

INCREMENT OF THE OBJECTIVE FUNCTION  
AND OPTIMALITY CRITERION FOR ONE  
NON-HOMOGENEOUS NETWORK FLOW  
PROGRAMMING PROBLEM

L.A. Pilipchuk<sup>1</sup>, A.S. Pilipchuk<sup>2</sup>, Y.H. Pesheva<sup>3 §</sup>

<sup>1</sup>Belarussian State University  
4, Nezalezhnosti Ave., Minsk, 220050, BELARUS  
e-mail: pilipchuk@bsu.by

<sup>2</sup>State Scientific Institution  
99, Academician A.K. Krasin Str., Minsk, 220109, BELARUS  
e-mail: pilip74@rambler.ru

<sup>3</sup>Department of Differential Equations  
Faculty of Applied Mathematics and Informatics  
Technical University of Sofia  
P.O. Box 384, Sofia, 1000, BULGARIA  
e-mail: yhp@vmei.acad.bg

**Abstract:** For an linear non-homogeneous flow programming problem with additional constraints of general kind are obtained the increment of the objective function using network properties of the problem and principles of decomposition of a support. Optimality conditions are received.

**AMS Subject Classification:** 65K05, 90C08, 90C35

**Key Words:** network flow, non-homogeneous problem, support, objective function, criterion of optimality, nonsingular plan

### 1. Statement of the Problem, Basic Concepts and Definitions

Consider the following mathematical model of inhomogeneous extreme network flow problem

$$\sum_{(i,j) \in U} \sum_{k \in K(i,j)} c_{ij}^k x_{ij}^k \longrightarrow \min, \quad (1)$$

$$\sum_{j \in I_i^+(U^k)} x_{ij}^k - \sum_{j \in I_i^-(U^k)} x_{ji}^k = a_i^k, \quad \text{for } i \in I^k, k \in K; \quad (2)$$

$$\sum_{(i,j) \in U} \sum_{k \in K(i,j)} \lambda_{ij}^{kp} x_{ij}^k = \alpha_p, \quad \text{for } p = \overline{1, l}; \quad (3)$$

$$\sum_{k \in K_0(i,j)} x_{ij}^k \leq d_{ij}^0, \quad \text{for } (i,j) \in U_0; \quad (4)$$

$$0 \leq x_{ij}^k \leq d_{ij}^k, \quad \text{for } k \in K_1(i,j), (i,j) \in U; \quad (5)$$

$$x_{ij}^k \geq 0, \quad \text{for } k \in K(i,j) \setminus K_1(i,j), (i,j) \in U, \quad (6)$$

where  $G = (I, U)$  – a finite orientated connected network without multiple arcs and loops,  $I$  is a set of nodes and  $U \subset I \times I$  is a set of arcs;  $K = \{1, \dots, |K|\}$  – a finite non-empty set of different products (commodities) is transported through the network  $G$ . For each  $k \in K$  there exists a connected subnetwork  $G^k = (I^k, U^k) \subseteq G$ , such that  $U^k \subseteq U$  is a non-empty set of arcs carrying the  $k$ -th product,  $I^k = I(U^k)$  – is the set of nodes used for transporting (producing/consuming/transiting) the  $k$ -th product. In order to distinguish the products, which can simultaneously pass through an arc  $(i, j) \in U$ , we introduce the set  $K(i, j) = \{k \in K : (i, j) \in U^k\}$ . Similarly,  $K(i) = \{k \in K : i \in I^k\}$  is the set of products simultaneously transported through a node  $i \in I$ . Now let us define a set  $U_0 \subseteq U$  as an arbitrary subset of multiarcs of the network  $G$ . Each multiarc  $(i, j) \in U_0$  has an aggregate capacity constraint for a total amount of transported products from a subset  $K_0(i, j) \subseteq K(i, j), |K_0(i, j)| > 1$ . For all arcs  $(i, j) \in U$  we assume the amount of each product  $k \in K(i, j)$  to be non-negative. Moreover, each arc  $(i, j) \in U$  can be equipped with carrying capacities for products from a set  $K_1(i, j)$ , where  $K_1(i, j) \subseteq K(i, j)$  is an arbitrary subset of products transported through the arc  $(i, j)$ .  $I_i^+(U^k) = \{j \in I^k : (i, j) \in U^k\}, I_i^-(U^k) = \{j \in I^k : (j, i) \in U^k\}; x_{ij}^k$  – amount of the  $k$ -th product transported through an arc  $(i, j)$ ;  $c_{ij}^k$  – transportation cost through an arc  $(i, j)$  of a unit of the  $k$ -th product;  $d_{ij}^k$  – carrying capacity of an arc  $(i, j)$  for the  $k$ -th product;  $d_{ij}^0$  – aggregate capacity of an arc  $(i, j) \in U_0$  for a total amount of

products  $K_0(i, j)$ ;  $\lambda_{ij}^{kp}$  – weight of a unit of the  $k$ -th product transported through an arc  $(i, j)$  in the  $p$ -th additional constraint;  $\alpha_p$  – total weighted amount of products imposed by the  $p$ -th additional constraint;  $a_i^k$  – intensity of a node  $i$  for the  $k$ -th product.

### 2. Formula for Increment of the Objective Function

Let  $x = (x_{ij}^k, (i, j) \in U, k \in K(i, j))$  be a plan [2] of the problem (1)-(6), i.e. components of the vector  $x$  meet the conditions (2)-(6). Along with the plan  $x$  let us define support plan  $\{x, U_S\}$  as a pair, containing of an arbitrary flow  $x$  and a support  $U_S = \{U_S^k, k \in K; U^*\}$   $U^* \subseteq \bar{U}_0, \bar{U}_0 = \{(i, j) \in U_0 : |K_S^0(i, j)| > 1\}$  of the problem (1)-(6) [2, 4]. Let us consider some other plan  $\bar{x} = (\bar{x}_{ij}^k : (i, j) \in U, k \in K(i, j)) = (x_{ij}^k + \Delta x_{ij}^k : (i, j) \in U, k \in K(i, j))$ . Then  $\Delta x = (\Delta x_{ij}^k, (i, j) \in U, k \in K(i, j))$  is the vector of flow increments along the arc  $(i, j) \in U$ .

Let us denote

$$\begin{aligned} z_{ij} &= \sum_{k \in K_0(i, j)} x_{ij}^k, \quad \bar{z}_{ij} = \sum_{k \in K_0(i, j)} \bar{x}_{ij}^k, \\ \Delta z_{ij} &= \bar{z}_{ij} - z_{ij} = \sum_{k \in K_0(i, j)} \Delta x_{ij}^k, \quad (i, j) \in U_0. \end{aligned} \tag{7}$$

Since the plan  $\bar{x}$  meets the conditions (2)-(6) then the following relations are true

$$\sum_{j \in I_i^+(U^k)} \bar{x}_{ij}^k - \sum_{j \in I_i^-(U^k)} \bar{x}_{ji}^k = a_i^k, \quad i \in I^k, \quad k \in K, \tag{8}$$

$$\sum_{(i, j) \in U} \sum_{k \in K(i, j)} \lambda_{ij}^{kp} \bar{x}_{ij}^k = \alpha^p, \quad p = \bar{1}, \bar{l}, \tag{9}$$

$$\sum_{k \in K_0(i, j)} \bar{x}_{ij}^k \leq d_{ij}^0, \quad \bar{x}_{ij}^k \geq 0, \quad k \in K_0(i, j), \quad (i, j) \in U^*, \tag{10}$$

where the constraints (4) are written down only for the support multiarcs  $U^*$ .

Subtracting from (8)-(10) the corresponding constraints (2)-(4), we obtain:

$$\sum_{j \in I_i^+(U^k)} \Delta x_{ij}^k - \sum_{j \in I_i^-(U^k)} \Delta x_{ji}^k = 0, \quad i \in I^k, \quad k \in K, \tag{11}$$

$$\sum_{(i, j) \in U} \sum_{k \in K(i, j)} \lambda_{ij}^k \Delta x_{ij}^k = 0, \quad p = \bar{1}, \bar{l}, \tag{12}$$

$$\sum_{k \in K_0(i,j)} \Delta x_{ij}^k = \Delta z_{ij}, \quad (i, j) \in U^*, \tag{13}$$

where  $\Delta z_{ij}$  is defined by formula (7).

Let us order components of solution of system (11)-(13) the following way:  $\Delta x' = (\Delta x'_T, \Delta x'_C, \Delta x'_N)$ , where  $\Delta x'_T = (\Delta x_{ij}^k, (i, j)^k \in U_T^k, k \in K)$ ,  $\Delta x'_C = (\Delta x_{ij}^k, (i, j)^k \in U_C^k, k \in K)$ ,  $\Delta x'_N = (\Delta x_{ij}^k, (i, j)^k \in U_N^k, k \in K)$ ,  $U_N^k = U^k \setminus (U_T^k \cup U_C^k)$ ,  $U_T^k$  – spanning tree of the graph  $G^k$ ,  $k \in K$ .

The general solution of the homogeneous system (11) is the following [4]:

$$\Delta x_{ij}^k = \sum_{(\tau,\rho)^k \in U^k \setminus U_T^k} \Delta x_{\tau\rho}^k \text{sign}(i, j)^{L_{t(\tau,\rho)}^k}, \quad (i, j)^k \in U_T^k, k \in K, \tag{14}$$

$$\text{sign}(i, j)^{L_t^k} = \begin{cases} 1, & \text{if } (i, j)^k \in L_t^{k+}; \\ -1, & \text{if } (i, j)^k \in L_t^{k-}; \\ 0, & \text{if } (i, j)^k \notin L_t^k. \end{cases}$$

Let us put the items, corresponding to components of the vector  $\Delta x'_T$ , together:

$$\Delta\varphi(x) = \sum_{k \in K} \sum_{(i,j)^k \in U^k} c_{ij}^k \Delta x_{ij}^k = \sum_{k \in K} \sum_{(i,j)^k \in U_T^k} c_{ij}^k \Delta x_{ij}^k + \sum_{k \in K} \sum_{(i,j)^k \in U^k \setminus U_T^k} c_{ij}^k \Delta x_{ij}^k. \tag{15}$$

Let us substitute (14) into (15):

$$\begin{aligned} \Delta\varphi(x) &= \sum_{k \in K} \sum_{(i,j)^k \in U_T^k} c_{ij}^k \left[ \sum_{(\tau,\rho)^k \in U^k \setminus U_T^k} \Delta x_{\tau\rho}^k \text{sign}(i, j)^{L_{t(\tau,\rho)}^k} \right] \\ &\quad + \sum_{k \in K} \sum_{(\tau,\rho)^k \in U^k \setminus U_T^k} c_{\tau\rho}^k \Delta x_{\tau\rho}^k \\ &= \sum_{k \in K} \sum_{(\tau,\rho)^k \in U^k \setminus U_T^k} \left[ c_{\tau\rho}^k + \sum_{(i,j)^k \in U_T^k} c_{ij}^k \text{sign}(i, j)^{L_{t(\tau,\rho)}^k} \right] \Delta x_{\tau\rho}^k. \end{aligned}$$

Let us denote  $\sum_{(i,j)^k \in L_{t(\tau,\rho)}^k} c_{ij}^k \text{sign}(i, j)^{L_{t(\tau,\rho)}^k}$ , with  $\tilde{\Delta}_{\tau\rho}^k$ . Then

$$\Delta\varphi(x) = \sum_{k \in K} \sum_{(\tau,\rho)^k \in U^k \setminus U_T^k} \tilde{\Delta}_{\tau\rho}^k \Delta x_{\tau\rho}^k. \tag{16}$$

Knowing that  $U^k \setminus U_T^k = U_C^k \cup U_N^k$ , we break the sum again:

$$\Delta\varphi(x) = \sum_{k \in K} \sum_{(\tau, \rho)^k \in U_C^k} \tilde{\Delta}_{\tau\rho}^k \Delta x_{\tau\rho}^k + \sum_{k \in K} \sum_{(\tau, \rho)^k \in U_N^k} \tilde{\Delta}_{\tau\rho}^k \Delta x_{\tau\rho}^k. \tag{17}$$

By analogy with [6], [7] we obtain the components of the vector  $\Delta x'_C$  for system (11)-(13):

$$\Delta x_{\tau\rho}^k = \sum_{p=1}^l \nu_{t(\tau, \rho)^k, p} \tilde{\beta}_p + \sum_{(i, j) \in U^*} \nu_{t(\tau, \rho)^k, l+\xi(i, j)} \tilde{\beta}_{l+\xi(i, j)},$$

$$(\tau, \rho)^k \in U_C^k, k \in K, \tilde{\beta} = \begin{pmatrix} \tilde{\beta}_p, p = \overline{1, l} \\ \tilde{\beta}_{\xi(i, j)}, (i, j) \in U_0 \end{pmatrix}. \tag{18}$$

The values of the components of the vectors  $\tilde{\beta}_p$  and  $\tilde{\beta}_{\xi(i, j)}$  are computed according to the following formulas:

$$\tilde{\beta}_p = - \sum_{k \in K} \sum_{(\tau, \rho)^k \in U_N^k} R_p(L_{t(\tau, \rho)}^k) \Delta x_{\tau\rho}^k, \quad p = \overline{1, l}, \tag{19}$$

$$\tilde{\beta}_{\xi(i, j)} = \Delta z_{ij} - \sum_{k \in K_0(i, j)} \sum_{(\tau, \rho)^k \in U_N^k} \delta_{\xi(i, j)}(L_{t(\tau, \rho)}^k) \Delta x_{\tau\rho}^k, \quad (i, j) \in U^*. \tag{20}$$

Taking into account the formula (14) we obtain:

$$\Delta\varphi(x) = \sum_{k \in K} \sum_{(\tau, \rho)^k \in U_C^k} \tilde{\Delta}_{\tau\rho}^k \left[ \sum_{p=1}^l \nu_{t(\tau, \rho)^k, p} \tilde{\beta}_p + \sum_{(i, j) \in U^*} \nu_{t(\tau, \rho)^k, l+\xi(i, j)} \tilde{\beta}_{l+\xi(i, j)} \right] + \sum_{k \in K} \sum_{(\tau, \rho)^k \in U_N^k} \tilde{\Delta}_{\tau\rho}^k \Delta x_{\tau\rho}^k. \tag{21}$$

Let us introduce the following denotation:

$$r_p = \sum_{k \in K} \sum_{(\tau, \rho)^k} \tilde{\Delta}_{\tau\rho}^k \nu_{t(\tau, \rho)^k, p}, \quad p = \overline{1, l},$$

$$r_{ij} = \sum_{k \in K} \sum_{(\tau, \rho)^k \in U_q^k} \tilde{\Delta}_{\tau\rho}^k \nu_{t(\tau, \rho)^k, l+\xi(i, j)}, \quad (i, j) \in U^*.$$

Taking into account the denotations made, we may represent  $\Delta\varphi(x)$  the following way:

$$\Delta\varphi(x) = \sum_{(i, j) \in U^*} r_{ij} \Delta z_{ij} + \sum_{k \in K} \sum_{(\tau, \rho)^k \in U_N^k} \left[ \tilde{\Delta}_{\tau\rho}^k \right]$$

$$\begin{aligned}
& - \sum_{p=1}^l r_p R_p(L_{t(\tau,\rho)}^k) - \sum_{(i,j) \in U^*} r_{ij} \delta_{\xi(i,j)}(L_{t(\tau,\rho)}^k) \Big] \Delta x_{\tau\rho}^k \\
& = \sum_{(i,j) \in U^*} \gamma_{ij} \Delta z_{ij} + \sum_{k \in K} \sum_{(\tau,\rho)^k \in U_N^k} \Delta_{\tau\rho}^k \Delta x_{\tau\rho}^k, \quad (22)
\end{aligned}$$

where  $\Delta z_{ij}$  is defined by formula (7),

$$\begin{aligned}
\Delta_{\tau\rho}^k = \tilde{\Delta}_{\tau\rho}^k - \sum_{p=1}^l r_p R_p(L_{t(\tau,\rho)}^k) - \sum_{(i,j) \in U^*} r_{ij} \delta_{\xi(i,j)}(L_{t(\tau,\rho)}^k), \\
(\tau,\rho)^k \in U_N^k, \quad k \in K, \quad \gamma_{ij} = r_{ij}. \quad (23)
\end{aligned}$$

### 3. Conditions of Optimality

**Definition 1.** A support plan  $\{x, U_S\}$  is called nonsingular if the following conditions are met:

$$\begin{aligned}
0 < x_{ij}^k < d_{ij}^k, \quad k \in K_S^1(i,j), \quad (i,j) \in U, \\
x_{ij}^k > 0, \quad k \in K_S^0(i,j), \quad (i,j) \in U_0, \\
x_{ij}^k > 0, \quad k \in K_S(i,j) \setminus K_S^1(i,j), \quad (i,j) \in U \setminus U_0, \\
0 < \sum_{k \in K_0(i,j)} x_{ij}^k < d_{ij}^0, \quad (i,j) \in U_0 \setminus U^*. \quad (24)
\end{aligned}$$

**Theorem 1.** Let  $\{x, U_S\}$  be a support plan. The following conditions are necessary for optimality of  $\{x, U_S\}$  and are also sufficient if  $\{x, U_S\}$  is nonsingular:

$$\begin{aligned}
x_{ij}^k &= 0 && \text{if } \Delta_{ij}^k > 0, \\
x_{ij}^k &= d_{ij}^k && \text{if } \Delta_{ij}^k < 0, \\
x_{ij}^k &\in [0, d_{ij}^k] && \text{if } \Delta_{ij}^k = 0, k \in K_N^1(i,j), (i,j) \in U; \quad (25)
\end{aligned}$$

$$\begin{aligned}
x_{ij}^k &= 0 && \text{if } \Delta_{ij}^k > 0, \\
x_{ij}^k &\geq 0 && \text{if } \Delta_{ij}^k = 0, k \in K_N^0(i,j), (i,j) \in U_0; \quad (26)
\end{aligned}$$

$$\begin{aligned}
x_{ij}^k &= 0 && \text{if } \Delta_{ij}^k > 0, \\
x_{ij}^k &\geq 0 && \text{if } \Delta_{ij}^k = 0, k \in K_N(i,j) \setminus K_N^1(i,j), (i,j) \in U \setminus U_0; \quad (27)
\end{aligned}$$

$$\begin{aligned}
 \sum_{k \in K_0(i,j)} x_{ij}^k &= 0 && \text{if } \gamma_{ij} > 0, \\
 \sum_{k \in \tilde{K}_0(i,j)} x_{ij}^k &= d_{ij}^0 && \text{if } \gamma_{ij} < 0, \\
 \sum_{k \in K_0(i,j)} x_{ij}^k &\in [0, d_{ij}^0] && \text{if } \gamma_{ij} = 0, (i, j) \in U^*;
 \end{aligned}
 \tag{28}$$

*Proof.* The proof is given in [4]. □

In the criterion of an optimality (25)–(28) we used the analytical formula for computing reduced costs  $\Delta_{\tau\rho}^k$ :

$$\Delta_{\tau\rho}^k = \tilde{\Delta}_{\tau\rho}^k - \sum_{p=1}^l r_p R_p(L_{t(\tau,\rho)}^k) - \sum_{(i,j) \in U^*} r_{ij} \delta_{\xi(i,j)}(L_{t(\tau,\rho)}^k),$$

$(\tau, \rho)^k \in U_N^k, \quad k \in K, \quad \gamma_{ij} = r_{ij}.$

For computing reduced costs  $\Delta_{\tau\rho}^k$  we can build the vector  $r = (r_p : p = \overline{1, l}; \gamma_{ij}, (i, j) \in U^*), u_i = (u_i^k, k \in K(i)), i \in I$  as a solution of the potential system [2, 4].

We compute the reduced costs  $\Delta_{ij}^k$  for the arcs  $(i, j)^k \in U_N^k, U_N^k = U^k \setminus U_S^k, k \in K$  and for the arcs  $(i, j)^k, k \in K_S^0(i, j), (i, j) \in U^*$  using the following formula:

$$\Delta_{ij}^k = c_{ij}^k - \left( u_i^k - u_j^k + \sum_{p=1}^l \lambda_{ij}^{kp} r_p \right).
 \tag{29}$$

One may check that the formulas (23) and (29) give the identical results for the problem (1)-(6). Strategy of application (23) or (29) are described in [1, 4, 6].

### References

- [1] Ravindra K. Ahuja, Thomas L. Magnanti, James B. Orlin, *Network Flows: Theory, Algorithms, and Applications*, New Jersey (1993).
- [2] R. Gabasov, F.M. Kirillova, O.I. Kostjukova, *Constructive Methods of Optimization. Part 3: Network Problems*, Minsk, BSU (1986), In Russian.
- [3] L.A. Pilipchuk, Y.V. Malakhouskaya, D.R. Kincaid, M. Lai, Algorithms of solving large sparse underdetermined linear systems with embedded network structure, *East-West J. of Mathematics*, **4**, No. 2 (2002), 191-202.

- [4] L.A. Pilipchuk, *Linear Inhomogeneous Problems of Networks Flows*, Minsk, BSU (2009), In Russian.
- [5] L.A. Pilipchuk, A.S. Pilipchuk, Y.H. Pesheva, Algorithms for construction of optimal and suboptimal solutions in network optimization problem, *IJPAM*, **54**, No. 2 (2009), 193-205.
- [6] L.A. Pilipchuk, I.V. Romanovski, Y.H. Pesheva, Inverse matrix updating in one inhomogeneous network flow programming problem, *Mathematica Balkanica*, New Series, **21**, No-s: 1-2 (2008), 329-338.
- [7] L.A. Pilipchuk, E.S. Vecharynski, Y.H. Pesheva, Solution of large linear systems with embedded network structure for a non-homogeneous network flow programming problem, *Mathematica Balkanica*, New Series, **22**, No-s: 3-4 (2008), 233-252.