International Journal of Pure and Applied Mathematics
Volume 60 No. 3 2010, 245-257

PRESERVATION OF THE RANK OF MATRICES FORCES THE LINEARITY

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Abstract: Operators preserving the rank of real matrices were studied in Basley [1] under assumption that the operator is linear. In the present paper the linearity of the operator is not assumed: we assume only that the operator is of the form $F = [f_{i,j}]$, where $f_{i,j} : \mathbb{R} \longrightarrow \mathbb{R}$ are functions for $i = 1, 2, \ldots, m$; $j = 1, 2, \ldots, n$. If the $\min\{m, n\} \geq 3$, then in the matrix space $M_{m,n}$ the operator preserving the rank of matrices must be linear as in Basley [1]. If the $\min\{m, n\} \leq 2$, then the operator may be nonlinear. In both cases the forms of the operator are presented.

AMS Subject Classification: 15A03
Key Words: rank of matrices, preservers

1. Introduction

Denote by $\mathbb{R}$ and $\mathbb{N}$ the set of real numbers and the set of positive integer numbers, respectively. Let $M_{m,n}$ be the set of $m \times n$ real matrices and let $M_n = M_{n,n}$, where $m, n \in \mathbb{N}$.

Let $m, n \in \mathbb{N}$ be fixed. First of all let us introduce

Definition 1. We say that an operator

$$F = [f_{i,j}], \quad \text{where functions } f_{i,j} : \mathbb{R} \longrightarrow \mathbb{R}, \tag{1}$$

for $i = 1, 2, \ldots, m$; $j = 1, 2, \ldots, n$; preserves the rank of matrices from $M_{m,n}$ if and only if for every matrix $A \in M_{m,n}$, the rank of the matrix A is equal to the

Received: February 11, 2010

rank of the matrix $F(A) := [f_{i,j}(a_{i,j})]$, where $i = 1, 2, \dots, m$; $j = 1, 2, \dots, n$.

In the paper we denote

$$c_{i,j} := f_{i,j}(1) \text{ for } i = 1, 2, \dots, m; \; j = 1, 2, \dots, n; \tag{2}$$

and

$$C := [c_{i,j}] \text{ for } i = 1, 2, \dots, m; \; j = 1, 2, \dots, n. \tag{3}$$

Let us prove the following lemmas.

Lemma 1. *Let an operator F of the form (1) preserves the rank of matrices from M_n for $n \in \mathbb{N}$. Then the equivalence*

$$x = 0 \iff f_{i,j}(x) = 0 \quad \text{for} \quad i, j = 1, 2, \dots, n \tag{4}$$

holds.

Proof. Let $B_1 \in M_n$ be the matrix with all entries equal to zero. Then the rank$(B_1) = 0$ and because F is an operator preserving the rank of matrices, we have the rank$(F(B_1)) = 0$. It follows that $f_{i,j}(0) = 0$ for $i, j = 1, 2, \dots, n$, i.e.

$$x = 0 \implies f_{i,j}(x) = 0 \quad \text{for} \quad i, j = 1, 2, \dots, n.$$

Now fix $r, s \in \{1, 2, \dots, n\}$. Consider a generalized permutation matrix B_2 with entries a_{i,j_i} such that $a_{i,j_i} = a \neq 0$ for $i = 1, 2, \dots, n$ and $j_r = s$. Then $\det(B_2) = \pm a^n \neq 0$ and the rank$(B_2) = n$. Then the rank$(F(B_2)) = n$ and hence $\det(F(B_2)) \neq 0$. From properties of determinants $\det(F(B_2)) = \pm \prod_{i=1}^{n} f_{i,j_i}(a) \neq 0$. From the above it follows that

$$a \neq 0 \implies f_{i,j}(a) \neq 0 \quad \text{for} \quad i, j = 1, 2, \dots, n.$$

Since the indices r, s were chosen arbitrary it is equivalent to the implication

$$f_{i,j}(x) = 0 \implies x = 0 \quad \text{for} \quad i, j = 1, 2, \dots, n,$$

which concludes the proof. □

Lemma 2. *If a matrix $A \in M_{m,n}$ has the non-zero entries $a_{i,j}$ for $i = 1, 2, \dots m$; $j = 1, 2, \dots, n$ and the* rank$(A) = 1$, *then there exist non-zero real numbers $u_1, u_2, \dots, u_m$ and $v_1, v_2, \dots, v_n$ such that for all entries $a_{i,j}$ of the matrix A the equalities*

$$a_{i,j} = u_i \cdot v_j \text{ for } i = 1, 2, \dots, m; \; j = 1, 2, \dots, n \tag{5}$$

are satisfied.

Proof. Let $A \in M_{m,n}$ be a matrix such that the rank$(A) = 1$ with entries $a_{i,j} \neq 0$ for $i = 1, 2, \dots, m$; $j = 1, 2, \dots, n$.

We define $u_i = \frac{a_{i,1}}{a_{1,1}}$ for $i = 1, 2, \dots, m$; then i-th row is the product of the number u_i and the first row. Dividing first, second, ... , m-th row by $u_1, u_2, \dots, u_m$, respectively, we obtain the matrix $A_1 \in M_{m,n}$ with all the identical rows. Next we define $v_j = a_{1,j}$ for $j = 1, 2, \ldots, n$. From the presented construction $a_{i,j} = u_i \cdot a_{1,j} = u_i \cdot v_j$ and condition (5) is fulfilled. □

Lemma 3. *Let an operator F of the form (1) preserves the rank of matrices from $M_{m,n}$ for $m, n \in \mathbb{N}$. Then*

$$\operatorname{rank}(C) = 1, \tag{6}$$

where C is defined by (3) and (2).

Proof. Let $B_3 \in M_n$ be the matrix with all entries equal to one. Then the $\operatorname{rank}(B_3) = 1$ and $F(B_3) = C$. Since F is an operator preserving the rank of matrices, then the $\operatorname{rank}(F(B_3)) = \operatorname{rank}(C) = 1$, which was to be shown. □

We prove the following

Theorem 1. *An operator F of the form (1) preserves the rank of matrices from M_2 if and only if there exist constants $c_{i,j} \neq 0$ for $i, j = 1, 2$ such that*

$$f_{i,j}(x) = c_{i,j} \cdot g(x), \quad x \in \mathbb{R}, \tag{7}$$

where coefficients $c_{i,j}$ for $i, j = 1, 2$ are defined by (2) and for the matrix C defined by (3) the equality (6) holds, and the function g is an injective solution of the multiplicative Cauchy functional equation

$$g(x \cdot y) = g(x) \cdot g(y) \quad \text{for} \quad x, y \in \mathbb{R}, \tag{8}$$

such that

$$g(x) = 0 \quad \Longleftrightarrow \quad x = 0. \tag{9}$$

Proof. Let F be an operator preserving the rank of matrices from M_2. Let us observe that by Lemma 1 it follows that constants $c_{i,j} \neq 0$ for $i, j = 1, 2$ defined by (2) are non-zero. By Lemma 3 equality (6) holds.

Consider the matrix

$$B_4 = \begin{bmatrix} 1 & a \\ b & a \cdot b \end{bmatrix}$$

for arbitrary $a, b \in \mathbb{R}$. Then the $\operatorname{rank}(F(B_4)) = 1$ and $\det(F(B_4)) = 0$. In other words we get

$$c_{1,1} \cdot f_{2,2}(a \cdot b) = f_{1,2}(a) \cdot f_{2,1}(b) \quad \text{for} \quad a, b \in \mathbb{R}. \tag{10}$$

From (10), for $a = 1$ and $b = x$ we have

$$f_{2,1}(x) = \frac{c_{1,1}}{c_{1,2}} \cdot f_{2,2}(x) \quad \text{for} \quad x \in \mathbb{R}. \tag{11}$$

Again from (10), for $a = x$ and $b = 1$ we have

$$f_{1,2}(x) = \frac{c_{1,1}}{c_{2,1}} \cdot f_{2,2}(x) \quad \text{for} \quad x \in \mathbb{R}; \tag{12}$$

and from (10) for $a = 1$ and $b = 1$ it follows that

$$c_{1,1} \cdot c_{2,2} = c_{1,2} \cdot c_{2,1}. \tag{13}$$

Substituting (11) and (12) to (10) with $a = x$, $b = y$ we obtain

$$c_{1,1} \cdot f_{2,2}(x \cdot y) = \frac{c_{1,1}}{c_{2,1}} \cdot f_{2,2}(x) \cdot \frac{c_{1,1}}{c_{1,2}} \cdot f_{2,2}(y) \quad \text{for} \quad x, y \in \mathbb{R}.$$

Dividing both sides of above equation by $c_{1,1} \cdot c_{2,2} \neq 0$ and using (13) we obtain

$$\frac{f_{2,2}(x \cdot y)}{c_{2,2}} = \frac{f_{2,2}(x)}{c_{2,2}} \cdot \frac{f_{2,2}(y)}{c_{2,2}} \quad \text{for} \quad x, y \in \mathbb{R}. \tag{14}$$

Define a function $g : \mathbb{R} \longrightarrow \mathbb{R}$ by

$$g(x) = \frac{f_{2,2}(x)}{c_{2,2}} \quad \text{for} \quad x \in \mathbb{R}. \tag{15}$$

By (14), g satisfies the multiplicative Cauchy functional equation (8). From definition (15) we have

$$f_{2,2}(x) = c_{2,2} \cdot g(x) \quad \text{for} \quad x \in \mathbb{R}.$$

Using (11) and (13) we receive

$$f_{2,1}(x) = \frac{c_{1,1}}{c_{1,2}} \cdot f_{2,2}(x) = c_{2,1} \cdot g(x) \quad \text{for} \quad x \in \mathbb{R}, \tag{16}$$

and using (12) and relation (13) we obtain

$$f_{1,2}(x) = \frac{c_{1,1}}{c_{2,1}} \cdot f_{2,2}(x) = c_{1,2} \cdot g(x) \quad \text{for} \quad x \in \mathbb{R}.$$

Now consider the matrix

$$B_5 = \begin{bmatrix} x & 1 \\ x & 1 \end{bmatrix}$$

for arbitrary $x \in \mathbb{R}$. Then the $\operatorname{rank}(F(B_5)) = 1$ and $\det(F(B_5)) = 0$. Hence

$$f_{1,1}(x) \cdot c_{2,2} = c_{1,2} \cdot f_{2,1}(x) \quad \text{for} \quad x \in \mathbb{R}. \tag{17}$$

From (16) and (17), using relation (13) we obtain

$$f_{1,1}(x) = \frac{c_{1,2}}{c_{2,2}} \cdot f_{2,1}(x) = c_{1,1} \cdot g(x) \quad \text{for} \quad x \in \mathbb{R}.$$

Thus functions $f_{i,j}$ for $i, j = 1, 2$ are of the form (7). To show injectivity of g, consider the matrix

$$B_6 = \begin{bmatrix} 1 & x \\ 1 & y \end{bmatrix}$$

for arbitrary $x, y \in \mathbb{R}$ with $x \neq 0$, $y \neq 0$ and $x \neq y$. Then the $\operatorname{rank}(B_6) = \operatorname{rank}(F(B_6)) = 2$ and $\det(F(B_6)) \neq 0$. Therefore we have

$$f_{1,2}(x) \cdot c_{2,1} \neq c_{1,1} \cdot f_{2,2}(y)$$

and in view of (7) we obtain

$$c_{1,2} \cdot g(x) \cdot c_{2,1} \cdot g(1) \neq c_{1,1} \cdot g(1) \cdot c_{2,2} \cdot g(y).$$

Using equality (13) and dividing both sides by $c_{1,2} \cdot c_{2,1} \cdot g(1) \neq 0$ we obtain

$$g(x) \neq g(y) \quad \text{for} \quad x \neq y.$$

Moreover, by (7) and Lemma 1 we obtain (9).

Now let F be of the form (1), with functions $f_{i,j}$ for $i, j = 1, 2$, satisfying with an injective solution g of the multiplicative Cauchy functional equation (8). Assume that (7) holds and the coefficients $c_{i,j} \neq 0$ for $i, j = 1, 2$ (cf. (2)), while C satisfies equality (6). We prove that F is an operator preserving the rank of matrices from M_2.

Let $D \in M_2$ is a real matrix with entries $d_{1,1}, d_{1,2}, d_{2,1}, d_{2,2}$.

If the rank$(D) = 0$, then the rank$(F(D)) = 0$ follows from (9).

Assume that the rank$(D) = 1$ and consider the determinant

$$\begin{aligned}\det(F(D)) &= f_{1,1}(d_{1,1}) \cdot f_{2,2}(d_{2,2}) - f_{1,2}(d_{1,2}) \cdot f_{2,1}(d_{2,1}) \\ &= c_{1,1} \cdot g(d_{1,1}) \cdot c_{2,2} \cdot g(d_{2,2}) - c_{1,2} \cdot g(d_{1,2}) \cdot c_{2,1} \cdot g(d_{2,1}).\end{aligned}$$

Using Lemma 2 for $A = C$ we obtain

$$\begin{aligned}\det(F(D)) &= u_1 \cdot v_1 \cdot g(d_{1,1}) \cdot u_2 \cdot v_2 \cdot g(d_{2,2}) - u_1 \cdot v_2 \cdot g(d_{1,2}) \cdot u_2 \cdot v_1 \cdot g(d_{2,1}) \\ &= u_1 \cdot u_2 \cdot v_1 \cdot v_2 \cdot g(d_{1,1}) \cdot g(d_{2,2}) - u_1 \cdot u_2 \cdot v_1 \cdot v_2 \cdot g(d_{1,2}) \cdot g(d_{2,1}) \\ &= u_1 \cdot u_2 \cdot v_1 \cdot v_2 \cdot (g(d_{1,1}) \cdot g(d_{2,2}) - g(d_{1,2}) \cdot g(d_{2,1})).\end{aligned}$$

Because g is a solution of the multiplicative Cauchy functional equation (8), then

$$\det(F(D)) = u_1 \cdot u_2 \cdot v_1 \cdot v_2 \cdot (g(d_{1,1} \cdot d_{2,2}) - g(d_{1,2} \cdot d_{2,1})).$$

Let us note that $c_{i,j} \neq 0$, $i, j = 1, 2, \ldots, n$ implies that $u_1 \cdot u_2 \cdot v_1 \cdot v_2 \neq 0$. Thus from the above equality and injectivity the function g an operator F preserves the rank equal to 1 or 2 of the matrix D.

This completes the proof of the Theorem 1. □

Remark 1. (M. Sablik, personal communication) Any invertible solution of (8) satisfies (9).

Proof. Setting $x = y = 1$ in (8) we get $g(1) = g(1)^2$ whence $g(1) \in \{0, 1\}$. Suppose that $g(1) = 0$, then for every $x \in K$ we have $g(x) = g(x) \cdot g(1) = 0$ whence it follows that $g = 0$ which contradicts invertibility of g. Now, setting $x = y = 0$ in (8) we obtain $g(0) = 0$ in view of the above observation and invertibility of g. By invertibility again, 0 is the only point at which g vanishes. This means that (9) holds. □

Next simple numerical examples illustrate the role of some assumptions in Theorem 1.

Example 1. An operator F defined by formulae $f_{1,1}(x) = x^3$, $f_{1,2}(x) = 2x^3$, $f_{2,1}(x) = 3x^3$, $f_{2,2}(x) = 6x^3$ for $x \in \mathbb{R}$ fulfills assumptions of Theorem 1. It is a nonlinear operator preserving the rank of matrices from M_2. □

Without the assumption that g is injective, the statement of Theorem 1 is no more true.

Example 2. Let $g(x) = x^2$, $x \in \mathbb{R}$. Then g is a non-injective solution of the multiplicative functional Cauchy equation (8). Define F by $f_{1,1}(x) = f_{1,2}(x) = f_{2,1}(x) = f_{2,2}(x) = x^2$. Then the matrix C satisfies (6). Consider the matrices

$$B_7 = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \quad \text{and} \quad F(B_7) = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}.$$

Observe, that the rank$(B_7) = 2$, while the rank$(F(B_7)) = 1$. The operator F does not preserve the rank of matrices from M_2. □

Let us observe that the result obtained in Theorem 1 for $n = 2$ is not true for $n = 3$. Consider the following

Example 3. Let $g(x) = x^3$, $x \in \mathbb{R}$. The function g is an injective solution of the multiplicative Cauchy functional equation (8). Let $f_{1,1}(x) = f_{1,2}(x) = f_{1,3}(x) = x^3$, $f_{2,1}(x) = f_{2,2}(x) = f_{2,3}(x) = 2x^3$, $f_{3,1}(x) = f_{3,2}(x) = f_{3,3}(x) = x^3$. Then C fulfills (6). Consider the matrices

$$B_8 = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 2 & 1 & 1 \end{bmatrix} \quad \text{and} \quad F(B_8) = \begin{bmatrix} 1 & 1 & 0 \\ 2 & 0 & 2 \\ 8 & 1 & 1 \end{bmatrix}.$$

Observe that the rank$(B_8) = 2$, while the rank$(F(B_8)) = 3$. The operator F does not preserve the rank of matrices from M_3. □

For $n = 3$ we prove

Theorem 2. *An operator F of the form (1) preserves the rank of matrices from M_3 if and only if there exists a matrix C with entries $c_{i,j} \neq 0$ for $i = 1, 2, 3$, fulfilling (6) and such that*

$$f_{i,j}(x) = c_{i,j} \cdot x, \quad x \in \mathbb{R} \tag{18}$$

holds.

Proof. Let F be an operator preserving the rank of matrices from M_3. Observe that by Lemma 1 it follows that constants $c_{i,j} \neq 0$ for $i, j = 1, 2, 3$ defined by (2) are non-zero. By Lemma 3 the equality (6) holds.

Let us consider the matrix

$$B_9 = \begin{bmatrix} x & 1 & 0 \\ y & 0 & 1 \\ x+y & 1 & 1 \end{bmatrix}$$

for $x, y \in \mathbb{R}$. Because the rank$(B_9) = 2$, we have the rank$(F(B_9)) = 2$ and hence $\det(F(B_9)) = f_{3,1}(x+y) \cdot c_{1,2} \cdot c_{2,3} - f_{1,1}(x) \cdot c_{2,3} \cdot c_{3,2} - f_{2,1}(y) \cdot c_{1,2} \cdot c_{3,3} = 0$. In other words

$$f_{3,1}(x+y) \cdot c_{1,2} \cdot c_{2,3} = f_{1,1}(x) \cdot c_{2,3} \cdot c_{3,2} + f_{2,1}(y) \cdot c_{1,2} \cdot c_{3,3} \tag{19}$$

is fulfilled for arbitrary $x, y \in \mathbb{R}$.

Consider the matrix

$$B_{10} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}.$$

Since the rank$(B_{10}) = 2$, we have the rank$(F(B_{10})) = 2$ and $\det(F(B_{10})) = -c_{2,3} \cdot (c_{1,1} \cdot c_{3,2} - c_{1,2} \cdot c_{3,1}) = 0$. It follows that

$$c_{3,1} \cdot c_{1,2} \cdot c_{2,3} = c_{1,1} \cdot c_{2,3} \cdot c_{3,2} \tag{20}$$

is true.

For the matrix

$$B_{11} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$

we have the rank$(B_{11}) = 2$, whence

$$\det(F(B_{11})) = -c_{1,2} \cdot (c_{2,1} \cdot c_{3,3} - c_{2,3} \cdot c_{3,1}) = 0.$$

We obtain therefore

$$c_{3,1} \cdot c_{1,2} \cdot c_{2,3} = c_{2,1} \cdot c_{1,2} \cdot c_{3,3}. \tag{21}$$

Divide both sides of equation (19) by $c_{3,1} \cdot c_{1,2} \cdot c_{2,3} \neq 0$ to obtain

$$\frac{f_{3,1}(x+y) \cdot c_{1,2} \cdot c_{2,3}}{c_{3,1} \cdot c_{1,2} \cdot c_{2,3}} = \frac{f_{1,1}(x) \cdot c_{2,3} \cdot c_{3,2}}{c_{3,1} \cdot c_{1,2} \cdot c_{2,3}} + \frac{f_{2,1}(y) \cdot c_{1,2} \cdot c_{3,3}}{c_{3,1} \cdot c_{1,2} \cdot c_{2,3}}. \tag{22}$$

Substituting (20) and (21) to (22) we derive

$$\frac{f_{3,1}(x+y)}{c_{3,1}} = \frac{f_{1,1}(x)}{c_{1,1}} + \frac{f_{2,1}(y)}{c_{2,1}} \quad \text{for } x, y \in \mathbb{R}. \tag{23}$$

Now define new functions $h_{i,1}$, $i = 1, 2, 3$, by

$$h_{i,1}(x) = \frac{f_{i,1}(x)}{c_{i,1}} \quad \text{for} \quad x \in \mathbb{R}. \tag{24}$$

The functions $h_{1,1}, h_{2,1}, h_{3,1}$ satisfy in view of (23) the additive Pexider type functional equation

$$h_{3,1}(x+y) = h_{1,1}(x) + h_{2,1}(y) \quad \text{for } x, y \in \mathbb{R}. \tag{25}$$

From Kuczma [2] (Theorem 1, p. 317), it follows that the solution of (25) is of the form

$$h_{3,1}(x) = h_1(x) + \alpha_{1,1} + \alpha_{2,1}, \quad x \in \mathbb{R},$$
$$h_{1,1}(x) = h_1(x) + \alpha_{1,1}, \quad x \in \mathbb{R},$$
$$h_{2,1}(x) = h_1(x) + \alpha_{2,1}, \quad x \in \mathbb{R},$$

where h_1 is an additive function on $\mathbb{R}$ and $\alpha_{1,1}, \alpha_{2,1} \in \mathbb{R}$ are constants. Because (4) is fulfilled, then for $x = 0$ we have $h_{3,1}(0) = h_{2,1}(0) = h_{1,1}(0) = 0$. Since h_1 is additive we have $h_1(0) = 0$, hence $\alpha_{1,1} = \alpha_{2,1} = 0$. From the definition (24) of functions $h_{i,1}$ we obtain $h_{3,1}(1) = h_{2,1}(1) = h_{1,1}(1) = 1$ and $h_1(1) = 1$. Summarizing, we get

$$h_{1,1}(x) = h_{2,1}(x) = h_{3,1}(x) = h_1(x) \quad \text{for } x \in \mathbb{R}. \tag{26}$$

Considering the matrices

$$B_{12} = \begin{bmatrix} 1 & x & 0 \\ 0 & y & 1 \\ 1 & x+y & 1 \end{bmatrix} \quad \text{and} \quad B_{13} = \begin{bmatrix} 1 & 0 & x \\ 0 & 1 & y \\ 1 & 1 & x+y \end{bmatrix}$$

for arbitrary $x, y \in \mathbb{R}$ and using definitions

$$h_{i,2}(x) = \frac{f_{i,2}(x)}{c_{i,2}} \quad \text{and} \quad h_{i,3}(x) = \frac{f_{i,3}(x)}{c_{i,3}} \quad \text{for} \quad x \in \mathbb{R}$$

for $i = 1, 2, 3$ similarly as above we get by Theorem 1 from Kuczma [2] p. 317

$$h_{1,2}(x) = h_{2,2}(x) = h_{3,2}(x) = h_2(x) \quad \text{for } x \in \mathbb{R} \tag{27}$$

and

$$h_{1,3}(x) = h_{2,3}(x) = h_{3,3}(x) = h_3(x) \quad \text{for } x \in \mathbb{R}, \tag{28}$$

where h_2 and h_3 are additive mappings from $\mathbb{R}$ to $\mathbb{R}$.

The matrix

$$B_{14} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ x & y & x+y \end{bmatrix}$$

has the rank equal to 2. Using (26)-(28) we have $h_i(0) = 0$, $h_i(1) = 1$, $i = 1, 2, 3$. It follows that

$$\det(F(B_{14})) = c_{1,1} \cdot c_{2,2} \cdot c_{3,3} \cdot h_3(x+y) - c_{1,3} \cdot c_{2,2} \cdot c_{3,1} \cdot h_1(x) - c_{1,1} \cdot c_{2,3} \cdot c_{3,2} \cdot h_2(y) = 0. \tag{29}$$

Consider the matrix

$$B_{15} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}.$$

Since the rank$(B_{15}) = 2$, we have the rank$(F(B_{15})) = 2$ and $\det(F(B_{15})) = c_{2,2} \cdot (c_{1,1} \cdot c_{3,3} - c_{1,3} \cdot c_{3,1}) = 0$. It follows that

$$c_{1,1} \cdot c_{2,2} \cdot c_{3,3} = c_{1,3} \cdot c_{2,2} \cdot c_{3,1} \tag{30}$$

is true.

For the matrix

$$B_{16} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

we have the rank$(B_{16}) = 2$, whence

$$\det(F(B_{16})) = c_{1,1} \cdot (c_{2,2} \cdot c_{3,3} - c_{2,3} \cdot c_{3,2}) = 0.$$

We obtain therefore

$$c_{1,1} \cdot c_{2,2} \cdot c_{3,3} = c_{1,1} \cdot c_{2,3} \cdot c_{3,2}. \tag{31}$$

Substituting (30) and (31) to (29) and dividing both sides by $c_{1,1} \cdot c_{2,2} \cdot c_{3,3} \neq 0$ we obtain that an additive Pexider type functional equation

$$h_3(x + y) = h_1(x) + h_2(y) \quad \text{for } x, y \in \mathbb{R}, \tag{32}$$

for unknown functions h_i, $i = 1, 2, 3$, is satisfied.

In virtue Kuczma [2], p. 317 functions

$$h_1(x) = h(x) + \alpha_1, \quad h_2(x) = h(x) + \alpha_2, \quad h_3(x) = h(x) + \alpha_1 + \alpha_2,$$

are solution of the equation, where h is an additive function with constants $\alpha_1, \alpha_2 \in \mathbb{R}$. For h it follows $h(0) = 0$ and from (4) we obtain $\alpha_1 = \alpha_2 = 0$. Then $h_1 = h_2 = h_3 = h$.

Let as consider the matrix

$$B_{17} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Because the rank$(B_{17}) = 2$, then the rank$(F(B_{17})) = 2$. Using $h(1) = 1$ we have $\det(F(B_{17})) = c_{3,3} \cdot (c_{1,1} \cdot c_{2,2} - c_{1,2} \cdot c_{2,1}) = 0$. Then equation

$$c_{1,1} \cdot c_{2,2} = c_{1,2} \cdot c_{2,1} \tag{33}$$

is true.

For any square matrix

$$B_{18} = \begin{bmatrix} 1 & x & 0 \\ y & x \cdot y & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

where $x, y \in \mathbb{R}$, $x \neq 0$, $y \neq 0$ we obtain the $\operatorname{rank}(B_{18}) = 2$. Then also the $\operatorname{rank}(F(B_{18})) = 2$. Thus,

$$\det(F(B_{18})) = c_{3,3} \cdot (c_{1,1} \cdot c_{2,2} \cdot h_2(x \cdot y) - c_{1,2} \cdot h_2(x) \cdot c_{2,1} \cdot h_1(y)) = 0.$$

From the above and $c_{3,3} \neq 0$ we obtain

$$c_{1,1} \cdot c_{2,2} \cdot h_2(x \cdot y) = c_{1,2} \cdot h_2(x) \cdot c_{2,1} \cdot h_1(y).$$

Using $h_1 = h_2 = h$ and (33), we see that the function h solves the multiplicative Cauchy functional equation

$$h(x \cdot y) = h(x) \cdot h(y) \quad \text{for} \quad \text{all} \quad x, y \in \mathbb{R}.$$

Now, from Kuczma [2] (Theorem 1, p. 356), it follows that the only functions satisfying simultaneously additive and multiplicative Cauchy functional equation are $h = 0$ or $h = \mathrm{id}$, where id denotes the identity function on $\mathbb{R}$. Since $h(1) = 1$, we see that in our case $h(x) = x$ for $x \in \mathbb{R}$.

By definitions (26)-(28) we obtain for $i, j = 1, 2, 3$ that

$$f_{i,j}(x) = f_{i,j}(1) \cdot h_i(x) = c_{i,j} \cdot h(x) = c_{i,j} \cdot x \quad \text{for all} \quad x \in \mathbb{R}.$$

We prove that h is the injective function. Consider the matrix

$$B_{19} = \begin{bmatrix} 1 & x & 0 \\ 1 & y & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Because the $\operatorname{rank}(B_{19}) = 3$, then the $\operatorname{rank}(F(B_{19})) = 3$ and $\det(F(B_{19})) \neq 0$. Using $h(1) = 1$ we have

$$\det(F(B_{19})) = c_{3,3} \cdot (c_{1,1} \cdot c_{2,2} \cdot h(y) - c_{2,1} \cdot c_{1,2} \cdot h(x)) \neq 0.$$

Because $c_{3,3} \neq 0$ we obtain

$$c_{1,1} \cdot c_{2,2} \cdot h(y) \neq c_{2,1} \cdot c_{1,2} \cdot h(x).$$

The relation (33) follows

$$h(x) \neq h(y),$$

i.e. h is the injective function on $\mathbb{R}$.

Let us verify now that F is an operator preserving the rank of matrices. Let $D \in M_3$ is a real matrix with entries $d_{i,j}$ for $i, j = 1, 2, 3$.

Because the function $h = \mathrm{id}$ is an injective solution of multiplicative Cauchy functional equation fulfilling (4), then by Theorem 1 the rank of matrices D is preserved if it belongs to $\{0, 1, 2\}$.

By definition of the determinant we have

$$\det(F(D)) = \sum_{p\in S_3} (-1)^{I_p} \cdot \prod_{i=1}^{3} f_{i,j_i}(d_{i,j_i}),$$

where the sum runs over the set S_3 all permutations $p = (i_1, i_2, i_3)$ of the integers $1, 2, 3$ and I_p denotes the number of inverses of the permutation p.

It follows (cf. (18)) that

$$\det(F(D)) = \sum_{p\in S_3} (-1)^{I_p} \cdot \prod_{i=1}^{3} (c_{i,j_i} \cdot d_{i,j_i}).$$

Using Lemma 2 we infer that

$$\det(F(D)) = \sum_{p\in S_3} (-1)^{I_p} \cdot \prod_{i=1}^{3} (u_i \cdot v_{j_i} \cdot d_{i,j_i}) = \sum_{p\in S_3} (-1)^{I_p} \cdot \prod_{i=1}^{3} u_i \cdot \prod_{i=1}^{3} v_i \cdot \prod_{i=1}^{3} d_{i,j_i}$$
$$= \prod_{i=1}^{3} u_i \cdot \prod_{i=1}^{3} v_i \cdot \sum_{p\in S_3} (-1)^{I_p} \cdot \prod_{i=1}^{3} d_{i,j_i} = \prod_{i=1}^{3} u_i \cdot \prod_{i=1}^{3} v_i \cdot \det(D).$$

Because $\prod_{i=1}^{3} u_i \cdot \prod_{i=1}^{3} v_i \neq 0$, the operator F preserves also the rank equal to 3 of nonsingular matrices D.

This completes the proof of the Theorem 2. □

Now, let us prove a theorem which describes all operators of the form (1) preserving the rank of real matrices from M_n, $n \geq 3$.

Theorem 3. *An operator F of the form (1) preserves the rank of matrices from M_n, $n \geq 3$ if and only if there exists the matrix C with entries $c_{i,j} \neq 0$ and fulfilling (6), such that (18) for $i = 1, 2, \ldots, m$, $j = 1, 2, \ldots, n$, holds.*

Proof. Assume that an operator F preserves the rank of a matrix D from M_n, where $n \geq 3$. Observe that by Lemma 1 it follows that constants $c_{i,j} \neq 0$ for $i, j = 1, 2, \ldots, n$ defined by (2) are non-zero. By Lemma 3 equality (6) holds.

For $n = 3$ the assertion has been proved in the Theorem 2.

For $n > 3$ we have

$$\det(F(D)) = \sum_{p\in S_n} (-1)^{I_p} \cdot \prod_{i=1}^{n} f_{i,j_i}(d_{i,j_i}),$$

where the sum runs over the set S_n all permutations $p = (i_1, i_2, \ldots, i_n)$ of the integers $1, 2, \ldots, n$ and I_p denote the number of inverses of the permutation p.

From definitions of functions $f_{i,j}$ for $i, j = 1, 2, \dots, n$ we have

$$\det(F(D)) = \sum_{p \in S_n} (-1)^{I_p} \cdot \prod_{i=1}^{n} (c_{i,j_i} \cdot d_{i,j_i}).$$

Using Lemma 2

$$\det(F(D)) = \sum_{p \in S_n} (-1)^{I_p} \cdot \prod_{i=1}^{n} (u_i \cdot v_{i_j} \cdot d_{i,j_i}) = \sum_{p \in S_n} (-1)^{I_p} \cdot \prod_{i=1}^{n} u_i \cdot \prod_{i=1}^{n} v_i \cdot \prod_{i=1}^{n} d_{i,j_i}$$
$$= \prod_{i=1}^{n} u_i \cdot \prod_{i=1}^{n} v_i \cdot \sum_{p \in S_n} (-1)^{I_p} \cdot \prod_{i=1}^{n} d_{i,j_i} = \prod_{i=1}^{n} u_i \cdot \prod_{i=1}^{n} v_i \cdot \det(D).$$

Because $\prod_{i=1}^{n} u_i \cdot \prod_{i=1}^{n} v_i \neq 0$, the operator F preserves the rank equal to n of matrices D.

This completes the proof of the Theorem 3. □

For matrices from $M_{m,n}$ in view of Lemma 1 and Theorems 1 and 3 we obtain the following.

Theorem 4. *If* $\min\{m, n\} = 1$, *then* F *of the form (1) is an operator preserving the rank of matrices from* $M_{m,n}$ *if and only if the equivalence (4) for all* i, j *is fulfilled.*

Theorem 5. *If* $\min\{m, n\} = 2$, *then* F *of the form (1) is an operator preserving the rank of matrices from* $M_{m,n}$ *if and only if there exists a matrix* C *with entries* $c_{i,j} \neq 0$ *defined by (2) and fulfilling (6), such that (7) holds for* $i = 1, 2, \dots, m$, $j = 1, 2, \dots, n$, *where* g *is an injective solution of the multiplicative Cauchy functional equation (8) fulfilling the equivalence (9).*

Theorem 6. *If* $\min\{m, n\} \geq 3$, *then operator* F *of the form (1) is an operator preserving the rank of matrices from* $M_{m,n}$ *if and only if there exists a matrix* C *with entries* $c_{i,j} \neq 0$ *defined by (2) and fulfilling (6), such that (18) holds for* $i = 1, 2, \dots, m$, $j = 1, 2, \dots, n$.

In the Li and Pierce [3] a similar problem and its result is presented. Let $\Phi : M_{m,n} \longrightarrow M_{m,n}$ be a linear operator. In virtue of [1, Theorem 3.1] we obtain that the $\operatorname{rank}(\Phi(A)) = \operatorname{rank}(A)$ for all $A \in M_{m,n}$ if and only if there exist invertible matrices $M \in M_m$ and $N \in M_n$ such that $\Phi(A) = MAN$ or, if $m = n$, $\Phi(A) = MA^tN$, where A^t denote transposition of the matrix A.

Let us observe that in case $\min\{m, n\} \geq 3$ the obtained result for operators of the form (1) in the Theorem 6 is the same. In this case $M \in M_m$, $N \in M_n$ are diagonal matrices and $M = \operatorname{diag}(u_1, u_2, \dots, u_m)$, $N = \operatorname{diag}(v_1, v_2, \dots, v_n)$, where entries $u_i \neq 0$, $i = 1, 2, \dots, m$ and $v_j \neq 0$, $j = 1, 2, \dots, n$ are as in Lemma 2. Also in the case $m = n$ by Theorem 3 there exist invertible matrices

$M, N \in M_n$ as in Li and Pierce [3].

Acknowledgments

I would like to express my thanks to the participants of the seminar on functional equations and inequalities conducted by professor Roman Ger for their valuable suggestions and remarks.

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