}{(2\pi)^{\frac{3}{2}}}ix + (I-Q)^{-1}Q\frac{e^{ikx}}{(2\pi)^{\frac{3}{2}}}ix + (I-Q)^{-1}(\nabla_k Q)(I-Q)^{-1}\frac{e^{ikx}}{(2\pi)^{\frac{3}{2}}} \tag{3.12}$$

and

$$\nabla_q\eta_q(y) = \frac{e^{iqy}}{(2\pi)^{\frac{3}{2}}}iy + (I-P)^{-1}P\frac{e^{iqy}}{(2\pi)^{\frac{3}{2}}}iy + (I-P)^{-1}(\nabla_q P)(I-P)^{-1}\frac{e^{iqy}}{(2\pi)^{\frac{3}{2}}},$$

with the operators

$$\nabla_k Q : L^\infty(\mathbb{R}^3) \to L^\infty(\mathbb{R}^3; \mathbb{C}^3) \quad \text{and} \quad \nabla_q P : L^\infty(\mathbb{R}^3) \to L^\infty(\mathbb{R}^3; \mathbb{C}^3)$$

having the integral kernels

$$\nabla_k Q(x,y,k) = -\frac{1}{4\pi} e^{i|k||x-y|} i \frac{k}{|k|} V(y)$$

and

$$\nabla_q P(y,z,q) = -\frac{1}{4\pi} e^{i|q||y-z|} i \frac{q}{|q|} U(z)$$

respectively. A trivial computation gives the upper bounds for their norms

$$\|\nabla_k Q\|_\infty \le \frac{1}{4\pi}\|V\|_{L^1(\mathbb{R}^3)} < \infty \quad \text{and} \quad \|\nabla_q P\|_\infty \le \frac{1}{4\pi}\|U\|_{L^1(\mathbb{R}^3)} < \infty$$

by means of the rate of decay of the potentials $V(x)$ and $U(y)$ stated explicitly in Assumption 1. By means of representation (3.12) the first term in (3.11) can be written as the sum of the three terms $T_1(k,q) + T_2(k,q) + T_3(k,q)$, where the first one is

$$T_1(k,q) := (f(x,y), \frac{e^{ikx}}{(2\pi)^{\frac{3}{2}}} ix\eta_q(y))_{L^2(\mathbb{R}^6)}.$$

Clearly we have the bound

$$|T_1(k,q)| \le \frac{1}{(2\pi)^3} \frac{1}{1-\|P\|_\infty} \int_{\mathbb{R}^3} dx \int_{\mathbb{R}^3} dy |x||f(x,y)| < \infty$$

due to Assumption 2. The second term

$$T_2(k,q) := (f(x,y), (I-Q)^{-1} Q \frac{e^{ikx}}{(2\pi)^{\frac{3}{2}}} ix\eta_q(y))_{L^2(\mathbb{R}^6)}$$

can be estimated as

$$|T_2(k,q)| \le \frac{1}{(2\pi)^3} \frac{1}{1-\|P\|_\infty} \frac{1}{1-\|Q\|_\infty} \|f\|_{L^1(\mathbb{R}^6)} \|Qe^{ikx}x\|_{L^\infty(\mathbb{R}^3)}.$$

We obtain the upper bound using the definition of the operator Q and Young's inequality

$$\begin{aligned}
|Qe^{ikx}x| &\le \frac{1}{4\pi}\int_{\mathbb{R}^3} \frac{|V(y)||y|}{|x-y|} dy \\
&= \frac{1}{4\pi}\{\Big(\frac{1}{|x|}\chi_{\{|x|\le 1\}}\Big) * |V(x)||x| + \Big(\frac{1}{|x|}\chi_{\{|x|>1\}}\Big) * |V(x)||x|\} \\
&\le \frac{1}{4\pi}\{\|V(y)y\|_{L^\infty(\mathbb{R}^3)} \int_0^1 4\pi r dr + \|\chi_{\{|x|>1\}}\frac{1}{|x|}\|_{L^4(\mathbb{R}^3)} \|V(x)x\|_{L^{\frac{4}{3}}(\mathbb{R}^3)}\} < \infty,
\end{aligned}$$

and k-independent since $V(x)x \in L^\infty(\mathbb{R}^3) \cap L^{\frac{4}{3}}(\mathbb{R}^3)$ by means of the rate of decay of the potential function $V(x)$ given in Assumption 1. Here and further down χ_A stands for the characteristic function of a set A. By means of Assumption 2 and Fact 2 of Appendix $f(x,y) \in L^1(\mathbb{R}^6)$. Hence $T_2(k,q) \in L^\infty(\mathbb{R}^6)$. The third term is given by

$$T_3(k,q) := (f(x,y), (I-Q)^{-1}(\nabla_k Q)(I-Q)^{-1}\frac{e^{ikx}}{(2\pi)^{\frac{3}{2}}}\eta_q(y))_{L^2(\mathbb{R}^6)}.$$

We estimate from above

$$|T_3(k,q)| \leq \frac{1}{4\pi(2\pi)^3}\|f\|_{L^1(\mathbb{R}^6)}\|V\|_{L^1(\mathbb{R}^3)}\frac{1}{(1-\|Q\|_\infty)^2}\frac{1}{1-\|P\|_\infty} < \infty,$$

uniformly for all $k, q \in \mathbb{R}^3$. Thus $T_3(k,q) \in L^\infty(\mathbb{R}^6)$. Similarly we express the second term in (3.11) as the sum of the terms $R_1(k,q) + R_2(k,q) + R_3(k,q)$ with

$$R_1(k,q) := (f(x,y), \varphi_k(x)\frac{e^{iqy}}{(2\pi)^{\frac{3}{2}}}iy)_{L^2(\mathbb{R}^6)}.$$

Obviously

$$|R_1(k,q)| \leq \frac{1}{(2\pi)^3}\frac{1}{1-\|Q\|_\infty}\int_{\mathbb{R}^3} dx \int_{\mathbb{R}^3} dy|y||f(x,y)| < \infty.$$

The next term is given by

$$R_2(k,q) := (f(x,y), \varphi_k(x)(I-P)^{-1}P\frac{e^{iqy}}{(2\pi)^{\frac{3}{2}}}iy)_{L^2(\mathbb{R}^6)},$$

such that

$$|R_2(k,q)| \leq \frac{1}{(2\pi)^3}\frac{1}{1-\|Q\|_\infty}\frac{1}{1-\|P\|_\infty}\|f\|_{L^1(\mathbb{R}^6)}\|Pe^{iqy}y\|_{L^\infty(\mathbb{R}^3)}.$$

The definition of the integral operator P along with Young's inequality yield

$$\begin{aligned}|Pe^{iqy}y| &\leq \frac{1}{4\pi}\int_{\mathbb{R}^3}\frac{|U(z)|}{|y-z|}|z|dz \\ &= \frac{1}{4\pi}\{\Big(\chi_{\{|y|\leq 1\}}\frac{1}{|y|}\Big) * |U(y)||y| + \Big(\chi_{\{|y|>1\}}\frac{1}{|y|}\Big) * |U(y)||y|\} \\ &\leq \frac{1}{4\pi}\{\|U(y)y\|_{L^\infty(\mathbb{R}^3)}\int_0^1 4\pi r dr + \|\chi_{\{|y|>1\}}\frac{1}{|y|}\|_{L^4(\mathbb{R}^3)}\|U(y)y\|_{L^{\frac{4}{3}}(\mathbb{R}^3)}\} < \infty,\end{aligned}$$

and q-independent since $U(y)y \in L^\infty(\mathbb{R}^3) \cap L^{\frac{4}{3}}(\mathbb{R}^3)$ due to the rate of decay of the potential function $U(y)$ stated explicitly in Assumption 1. Thus $R_2(k,q) \in L^\infty(\mathbb{R}^6)$. Therefore, it remains to estimate the term

$$R_3(k,q) := (f(x,y), \varphi_k(x)(I-P)^{-1}(\nabla_q P)(I-P)^{-1}\frac{e^{iqy}}{(2\pi)^{\frac{3}{2}}})_{L^2(\mathbb{R}^6)}$$

for which we easily obtain the bound uniform in $k, q \in \mathbb{R}^3$

$$|R_3(k,q)| \leq \frac{1}{4\pi(2\pi)^3}\|f\|_{L^1(\mathbb{R}^6)}\|U\|_{L^1(\mathbb{R}^3)}\frac{1}{(1-\|P\|_\infty)^2}\frac{1}{1-\|Q\|_\infty} < \infty. \quad \square$$

Armed with the technical lemma above we proceed to prove Theorem 3.

Proof of Theorem 3. Assume first that equation (2.2) admits two solutions $u_1,\ u_2 \in L^2(\mathbb{R}^6)$. Then their difference $w(x,y) := u_1(x,y) - u_2(x,y) \in L^2(\mathbb{R}^6)$ satisfies the homogeneous equation

$$H_a w = 0.$$

Since the operator H_a on $L^2(\mathbb{R}^6)$ is unitarily equivalent to $-\Delta_x - \Delta_y - a$, it has no nontrivial square integrable zero modes, and therefore $w(x,y)$ vanishes a.e. The analogous uniqueness argument works for the square integrable solutions of equation (2.3). By applying the generalized Fourier transform with respect to the functions of the continuous spectrum $\varphi_k(x)$ and $\eta_q(y)$ to equation (2.2) we easily obtain

$$\tilde{\tilde{u}}(k,q) = \frac{\tilde{\tilde{f}}(k,q)}{k^2+q^2-a}.$$

For technical purposes we will be using the spherical layer in the space of six dimensions

$$A_\sigma := \{(k,q) \in \mathbb{R}^6 : \sqrt{a} - \sigma \leq \sqrt{k^2+q^2} \leq \sqrt{a} + \sigma\},\ 0 < \sigma < \sqrt{a},$$

such that χ_{A_σ} and $\chi_{A_\sigma{}^c}$ denote the characteristic functions of the layer and its complement respectively. Thus we arrive at

$$\tilde{\tilde{u}}(k,q) = \frac{\tilde{\tilde{f}}(k,q)}{k^2+q^2-a}\chi_{A_\sigma} + \frac{\tilde{\tilde{f}}(k,q)}{k^2+q^2-a}\chi_{A_\sigma{}^c}. \tag{3.13}$$

For the second term in the right side of the identity above we have the estimate

$$\Big|\frac{\tilde{\tilde{f}}(k,q)}{k^2+q^2-a}\chi_{A_\sigma{}^c}\Big| \leq \frac{|\tilde{\tilde{f}}(k,q)|}{\sigma\sqrt{a}} \in L^2(\mathbb{R}^6).$$

Clearly, we have the representation $\tilde{\tilde{f}}(k,q) = \tilde{\tilde{f}}(\sqrt{a},\omega) + \int_{\sqrt{a}}^{\sqrt{k^2+q^2}} \frac{\partial \tilde{\tilde{f}}(|s|,\omega)}{\partial |s|} d|s|$. Here and further down ω stands for the angle variables on the sphere and $d\omega$ denotes integration with respect to these variables. This enables us to split the first term in the right side of (3.13) into the sum $\tilde{\tilde{u}}_1(k,q) + \tilde{\tilde{u}}_2(k,q)$ with

$$\tilde{\tilde{u}}_1(k,q) = \frac{\tilde{\tilde{f}}(\sqrt{a},\omega)}{k^2+q^2-a}\chi_{A_\sigma}, \quad \tilde{\tilde{u}}_2(k,q) = \frac{\int_{\sqrt{a}}^{\sqrt{k^2+q^2}} \frac{\partial \tilde{\tilde{f}}(|s|,\omega)}{\partial |s|} d|s|}{k^2+q^2-a}\chi_{A_\sigma}.$$

By means of Lemma 11

$$|\tilde{\tilde{u}}_2(k,q)| \leq C\frac{\chi_{A_\sigma}}{\sqrt{a}} \in L^2(\mathbb{R}^6).$$

Therefore, it remains to estimate the norm $\|\tilde{\tilde{u}}_1(k,q)\|^2_{L^2(\mathbb{R}^6)}$ equal to

$$\int_{\sqrt{a}-\sigma}^{\sqrt{a}+\sigma} d(\sqrt{k^2+q^2})\frac{(\sqrt{k^2+q^2})^5}{(\sqrt{k^2+q^2}-\sqrt{a})^2(\sqrt{k^2+q^2}+\sqrt{a})^2}\int_{S^6} d\omega|\tilde{\tilde{f}}(\sqrt{a},\omega)|^2 \ ,$$

which is finite if and only if $(f(x,y),\varphi_k(x)\eta_q(y))_{L^2(\mathbb{R}^6)} = 0$ for $(k,q) \in S^6_{\sqrt{a}}$ a.e, which completes the proof of part a) of the Theorem. Similarly for equation (2.3)

$$\tilde{\tilde{u}}(k,q) = \frac{\tilde{\tilde{f}}(k,q)}{k^2+q^2}.$$

Again we split the expression into the nonsingular and the singular parts, such that

$$\tilde{\tilde{u}}(k,q) = \frac{\tilde{\tilde{f}}(k,q)}{k^2+q^2}\chi_{\{k^2+q^2>1\}} + \frac{\tilde{\tilde{f}}(k,q)}{k^2+q^2}\chi_{\{k^2+q^2\leq 1\}}.$$

Obviously

$$\left|\frac{\tilde{\tilde{f}}(k,q)}{k^2+q^2}\chi_{\{k^2+q^2>1\}}\right| \leq |\tilde{\tilde{f}}(k,q)| \in L^2(\mathbb{R}^6).$$

The remaining expression will be estimated with the help of the identity

$$\tilde{\tilde{f}}(k,q) = \tilde{\tilde{f}}(0) + \int_0^{\sqrt{k^2+q^2}} \frac{\partial\tilde{\tilde{f}}(|s|,\omega)}{\partial|s|}d|s|.$$

Lemma 11 yields

$$\left|\frac{\int_0^{\sqrt{k^2+q^2}}\frac{\partial\tilde{\tilde{f}}(|s|,\omega)}{\partial|s|}d|s|}{k^2+q^2}\chi_{\{k^2+q^2\leq 1\}}\right| \leq \frac{C}{\sqrt{k^2+q^2}}\chi_{\{k^2+q^2\leq 1\}} \in L^2(\mathbb{R}^6).$$

Thus it remains to estimate

$$\left\|\frac{\tilde{\tilde{f}}(0)}{k^2+q^2}\chi_{\{k^2+q^2\leq 1\}}\right\|^2_{L^2(\mathbb{R}^6)} = |\tilde{\tilde{f}}(0)|^2\int_0^1 rdr|S^6| < \infty,$$

since $|\tilde{\tilde{f}}(k,q)| \leq \|f\|_{L^1(\mathbb{R}^6)}\|\varphi_k\|_{L^\infty(\mathbb{R}^3)}\|\eta_q\|_{L^\infty(\mathbb{R}^3)} < \infty, \quad k,q \in \mathbb{R}^3$ which means that no orthogonality conditions are required for part b) of the theorem. ☐

4. Properties of the Operator L_0 and Proof of Theorem 6

The operator L_0 is the sum of two operators. The first one is $-\Delta_x$ acting on $L^2(\mathbb{R}^n),\ n \in \mathbb{N}$ and the functions of its continuous spectrum are the Fourier harmonics $\frac{e^{ikx}}{(2\pi)^{\frac{n}{2}}},\ k \in \mathbb{R}^n$. The second one is the Schrödinger operator $-\Delta_y + U(y)$ on $L^2(\mathbb{R}^3)$ unitarily equivalent to $-\Delta_y$ via the wave operators which is discussed in Section 3 and the functions of its continuous spectrum are $\eta_q(y),\ q \in \mathbb{R}^3$. Let $\tilde{\hat{\phi}}(k, q)$ denote the generalized Fourier transform with respect to the products of the functions of the continuous spectrum $\frac{e^{ikx}}{(2\pi)^{\frac{n}{2}}}\eta_q(y)$ forming the complete system in $L^2(\mathbb{R}^{n+3})$, namely

$$\tilde{\hat{\phi}}(k, q) := (\phi(x, y), \frac{e^{ikx}}{(2\pi)^{\frac{n}{2}}}\eta_q(y))_{L^2(\mathbb{R}^{n+3})},\ k \in \mathbb{R}^n,\ q \in \mathbb{R}^3.$$

We establish the following auxiliary statement.

Lemma 12. *Let Assumptions 1 and 5 hold. Then*

$$(\nabla_k + \nabla_q)\tilde{\hat{\phi}}(k, q) \in L^\infty(\mathbb{R}^{n+3}), \quad n \in \mathbb{N}.$$

Proof. Obviously we need to take care of the following sum

$$(\phi(x, y), \nabla_k \frac{e^{ikx}}{(2\pi)^{\frac{n}{2}}}\eta_q(y))_{L^2(\mathbb{R}^{n+3})} + (\phi(x, y), \frac{e^{ikx}}{(2\pi)^{\frac{n}{2}}}\nabla_q\eta_q(y))_{L^2(\mathbb{R}^{n+3})}. \tag{4.14}$$

For the first term by means of Assumption 5 we have the finite and k, q independent upper bound

$$\left|(\phi(x, y), \frac{e^{ikx}ix}{(2\pi)^{\frac{n}{2}}}\eta_q(y))_{L^2(\mathbb{R}^{n+3})}\right| \le \frac{1}{(2\pi)^{\frac{n+3}{2}}}\frac{1}{1 - \|P\|_\infty}\|x\phi\|_{L^1(\mathbb{R}^{n+3})}.$$

The second term in (4.14) can be written as the sum of three expressions $S_1(k, q) + S_2(k, q) + S_3(k, q)$, such that the first one

$$S_1(k, q) := (\phi(x, y), \frac{e^{ikx}}{(2\pi)^{\frac{n}{2}}}\frac{e^{iqy}}{(2\pi)^{\frac{3}{2}}}iy)_{L^2(\mathbb{R}^{n+3})}$$

can be easily estimated as

$$|S_1(k, q)| \le \frac{1}{(2\pi)^{\frac{n+3}{2}}}\|y\phi\|_{L^1(\mathbb{R}^{n+3})} < \infty$$

for all $k \in \mathbb{R}^n$, $q \in \mathbb{R}^3$ due to Assumption 5. For the second expression

$$S_2(k, q) := (\phi(x, y), \frac{e^{ikx}}{(2\pi)^{\frac{n}{2}}}(I - P)^{-1}P\frac{e^{iqy}}{(2\pi)^{\frac{3}{2}}}iy)_{L^2(\mathbb{R}^{n+3})},$$

we obtain the upper bound

$$|S_2(k,q)| \le \frac{1}{(2\pi)^{\frac{n+3}{2}}} \|\phi\|_{L^1(\mathbb{R}^{n+3})} \frac{1}{1-\|P\|_\infty} \|Pe^{iqy}y\|_{L^\infty(\mathbb{R}^3)}.$$

The function $\phi(x,y) \in L^1(\mathbb{R}^{n+3})$ via Assumption 5 and Fact 2 of Appendix. The estimate on the term $\|Pe^{iqy}y\|_{L^\infty(\mathbb{R}^3)}$ established in the proof of Lemma 11 yields $S_2(k,q) \in L^\infty(\mathbb{R}^{n+3})$. We complete the proof of the lemma with the estimate on the third expression

$$S_3(k,q) := (\phi(x,y), \frac{e^{ikx}}{(2\pi)^{\frac{n}{2}}}(I-P)^{-1}(\nabla_q P)(I-P)^{-1}\frac{e^{iqy}}{(2\pi)^{\frac{3}{2}}})_{L^2(\mathbb{R}^{n+3})},$$

such that

$$|S_3(k,q)| \le \frac{1}{4\pi}\frac{1}{(2\pi)^{\frac{n+3}{2}}}\|U\|_{L^1(\mathbb{R}^3)}\frac{1}{(1-\|P\|_\infty)^2}\|\phi\|_{L^1(\mathbb{R}^{n+3})}.$$

Hence $S_3(k,q) \in L^\infty(\mathbb{R}^{n+3})$. ☐

Establishing Lemma 12 enables us to prove Theorem 6.

Proof of Theorem 6. Assume that equation (2.5) admits two solutions $u_1,\ u_2 \in L^2(\mathbb{R}^{n+3})$ such that their difference $w(x,y) := u_1(x,y) - u_2(x,y) \in L^2(\mathbb{R}^{n+3})$ is a solution to the homogeneous equation

$$L_a w = 0.$$

Since the operator L_a on $L^2(\mathbb{R}^{n+3})$ is unitarily equivalent to $-\Delta_x - \Delta_y - a$, it possesses no nontrivial zero modes $w \in L^2(\mathbb{R}^{n+3})$, we arrive at $w(x,y) = 0$ a.e. Analogously the uniqueness argument works for the square integrable solutions of problem (2.6). We apply the generalized Fourier transform with respect to the functions of the continuous spectrum $\frac{e^{ikx}}{(2\pi)^{\frac{n}{2}}},\ k \in \mathbb{R}^n$ and $\eta_q(y),\ q \in \mathbb{R}^3$ to equation (2.5) and obtain

$$\tilde{\hat{u}}(k,q) = \frac{\tilde{\hat{\phi}}(k,q)}{k^2+q^2-a}.$$

Let us introduce the auxiliary spherical layer in the space of $n+3$ dimensions

$$B_\sigma = \{(k,q) \in \mathbb{R}^{n+3}:\ \sqrt{a} - \sigma \le \sqrt{k^2+q^2} \le \sqrt{a} + \sigma\},\ 0 < \sigma < \sqrt{a},$$

such that χ_{B_σ} and $\chi_{B_\sigma^c}$ denote the characteristic functions of the layer and of its complement respectively. This enables us to split the right side of the equality above into the singular and the regular parts, such that

$$\tilde{\hat{u}}(k,q) = \frac{\tilde{\hat{\phi}}(k,q)}{k^2+q^2-a}\chi_{B_\sigma} + \frac{\tilde{\hat{\phi}}(k,q)}{k^2+q^2-a}\chi_{B_\sigma^c}. \tag{4.15}$$

Obviously

$$\Big|\frac{\widehat{\tilde{\phi}}(k,q)}{k^2+q^2-a}\chi_{B_\sigma^c}\Big| \le \frac{|\widehat{\tilde{\phi}}(k,q)|}{\sqrt{a}\sigma} \in L^2(\mathbb{R}^{n+3}).$$

We will make use of the representation

$$\widehat{\tilde{\phi}}(k,q) = \widehat{\tilde{\phi}}(\sqrt{a},\omega) + \int_{\sqrt{a}}^{\sqrt{k^2+q^2}} \frac{\partial\widehat{\tilde{\phi}}(|s|,\omega)}{\partial|s|} d|s|.$$

Thus the first term in the right side of (4.15) can be written as $\widehat{\tilde{u_1}}(k,q)+\widehat{\tilde{u_2}}(k,q)$ with

$$\widehat{\tilde{u_1}}(k,q) := \frac{\widehat{\tilde{\phi}}(\sqrt{a},\omega)}{k^2+q^2-a}\chi_{B_\sigma} \quad , \quad \widehat{\tilde{u_2}}(k,q) := \frac{\int_{\sqrt{a}}^{\sqrt{k^2+q^2}} \frac{\partial\widehat{\tilde{\phi}}(|s|,\omega)}{\partial|s|} d|s|}{k^2+q^2-a}\chi_{B_\sigma}.$$

By means of Lemma 12 $|\widehat{\tilde{u_2}}(k,q)| \le C\frac{\chi_{B_\sigma}}{\sqrt{a}} \in L^2(\mathbb{R}^{n+3})$. Finally we estimate

$$\|\widehat{\tilde{u_1}}(k,q)\|^2_{L^2(\mathbb{R}^{n+3})} = \int_{\sqrt{a}-\sigma}^{\sqrt{a}+\sigma} dr \frac{r^{n+2}}{(r-\sqrt{a})^2(r+\sqrt{a})^2} \int_{S^{n+3}} d\omega |\widehat{\tilde{\phi}}(\sqrt{a},\omega)|^2 < \infty$$

if and only if $\widehat{\tilde{\phi}}(\sqrt{a},\omega)=0$ a.e. on $S^{n+3}_{\sqrt{a}}$ which is equivalent to the orthogonality condition $(\phi(x,y), \frac{e^{ikx}}{(2\pi)^{\frac{n}{2}}}\eta_q(y))_{L^2(\mathbb{R}^{n+3})} = 0$ for $(k,q) \in S^{n+3}_{\sqrt{a}}$ a.e., $n \in \mathbb{N}$ as stated in part a) of the theorem.

Then we analogously apply the generalized Fourier transform with respect to the functions $\frac{e^{ikx}}{(2\pi)^{\frac{n}{2}}}$, $k \in \mathbb{R}^n$ and $\eta_q(y)$, $q \in \mathbb{R}^3$ to equation (2.6) which yields

$$\widehat{\tilde{u}}(k,q) = \frac{\widehat{\tilde{\phi}}(k,q)}{k^2+q^2}\chi_{\{k^2+q^2>1\}} + \frac{\widehat{\tilde{\phi}}(k,q)}{k^2+q^2}\chi_{\{k^2+q^2\le 1\}}.$$

The first term in the right side of the identity above is nonsingular and can be easily estimated as

$$\Big|\frac{\widehat{\tilde{\phi}}(k,q)}{k^2+q^2}\chi_{\{k^2+q^2>1\}}\Big| \le |\widehat{\tilde{\phi}}(k,q)| \in L^2(\mathbb{R}^{n+3}).$$

We write $\widehat{\tilde{\phi}}(k,q) = \widehat{\tilde{\phi}}(0) + \int_0^{\sqrt{k^2+q^2}} \frac{\partial\widehat{\tilde{\phi}}(|s|,\omega)}{\partial|s|} d|s|$. Hence it remains to study the square integrability of the sum of two terms $\widehat{\tilde{u}}_3(k,q)+\widehat{\tilde{u}}_4(k,q)$ such that

$$\widehat{\tilde{u}}_3(k,q) := \frac{\widehat{\tilde{\phi}}(0)}{k^2+q^2}\chi_{\{k^2+q^2\le 1\}}, \quad \widehat{\tilde{u}}_4(k,q) := \frac{\int_0^{\sqrt{k^2+q^2}} \frac{\partial\widehat{\tilde{\phi}}(|s|,\omega)}{\partial|s|} d|s|}{k^2+q^2}\chi_{\{k^2+q^2\le 1\}}.$$

Using Lemma 12 we arrive at $|\widehat{\tilde{u}}_4(k,q)| \le \frac{C}{\sqrt{k^2+q^2}}\chi_{\{k^2+q^2\le 1\}} \in L^2(\mathbb{R}^{n+3})$. Finally we compute

$$\|\widehat{\tilde{u}}_3\|^2_{L^2(\mathbb{R}^{n+3})} = |\widehat{\tilde{\phi}}(0)|^2 |S^{n+3}| \int_0^1 r^{n-2} dr.$$

Note that $|\widehat{\tilde{\phi}}(k,q)| \leq \frac{1}{(2\pi)^{\frac{n}{2}}} \|\eta_q\|_{L^\infty(\mathbb{R}^3)} \|\phi\|_{L^1(\mathbb{R}^{n+3})} < \infty,\ k \in \mathbb{R}^n,\ q \in \mathbb{R}^3$. The L^2 norm above is finite in all dimensions $n \geq 2$. When dimension $n = 1$ this norm is finite if and only if $\widehat{\tilde{\phi}}(0) = 0$ which is equivalent to the orthogonality condition $(\phi(x,y), \eta_0(y))_{L^2(\mathbb{R}^4)} = 0$. □

5. Properties of the Operator $\mathcal{L}$ and Proof of Theorem 10

By means of the spectral theorem the identity operator on $L^2(\mathbb{R}^m)$ can be written as $I = P_+ + P_0 + P_-$, where $P_\pm$ and P_0 denote the orthogonal projections onto the positive, negative and zero subspaces of the operator h. Then problem (2.7) can be easily related to the equivalent system of three equations

$$\mathcal{L}_+ u_+ = g_+, \tag{5.16}$$

$$\mathcal{L}_- u_- = g_-, \tag{5.17}$$

$$\mathcal{L}_0 u_0 = g_0, \tag{5.18}$$

with the operators $\mathcal{L}_\pm = P_\pm \mathcal{L} P_\pm$ and $\mathcal{L}_0 = P_0 \mathcal{L} P_0$ applied to the functions $u_\pm = P_\pm u$ and $u_0 = P_0 u$ respectively and the right sides of these equations $g_\pm = P_\pm g$ and $g_0 = P_0 g$. For the first of the equations above we have the following statement.

Lemma 13. *Let assumptions of Theorem 10 hold. Then equation (5.16) admits a solution* $u_+ \in L^2(\mathbb{R}^{3+m}),\ m \in \mathbb{N}$.

Proof. The orthogonal decomposition $g = g_+ + g_0 + g_-$ yields $\|g_+\|_{L^2(\mathbb{R}^{3+m})} \leq \|g\|_{L^2(\mathbb{R}^{3+m})}$. For the operator $\mathcal{L}_+$ on $L^2(\mathbb{R}^3) \otimes \mathrm{Ran}(P_+)$ we have the following lower bound in the sense of quadratic forms

$$\mathcal{L}_+ \geq P_+ h P_+ \geq e_{N+1} > 0,$$

where e_{N+1} stands for either the smallest positive eigenvalue of the operator h or the bottom of its essential spectrum, whichever is smaller. $\mathrm{Ran}(P_+)$ denotes the range of the projection operator P_+. Hence the self-adjoint operator $\mathcal{L}_+$ has the inverse $\mathcal{L}_+^{-1} : L^2(\mathbb{R}^3) \otimes \mathrm{Ran}(P_+) \to L^2(\mathbb{R}^{3+m})$ with the norm estimated from above by $\frac{1}{e_{N+1}}$. Therefore, equation (5.16) admits the solution $u_+ = \mathcal{L}_+^{-1} g_+$ such that

$$\|u_+\|_{L^2(\mathbb{R}^{3+m})} \leq \frac{1}{e_{N+1}}\|g\|_{L^2(\mathbb{R}^{3+m})} < \infty. \qquad \square$$

Next we turn our attention to equation (5.18). Without loss of generality it is being assumed that $g_0(x,y) = v_0(x)\psi_N^1(y) = (g(x,y), \psi_N^1(y))_{L^2(\mathbb{R}^m)}\psi_N^1(y)$.

Lemma 14. *Let assumptions of Theorem 10 hold. Then equation (5.18) possesses a solution* $u_0 \in L^2(\mathbb{R}^{3+m}),\ m \in \mathbb{N}$ *if and only if*

$$(g(x,y), \varphi_0(x)\psi_N^k(y))_{L^2(\mathbb{R}^{3+m})} = 0,\ 1 \leq k \leq m_N.$$

Proof. Since the operator $-\Delta_x + V(x)$ has no nontrivial square integrable zero modes (see Section 3), we have $u_0(x,y) = \xi(x)\psi_N^1(y)$. This allows us to relate equation (5.18) to the problem

$$(-\Delta_x + V(x))\xi(x) = v_0(x). \tag{5.19}$$

For its right hand side using Schwarz inequality, we establish the square integrability

$$\|v_0\|^2_{L^2(\mathbb{R}^3)} = \int_{\mathbb{R}^3} |(g(x,y), \psi_N^1(y))_{L^2(\mathbb{R}^m)}|^2 dx \leq \|g\|^2_{L^2(\mathbb{R}^{3+m})} < \infty.$$

To show the finiteness of another norm Schwarz inequality is being used as well

$$\|xv_0\|_{L^1(\mathbb{R}^3)} = \int_{\mathbb{R}^3} |x||(g(x,y), \psi_N^1(y))_{L^2(\mathbb{R}^m)}|dx$$

$$\leq \int_{\mathbb{R}^3} dx \frac{|x|}{\sqrt{1+|x|^\alpha}}\sqrt{1+|x|^\alpha}\sqrt{\int_{\mathbb{R}^m} dy|g(x,y)|^2}$$

with some $\alpha > 5$ such that $|x|^{\frac{\alpha}{2}}g(x,y) \in L^2(\mathbb{R}^{3+m})$. By means of Schwarz inequality this can be estimated from above as

$$\sqrt{\int_0^\infty \frac{4\pi r^4}{1+r^\alpha}dr}\sqrt{\|g\|^2_{L^2(\mathbb{R}^{3+m})} + \||x|^{\frac{\alpha}{2}}g\|^2_{L^2(\mathbb{R}^{3+m})}} < \infty.$$

Hence Assumption 1.1 of [23] holds for equation (5.19) and therefore by means of Theorem 1 of [23] we obtain the necessary and sufficient solvability conditions in $L^2(\mathbb{R}^3)$ for it, namely $(v_0(x), \varphi_0(x))_{L^2(\mathbb{R}^3)} = 0$, which yields the statement of the lemma. $\square$

Lemma 15. *Let assumptions of Theorem 10 hold. Then equation (5.17) possesses a solution* $u_- \in L^2(\mathbb{R}^{3+m}),\ m \in \mathbb{N}$ *if and only if*

$$(g(x,y), \varphi_k(x)\psi_j^l(y))_{L^2(\mathbb{R}^{3+m})} = 0, \quad 1 \leq l \leq m_j, \quad k \in S^3_{\sqrt{-e_j}} \quad \text{a.e.},$$

$$1 \leq j \leq N-1.$$

Proof. We denote the orthogonal projections onto the eigenspaces correspondent to the negative eigenvalues $\{e_j\}_{j=1}^{N-1}$ of the operator h as $\{P_{-,j}\}_{j=1}^{N-1}$, such that

$$P_- = \sum_{j=1}^{N-1} P_{-,j}, \quad P_{-,j}P_{-,m} = P_{-,j}\delta_{j,m}, \quad 1 \le j,\ m \le N-1.$$

Application of these projections to equation (5.17) easily yields the system of equations equivalent to it

$$(-\Delta_x + V(x) + h)u_{-,j} = g_{-,j}, \quad 1 \le j \le N-1 \tag{5.20}$$

with $P_{-,j}u_- = u_{-,j}$ and $P_{-,j}g_- = g_{-,j}$ such that $u_- = \sum_{j=1}^{N-1} u_{-,j}$ and $g_- = \sum_{j=1}^{N-1} g_{-,j}$. We assume that

$$g_{-,j}(x,y) = v_j(x)\psi_j^1(y) = (g(x,y), \psi_j^1(y))_{L^2(\mathbb{R}^m)}\psi_j^1(y),\ 1 \le j \le N-1.$$

Then equation (5.20) becomes

$$(-\Delta_x + V(x) + e_j)u_{-,j} = v_j(x)\psi_j^1(y), \quad 1 \le j \le N-1.$$

We write $u_{-,j}(x,y) = \xi_j(x)\psi_j^1(y)$ since the operator $-\Delta_x + V(x)$ has no positive eigenvalues with corresponding eigenfunctions in $L^2(\mathbb{R}^3)$ (see Section 3), which yields

$$(-\Delta_x + V(x) + e_j)\xi_j(x) = v_j(x), \quad 1 \le j \le N-1. \tag{5.21}$$

For the right hand side of the equation above by means of Schwarz inequality

$$\|v_j\|^2_{L^2(\mathbb{R}^3)} = \int_{\mathbb{R}^3} |(g(x,y), \psi_j^1(y))_{L^2(\mathbb{R}^m)}|^2 dx \le \|g\|^2_{L^2(\mathbb{R}^{3+m})} < \infty.$$

To estimate the norm below Schwarz inequality is being used as well

$$\|xv_j(x)\|_{L^1(\mathbb{R}^3)} = \int_{\mathbb{R}^3} dx|x||(g(x,y), \psi_j^1(y))_{L^2(\mathbb{R}^m)}|$$
$$\le \int_{\mathbb{R}^3} dx \frac{|x|}{\sqrt{1+|x|^\alpha}}\sqrt{1+|x|^\alpha}\sqrt{\int_{\mathbb{R}^m}|g(x,y)|^2 dy}$$

with some $\alpha > 5$ such that $|x|^{\frac{\alpha}{2}}g(x,y) \in L^2(\mathbb{R}^{3+m})$. Schwarz inequality yields the following upper bound for the expression above

$$\sqrt{\int_0^\infty \frac{4\pi r^4}{1+r^\alpha}dr}\sqrt{\|g\|^2_{L^2(\mathbb{R}^{3+m})} + \||x|^{\frac{\alpha}{2}}g\|^2_{L^2(\mathbb{R}^{3+m})}} < \infty.$$

Therefore, Assumption 1.1 of [23] is satisfied for equation (5.21) and Theorem 1 of [23] yields the necessary and sufficient solvability condition in $L^2(\mathbb{R}^3)$ for it, namely

$$(v_j(x), \varphi_k(x))_{L^2(\mathbb{R}^3)} = 0 \ \text{ for } \ k \in S^3_{\sqrt{-e_j}}$$

a.e. which implies the statement of the lemma. □

Proof of Theorem 10. The solution of equation (2.7) is being constructed as $u := u_+ + u_0 + u_-$, such that the existence of $u_+, u_0, u_- \in L^2(\mathbb{R}^{3+m})$ is proven in Lemmas 13, 14 and 15, respectively.

Suppose problem (2.7) possesses two solutions $u_1, u_2 \in L^2(\mathbb{R}^{3+m})$. Then the function $v := u_1 - u_2 \in L^2(\mathbb{R}^{3+m})$ is a solution of the homogeneous problem with separation of variables

$$\mathcal{L}v = 0$$

which solutions are linear combinations of functions

$$c_j^l(x)\psi_j^l(y), \quad 1 \le l \le m_j, \quad 1 \le j \le N$$

since equation (5.16) with vanishing right hand side admits only the trivial solution in $L^2(\mathbb{R}^{3+m})$ (see Lemma 13). But $c_j^l(x), \quad 1 \le l \le m_j, \quad 1 \le j \le N$ vanish a.e. since the operator $-\Delta_x + V(x)$ does not have nonnegative eigenvalues with corresponding eigenfunctions in $L^2(\mathbb{R}^3)$ (see Section 3). □

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Appendix

Fact 1. *Let* $f(x) \in L^2(\mathbb{R}^n)$ *and* $|x|f(x) \in L^1(\mathbb{R}^n)$, $n \in \mathbb{N}$. *Then* $f(x) \in L^1(\mathbb{R}^n)$.

Proof. We estimate the norm $\|f\|_{L^1(\mathbb{R}^n)}$ from above using Schwarz inequality by

$$\sqrt{\int_{|x|\leq 1} |f(x)|^2 dx}\sqrt{\int_{|x|\leq 1} dx} + \int_{|x|>1} |x||f(x)|dx$$
$$\leq \|f\|_{L^2(\mathbb{R}^n)}\sqrt{|B_n|} + \|xf\|_{L^1(\mathbb{R}^n)} < \infty,$$

where $|B_n|$ denotes the Lebesgue measure of a unit ball in the space of n dimensions. ☐

Fact 2. *Let* $f(x,y) \in L^2(\mathbb{R}^{n+m})$, $|x|f(x,y) \in L^1(\mathbb{R}^{n+m})$ *and* $|y|f(x,y) \in L^1(\mathbb{R}^{n+m})$ *with* $x \in \mathbb{R}^n$ *and* $y \in \mathbb{R}^m$, $n, m \in \mathbb{N}$. *Then* $f(x,y) \in L^1(\mathbb{R}^{n+m})$.

Proof. It can be easily estimated that the expression

$$\int_{\mathbb{R}^{n+m}} \sqrt{|x|^2 + |y|^2}|f(x,y)|dxdy$$

is bounded above by

$$\int_{\mathbb{R}^{n+m}} |x||f(x,y)|dxdy + \int_{\mathbb{R}^{n+m}} |y||f(x,y)|dxdy < \infty.$$

Therefore, $f(x,y) \in L^1(\mathbb{R}^{n+m})$ by means of Fact 1. $\square$