

DECOMPOSITION OF THE NETWORK SUPPORT FOR
ONE NON-HOMOGENEOUS NETWORK FLOW
PROGRAMMING PROBLEM

L.A. Pilipchuk¹, A.S. Pilipchuk², Y.H. Pesheva^{3 §}

¹Belarussian State University
4, Nezalezhnosti Ave., Minsk, 220050, BELARUS
e-mail: pilipchuk@bsu.by

²State Scientific Institution
99, Academician A.K. Krasin Str., Minsk, 220109, BELARUS
e-mail: pilip74@rambler.ru

³Department of Differential Equations
Faculty of Applied Mathematics and Informatics
Technical University of Sofia
P.O. Box 384, Sofia, 1000, BULGARIA
e-mail: yhp@vmei.acad.bg

Abstract: We consider one extremal linear non-homogeneous problem of flow programming with additional constraints of general kind. We use the network properties of the non-homogeneous problem for the decomposition of a network support into trees and cyclic parts.

AMS Subject Classification: 65K05, 90C08, 90C35, 15A06

Key Words: multicommodity network flow programming, transportation problem, sparse matrices, decomposition of support, fundamental system of cycles, direct method, spanning tree, cyclic arc

1. Mathematical Model

Let $G = (I, U)$ be a finite oriented connected network without multiple arcs and loops, where I is a set of nodes and $U \subset I \times I$ is a set of arcs. Assume that a finite non-empty set $K = \{1, \dots, |K|\}$ of different products (commodities) is

Received: March 20, 2010

© 2010 Academic Publications

[§]Correspondence author

transported through the network G . For each $k \in K$ there exists a connected subnetwork $G^k = (I^k, U^k) \subseteq G$, such that $U^k \subseteq U$ is a non-empty set of arcs carrying the k -th product, $I^k = I(U^k)$, $I(U^k)$ – is the set of nodes used for transporting (producing/consuming/transiting) the k -th product. In order to distinguish the products, which can simultaneously pass through an arc $(i, j) \in U$, we introduce the set $K(i, j) = \{k \in K : (i, j) \in U^k\}$. Similarly, $K(i) = \{k \in K : i \in I^k\}$ is the set of products simultaneously transported through a node $i \in I$.

Now let us define a set U_0 as an arbitrary subset of multiarcs of the network G , $U_0 \subseteq U$. Each multiarc $(i, j) \in U_0$ has an aggregate capacity constraint for a total amount of transported products from a subset $K_0(i, j) \subseteq K(i, j)$, $|K_0(i, j)| > 1$. For all arcs $(i, j) \in U$ we assume the amount of each product $k \in K(i, j)$ to be non-negative. Moreover, each arc $(i, j) \in U$ can be equipped with carrying capacities for products from a set $K_1(i, j)$, where $K_1(i, j) \subseteq K(i, j)$ is an arbitrary subset of products transported through the arc (i, j) .

Thus, given transportation costs through arcs $(i, j) \in U$ for all products, we consider the min-cost flow problem on the described network G , where the products are, additionally, related by equality constraints of a general kind.

$$\sum_{(i,j) \in U} \sum_{k \in K(i,j)} c_{ij}^k x_{ij}^k \longrightarrow \min, \quad (1)$$

$$\sum_{j \in I_i^+(U^k)} x_{ij}^k - \sum_{j \in I_i^-(U^k)} x_{ji}^k = a_i^k, \quad \text{for } i \in I^k, k \in K; \quad (2)$$

$$\sum_{(i,j) \in U} \sum_{k \in K(i,j)} \lambda_{ij}^{kp} x_{ij}^k = \alpha_p, \quad \text{for } p = \overline{1, l}; \quad (3)$$

$$\sum_{k \in K_0(i,j)} x_{ij}^k \leq d_{ij}^0, \quad \text{for } (i, j) \in U_0; \quad (4)$$

$$0 \leq x_{ij}^k \leq d_{ij}^k, \quad \text{for } k \in K_1(i, j), (i, j) \in U; \quad (5)$$

$$x_{ij}^k \geq 0, \quad \text{for } k \in K(i, j) \setminus K_1(i, j), (i, j) \in U, \quad (6)$$

where $I_i^+(U^k) = \{j \in I^k : (i, j) \in U^k\}$, $I_i^-(U^k) = \{j \in I^k : (j, i) \in U^k\}$; x_{ij}^k – amount of the k -th product transported through an arc (i, j) ; c_{ij}^k – transportation cost through an arc (i, j) of a unit of the k -th product; d_{ij}^k – carrying capacity of an arc (i, j) for the k -th product; d_{ij}^0 – aggregate capacity of an arc $(i, j) \in U_0$ for a total amount of products $K_0(i, j)$; λ_{ij}^{kp} – weight of a unit of

the k -th product transported through an arc (i, j) in the p -th additional constraint; α_p – total weighted amount of products imposed by the p -th additional constraint; a_i^k – intensity of a node i for the k -th product.

Definition 1. The vector $x = (x_{ij}^k, (i, j) \in U, k \in K(i, j))$ is a non-homogeneous flow on the network G , if it satisfies the constraints (2)-(6).

For brevity, further in the paper, we will call a “non-homogeneous flow”, simply, “a flow” and a “multinetwork”, simply “a network”.

Definition 2. The flow $x^0 = (x_{ij}^{0k}, (i, j) \in U, k \in K(i, j)), x^0 \in X$, where X is a set of all flows, is optimal if

$$\sum_{(i,j) \in U} \sum_{k \in K(i,j)} c_{ij}^k x_{ij}^{0k} = \min_{x \in X} \sum_{(i,j) \in U} \sum_{k \in K(i,j)} c_{ij}^k x_{ij}^k.$$

Definition 3. We call (2)-(4) the system of main constraints of the problem (1)-(6). Constraints (5)-(6) are the direct constraints of the problem (1)-(6).

Let us name, traditionally, different parts of the system of main constraints.

Definition 4. We call (2) the network part of the system of main constraints of the problem (1)-(6). Equations (3) are the additional part of the system of main constraints (additional constraints) of the problem (1)-(6). Inequalities (4) form the sparse part of the system of main constraints of the problem (1)-(6).

2. Example

In this section we introduce an example of the problem (1)-(6) for the multinetwork $G = (I, U), I = \{1, 2, 3, 4\}, U = \{(1, 3), (1, 4), (2, 1), (2, 4), (3, 2), (4, 3)\}$. Let $K = \{1, 2, 3, 4, 5\}$ be the set of transported products (Figure 1). Characteristics of the structure of the network G are provided in Table 1. The mathematical model of the problem (1)-(6) for the multinetwork $G = (I, U)$ (Figure 1) can be represented as (1a)-(6a).

$$2x_{13}^1 + x_{13}^2 + 3x_{13}^3 + 4x_{13}^5 + x_{14}^1 + 8x_{14}^4 + 5x_{21}^2 + x_{21}^4 + x_{24}^1 + 6x_{24}^2 + 2x_{24}^3 + 3x_{32}^2 + 2x_{32}^4 + x_{32}^5 + 4x_{43}^2 + x_{43}^3 \rightarrow \min, \tag{1a}$$

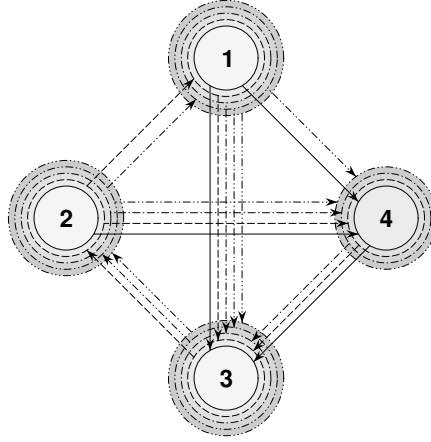


Figure 1: Multinetwork $G = (I, U)$

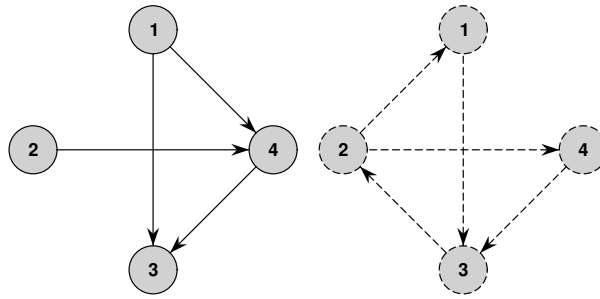


Figure 2: Networks $G^k = (I^k, U^k), k = 1, 2.$

$$\begin{aligned}
 x_{13}^1 + x_{14}^1 &= 1, & x_{13}^2 - x_{21}^2 &= -0.75, \\
 x_{24}^1 &= 0.5, & x_{21}^2 + x_{24}^2 - x_{32}^2 &= 0.25, \\
 -x_{13}^1 - x_{43}^1 &= -1, & x_{32}^2 - x_{13}^2 - x_{43}^2 &= 0.5, \\
 x_{43}^1 - x_{14}^1 - x_{24}^1 &= -0.5, & x_{43}^2 - x_{24}^2 &= 0, \\
 x_{13}^3 &= 0, & x_{13}^4 + x_{14}^4 - x_{21}^4 &= 0.5, \\
 x_{24}^3 &= 1, & x_{21}^4 + x_{24}^4 - x_{32}^4 &= -0.25, \\
 -x_{13}^3 - x_{43}^3 &= -1, & x_{32}^4 - x_{13}^4 &= 0.75, \\
 x_{43}^3 - x_{24}^3 &= 0, & -x_{14}^4 - x_{24}^4 &= -1, \\
 & & x_{13}^5 &= 1.5, \\
 & & -x_{32}^5 &= -0.5, \\
 & & x_{32}^5 - x_{13}^5 &= -1,
 \end{aligned} \tag{2a}$$

(i, j)	(1, 3)					(1, 4)				
k	1	2	3	4	5	1	2	3	4	5
U^k	+	+	+	+	+	+			+	
U_0	+									
$K(i, j)$	{1, 2, 3, 4, 5}					{1, 4}				
$K_1(i, j)$	{1, 2, 5}					{4}				
$K_0(i, j)$	{3, 4}									
(i, j)	(2, 1)					(2, 4)				
k	1	2	3	4	5	1	2	3	4	5
U^k		+		+		+	+	+	+	
U_0										
$K(i, j)$	{2, 4}					{1, 2, 3, 4}				
$K_1(i, j)$	{2, 4}					{1, 3}				
$K_0(i, j)$										
(i, j)	(3, 2)					(4, 3)				
k	1	2	3	4	5	1	2	3	4	5
U^k		+		+	+	+	+	+		
U_0						+				
$K(i, j)$	{2, 4, 5}					{1, 2, 3}				
$K_1(i, j)$	{2, 4, 5}					{3}				
$K_0(i, j)$						{1, 2}				

Table 1: Characteristics of the network structure

$$\begin{aligned}
 & 2x_{13}^1 + 3x_{13}^2 + x_{13}^3 + x_{13}^4 + 2x_{13}^5 + x_{14}^1 \\
 & + 3x_{14}^4 + 2x_{21}^2 + x_{21}^4 + x_{24}^1 + x_{24}^2 + 4x_{24}^3 \\
 & + x_{24}^4 + 2x_{32}^2 + 2x_{32}^4 + 3x_{32}^5 + x_{43}^1 + x_{43}^2 + 3x_{43}^3 = 42, \\
 & 2x_{13}^1 + 2x_{13}^2 + 3x_{13}^3 + x_{13}^4 + 2x_{13}^5 + 3x_{14}^1 \\
 & + x_{14}^4 + 2x_{21}^2 + x_{21}^4 + x_{24}^1 + 2x_{24}^2 + 4x_{24}^3 \\
 & + 3x_{24}^4 + 2x_{32}^2 + 4x_{32}^4 + x_{32}^5 + x_{43}^1 + 3x_{43}^2 + 3x_{43}^3 = 58,
 \end{aligned} \tag{3a}$$

$$\begin{aligned}
 & x_{13}^3 + x_{13}^4 \leq 6, \quad x_{13}^3 \geq 0, \quad x_{13}^4 \geq 0, \\
 & x_{43}^1 + x_{43}^2 \leq 7, \quad x_{43}^1 \geq 0, \quad x_{43}^2 \geq 0,
 \end{aligned} \tag{4a}$$

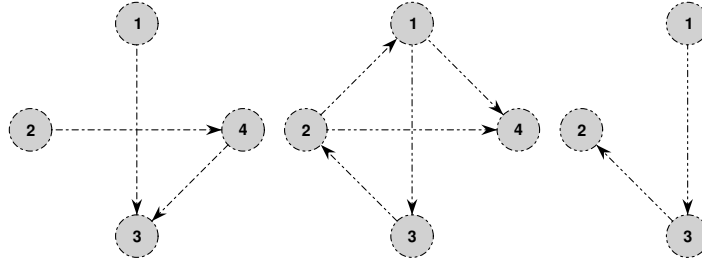


Figure 3: Networks $G^k = (I^k, U^k), k = 3, 4, 5$.

$$\begin{aligned}
 0 \leq x_{13}^1 \leq 4, \quad 0 \leq x_{14}^4 \leq 4, \quad 0 \leq x_{24}^1 \leq 2, \quad 0 \leq x_{32}^4 \leq 3, \\
 0 \leq x_{13}^2 \leq 2, \quad 0 \leq x_{21}^2 \leq 5, \quad 0 \leq x_{24}^3 \leq 4, \quad 0 \leq x_{32}^5 \leq 2, \\
 0 \leq x_{13}^5 \leq 6, \quad 0 \leq x_{21}^4 \leq 2, \quad 0 \leq x_{32}^2 \leq 4, \quad 0 \leq x_{43}^3 \leq 4,
 \end{aligned} \tag{5a}$$

$$x_{14}^1 \geq 0, \quad x_{24}^2 \geq 0, \quad x_{24}^4 \geq 0. \tag{6a}$$

3. Decomposition of the Network Support, Determinants of Cycles

In this section we define a support of the network $G = (I, U)$ for the problem (1)-(6). In order to give the definition we will need to introduce some more sets.

Let $U_S^k \subseteq U^k, k \in K$ be arbitrary subsets of arcs of the networks G^k . For each arc $(i, j) \in U$ we define a subset of products $K_S(i, j) \subseteq K(i, j), K_S(i, j) = \{k \in K : (i, j)^k \in U_S^k\}$, transported through an arc (i, j) . Also, for arcs from the set U_0 we introduce a subset U^* , such that $U^* \subseteq \bar{U}_0 \subseteq U_0, \bar{U}_0 = \{(i, j) \in U_0 : |K_S^0(i, j)| > 1\}$, where $K_S^0(i, j) = K_S(i, j) \cap K_0(i, j), (i, j) \in U_0$.

Let $U_S^k \subseteq U^k, k \in K$ be arbitrary subsets of arcs of the subnetworks G^k . Thus, for each arc $(i, j) \in U$ we can define a subset of products $K_S(i, j) \subseteq K(i, j), K_S(i, j) = \{k \in K : (i, j)^k \in U_S^k\}$, transported through an arc (i, j) . For arcs from the set U_0 we introduce a subset U^* , so that $U^* \subseteq U_0$. Also, we denote $K_S^0(i, j) = K_S(i, j) \cap K_0(i, j), (i, j) \in U_0$.

Definition 5. The aggregate of sets $U_S = \{U_S^k, k \in K; U^*\} U^* \subseteq \bar{U}_0, \bar{U}_0 = \{(i, j) \in U_0 : |K_S^0(i, j)| > 1\}$, is a support of the network G (or, network support) for the problem (1)-(6), if the system

$$\sum_{j \in I_i^+(\hat{U}^k)} x_{ij}^k - \sum_{j \in I_i^-(\hat{U}^k)} x_{ji}^k = 0, \quad i \in I^k, \quad k \in K, \tag{7}$$

$$\sum_{k \in K} \sum_{(i,j) \in \hat{U}^k} \lambda_{ij}^{kp} x_{ij}^k = 0, \quad p = \overline{1, l}, \quad (8)$$

$$\sum_{k \in K_S^0(i,j)} x_{ij}^k = 0, \quad (i,j) \in \hat{U}^*, \quad (9)$$

has only the trivial solution $x_{ij}^k = 0, (i,j) \in \hat{U}^k, k \in K$, when $\hat{U}^k = U_S^k$ and $\hat{U}^* = U^*$, $\hat{K}_S^0(i,j) = K_S^0(i,j)$, but has a non-trivial solution if either:

$$1) \hat{U}^k = U_S^k, k \in K; \hat{U}^* = U^* \setminus (i_0, j_0), (i_0, j_0) \in U^*, \text{ or}$$

$$2) \hat{U}^k = U_S^k, k \in K \setminus k_0, \hat{U}_S^{k_0} = U_S^{k_0} \cup (i_0, j_0)^{k_0}, (i_0, j_0)^{k_0} \notin U_S^{k_0}, k_0 \in K; \hat{U}^* = U^*,$$

$$\hat{K}_S^0(i,j) = \begin{cases} K_S^0(i,j), & \text{if } (i,j) \neq (i_0, j_0), \\ K_S^0(i,j) \cup (K^0 \cap K^0(i,j)), & \text{if } (i,j) = (i_0, j_0). \end{cases}$$

Consider the support structure $U_S = \{U_S^k, k \in K; U^*\}, U_S^k \subset U^k, k \in K; U^* \subset \overline{U}_0, \overline{U}_0 = \{(i,j) \in U_0 : |K_S^0| > 1\}, K_S(i,j) = \{k \in K(i,j) : (i,j)^k \in U_S^k\}, (i,j) \in U, K_S^0(i,j) = K_S(i,j) \cap K_0(i,j), (i,j) \in U_0$. Let the set U_S^k contain l_k arcs $(i,j)^k$ such that removing them from the set U_S^k gives the set U_T^k such that the network $(I(U_T^k), U_T^k)$ contains no cycles but every network $(I(U_T^k), U_T^k \cup (i,j)^k)$ contains a cycle. Let us denote $U_C^k = U_S^k \setminus U_T^k, k \in K$.

Definition 6. The elements of the set $U_C = \bigcup_{k \in K} U_C^k$ are called cyclic arcs.

The elements of the set $U_T = \bigcup_{k \in K} U_T^k$ are called tree arcs.

Let K be the set $\{1, 2, \dots, |K|\}$. Let us suppose that U_S^k contain l_k independent cycles, $k \in K$. The cycles $Z_k = \{L_{\tau\rho}^k, (\tau, \rho)^k \in U_C^k\}$ of the network G are called independent if every cycle has at least one edge that does not belong to other cycles and the unitary circulations [3], [5] form a linearly-independent system of vectors, $k \in K$.

Every arc $(\tau, \rho)^k \in U_C^k$ belongs to some cycle $L_{\tau\rho}^k$ from the set U_S^k . If the network $G_S^k = (I(U_S^k), U_S^k)$ is connected then the cycles $Z_k = \{L_{\tau\rho}^k, (\tau, \rho)^k \in U_C^k\}, k \in K$ introduced the same way form the fundamental set of cycles. We introduce arbitrary numbering of arcs within the set and $U_C = \bigcup_{k \in K} U_C^k$. Let

us mark every arc $(\tau\rho)^k \in U_C^k$ with the number $t = t(\tau\rho)^k$. Let us consider an arbitrary cycle $L_t^k, t = t(\tau, \rho)^k$. We choose the direction of cycle detour such that the arc $(\tau, \rho)^k$ is a forward one. Let L_t^{k+}, L_t^{k-} be sets of forward and backward arcs of the cycle L_t^k correspondingly.

Definition 7. The number $R_p(L_t^k) = \sum_{(i,j)^k \in L_t^k} \lambda_{ij}^{kp} \text{sign}(i,j)^{L_t^k}$ is called the determinant of the cycle L_t^k with respect to the additional restriction (3) with the number p , where

$$\text{sign}(i,j)^{L_t^k} = \begin{cases} 1, & \text{if } (i,j)^k \in L_t^{k+}; \\ -1, & \text{if } (i,j)^k \in L_t^{k-}; \\ 0, & \text{if } (i,j)^k \notin L_t^k. \end{cases}$$

Let us put the arcs of the set U^* in arbitrary order. Let $\xi = \xi(i,j)$ be the serial number of the multiarc (i,j) in the set U^* , $1 \leq \xi \leq m, m = |U^*|$. We build the matrix $D = \begin{pmatrix} D_1 \\ D_2 \end{pmatrix}$, where $D_1 = (R_p(L_t^k), p = \overline{1, l}, t = \overline{1, \tilde{t}}), \tilde{t} = |U_c|$ is a matrix of order $l \times \tilde{t}$ consisting of determinants $R_p(L_t^k) = R_p(L_{(\tau,\rho)}^k), t = t(\tau, \rho)$ with respect to the restrictions (3), $D_2 = (\delta_{\xi(i,j)}(L_t^k), \xi(i,j) = \overline{1, m}, t = \overline{1, \tilde{t}})$ is an $m \times \tilde{t}$ -matrix that consists of the following elements:

$$\delta_{\xi(i,j)}(L_t^k) = \begin{cases} 1, & \text{if } (i,j)^k \in L_{t(\tau,\rho)}^{k+}, k \in K_0(i,j), (i,j) \in U_0; \\ -1, & \text{if } (i,j)^k \in L_{t(\tau,\rho)}^{k-}, k \in K_0(i,j), (i,j) \in U_0; \\ 0, & \text{if } (i,j)^k \notin L_{t(\tau,\rho)}^k, k \in K_0(i,j), (i,j) \in U_0; \\ 0, & \text{if } (i,j)^k \notin L_{t(\tau,\rho)}^k, k \in K(i,j) \setminus K_0(i,j), \\ & (i,j) \in U_0; \end{cases}$$

$\xi = \xi(i,j), (i,j) \in U^*.$

If $\tilde{t} \neq l + m, m = |U^*|$ we supply the matrix D with zeros to make it a square matrix of order $\max\{\tilde{t}, l + |U^*|\}$. Let $R(U_S) = \det D$.

4. Theoretical – Graphical Properties

Theorem 1. (Criterion of Support) *The set of arcs $U_S = \{U_S^k, k \in K, U^*\}, U_S^k \subset U^k, k \in K; U^* \subset \overline{U}_0, \overline{U}_0 = \{(i,j) \in U_0 : |K_S^0| > 1\}, K_S(i,j) = \{k \in K(i,j) : (i,j)^k \in U_S^k\}, (i,j) \in U, K_S^0(i,j) = K_S(i,j) \cap K_0(i,j), (i,j) \in U_0$ is a support of the network G iff the following conditions are met:*

1. $I(U_S^k) = I^k, k \in K;$
2. U_S^k is a connected set, $k \in K;$
3. $R(U_S) \neq 0.$

Proof. Is made [5]. □

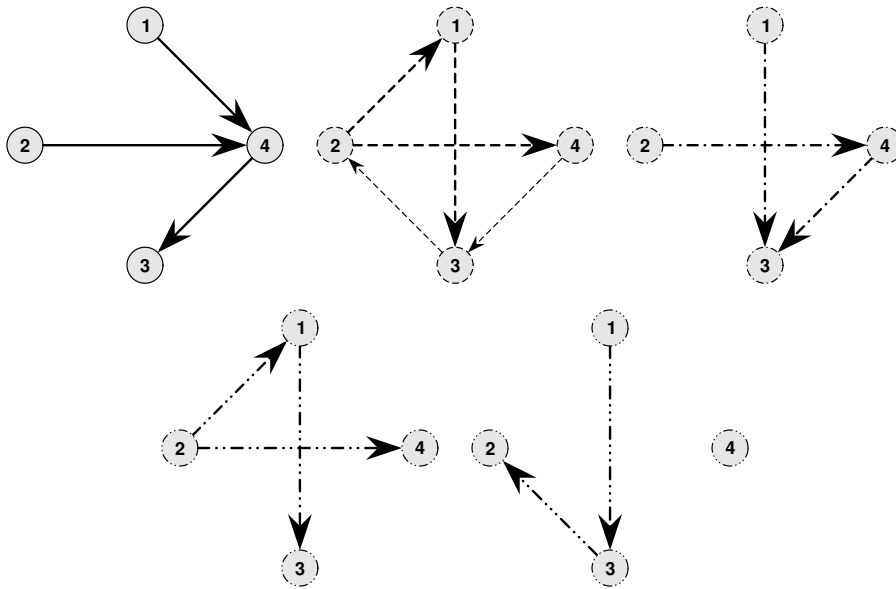


Figure 4: A support for the problem (1a)-(6a): bold arrows mean tree arcs, thin arrows mean cyclic arcs

(i, j)	(1, 3)	(1, 4)	(2, 1)
$K_S(i, j)$	{2, 3, 4, 5}	{1}	{2, 4}
$K_S^1(i, j)$	{2, 5}	\emptyset	{2, 4}
$K_S^0(i, j)$	{3, 4}		
$K_N(i, j)$	{1}	{4}	\emptyset
$K_N^1(i, j)$	{1}	{4}	\emptyset
$K_N^0(i, j)$	\emptyset		
(i, j)	(2, 4)	(3, 2)	(4, 3)
$K_S(i, j)$	{1, 2, 3, 4}	{2, 5}	{2, 1, 3}
$K_S^1(i, j)$	{1, 3}	{2, 5}	{3}
$K_S^0(i, j)$			{2, 1}
$K_N(i, j)$	\emptyset	{4}	\emptyset
$K_N^1(i, j)$	\emptyset	{4}	\emptyset
$K_N^0(i, j)$			\emptyset

Table 2: Characteristics of the support for the problem (1a)-(6a)

Let $D = D(U_S)$ be the matrix of determinants that corresponds to the support $U_S = \{U_S^k, k \in K, U^*\}$.

An example of a support for the problem (1a)-(6a) can be found at Figure 4. The characteristics of the support are represented in Table 2, where

$$K_S^1(i, j) = K_S(i, j) \cap K_1(i, j), (i, j) \in U;$$

$$K_S^0(i, j) = K_S(i, j) \cap K_0(i, j), (i, j) \in U_0;$$

$$K_N(i, j) = K(i, j) \setminus K_S(i, j);$$

$$K_N^1(i, j) = K_N(i, j) \cap K_1(i, j), (i, j) \in U;$$

$$K_N^0(i, j) = K_N(i, j) \cap K_0(i, j), (i, j) \in U_0.$$

$$U^* = \emptyset, D = \begin{bmatrix} R_1(L_{32}^2) & R_1(L_{43}^2) \\ R_2(L_{32}^2) & R_2(L_{43}^2) \end{bmatrix} = \begin{bmatrix} 7 & -3 \\ 6 & 1 \end{bmatrix}, \det D \neq 0.$$

References

- [1] Ravindra K. Ahuja, Thomas L. Magnanti, James B. Orlin, *Network Flows: Theory, Algorithms, and Applications*, New Jersey (1993).
- [2] L.A. Pilipchuk, A.S. Pilipchuk, Y.H. Pesheva, Algorithms for construction of optimal and suboptimal solutions in network optimization problem, *IJPAM*, **54**, No. 2 (2009), 193-205.
- [3] L.A. Pilipchuk, E.S. Vecharynski, Y.H. Pesheva, Solution of large linear systems with embedded network structure for a non-homogeneous network flow programming problem, *Mathematica Balkanica, New Series*, **22**, No-s: 3-4 (2008), 233-252.
- [4] L.A. Pilipchuk, *Linear Inhomogeneous Problems of Networks Flows*, Minsk, BSU (2009), In Russian.