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ON CERTAIN CLASS OF SZÁSZ-MIRAKYAN OPERATORS IN EXPONENTIAL WEIGHT SPACES

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Abstract: Applying the Borel methods of summability of sequences, we introduce the class of Szász-Mirakyan operators acting from the exponential weight space C_q to C_q and we give approximation theorems for them.

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1. Introduction

1.1. The approximation properties of the Szász-Mirakyan operators

$$S_n(f;x) := e^{-nx}\sum_{k=0}^{\infty}\frac{(nx)^k}{k!}f\left(\frac{k}{n}\right), \qquad x\in\mathbb{R}_0, n\in\mathbb{N}, \tag{1}$$

$(\mathbb{R}_0 = [0,\infty), \mathbb{N} = \{1,2,\ldots\})$ and their various modifications were examined in many papers and monographs (e.g. [1]-[5], [8]-[10], [12], [13]).

In the paper [2] the operators S_n were investigated in the exponential weight space C_q, $q = \text{const.} > 0$, with the weight function $v_q(x) = e^{-qx}$, $x \in \mathbb{R}_0$. There C_q is the set of all functions $f : \mathbb{R}_0 \to \mathbb{R}$ for which $v_q f$ is uniformly continuous and bounded on $\mathbb{R}_0$ and the norm is defined by

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$$\|f\|_q \equiv \|f(\cdot)\|_q := \sup_{x\in\mathbb{R}_0} v_q(x)|f(x)|. \tag{2}$$

In [2] it was proved that S_n is a positive linear operator acting from the space C_q to C_r provided that $r > q > 0$ and $n > n_0$, where $n_0 > q/\ln(r/q)$ is a fixed integer.

1.2. In the paper [8] the following modified Szász-Mirakyan operators:

$$S_n(f;q;x) := e^{-nx}\sum_{k=0}^{\infty}\frac{(nx)^k}{k!}f\left(\frac{k}{n+q}\right), \quad x\in\mathbb{R}_0, n\in\mathbb{N}, \tag{3}$$

were introduced for $f \in C_q$. In [8] was proved that $S_n(q)$ is a positive linear operator acting from the space C_q to C_q and

$$\|S_n(f;q)\|_q \le \|f\|_q \quad \text{for} \quad f\in C_q, n\in\mathbb{N}, \tag{4}$$

and approximation theorems were given for them.

1.3. In this paper (Section 2) we generalize the formulas (1) and (3) for S_n, applying the Borel methods of summability of sequences. We introduce the class of Szász-Mirakyan operators in the exponential weight space C_q and we examine approximation properties these operators.

The sequence $(a_n)_0^\infty$ of numbers $a_n \in \mathbb{R}$ is summable to g by the Borel method B_r, $r \in \mathbb{N}$, ([6] and [7]) if the series $\sum_{k=0}^{\infty}\frac{x^{rk}}{(rk)!}a_k$ is convergent on $\mathbb{R}_0$ and

$$\lim_{x\to\infty} re^{-x}\sum_{k=0}^{\infty}\frac{x^{rk}}{(rk)!}a_k = g.$$

In the paper [10] was defined the class of Szász-Mirakyan operators $S_{n;r}$.

$$S_{n;r}(f;x) := \frac{1}{A_r(nx)}\sum_{k=0}^{\infty}\frac{(nx)^{rk}}{(rk)!}f\left(\frac{rk}{n}\right), \qquad x\in\mathbb{R}_0, n\in\mathbb{N}, r\in\mathbb{N}, \tag{5}$$

$$A_r(t) := \sum_{k=0}^{\infty}\frac{t^{rk}}{(rk)!}, \quad t\in\mathbb{R}_0, \tag{6}$$

for functions f belonging to the polynomial weight spaces. By (1), (5) and (6) it is obvious that $S_{n;1}(f) \equiv S_n(f)$ for $n \in \mathbb{N}$.

Now we shall consider certain modification of the operators $S_{n;r}$ in exponential weight spaces C_q.

2. Definition and Auxiliary Results

2.1. Let $r \in \mathbb{N}$ and $q > 0$ be fixed numbers. For $f \in C_q$ we define the opertaors:

$$S_{n;r,q}(f;x) := \frac{1}{A_r(nx)} \sum_{k=0}^{\infty} \frac{(nx)^{rk}}{(rk)!} f\left(\frac{rk}{n+q}\right), \qquad x \in \mathbb{R}_0, n \in \mathbb{N}, \tag{7}$$

where A_r is the function given by (6). We shall denote by Ω_q, $q > 0$, the class of all operators $S_{n;r,q}$ with $n, r \in \mathbb{N}$.

By (7) and (3) it is obvious that $S_{n;1,q}(f;x) \equiv S_n(f;q;x)$ for $f \in C_q$, $x \in \mathbb{R}_0$ and $n \in \mathbb{N}$.

Let $f_k(x) = x^k$ for $k = 0, 1, 2$. We immediately obtain from (7) and (6):

$$S_{n;r,q}(f_0;x) = 1, \qquad S_{n;r,q}(f_1;x) = \frac{x}{n+q}\frac{A_r'(nx)}{A_r(nx)}, \tag{8}$$

$$S_{n;r,q}(f_2;x) = \frac{x^2}{(n+q)^2}\frac{A_r''(nx)}{A_r(nx)} + \frac{x}{(n+q)^2}\frac{A_r'(nx)}{A_r(nx)}, \tag{9}$$

for $x \in \mathbb{R}_0$ and $n \in \mathbb{N}$. Moreover, we have

$$S_{n;r,q}(f;0) = f(0) \qquad \text{for} \qquad f \in C_q, n \in \mathbb{N}. \tag{10}$$

2.2. In the paper [10] the following lemma was proved.

Lemma 1. *The function A_r, $3 \le r \in \mathbb{N}$, defined by (6) can be rewritten in the following form:*

$$A_{2m}(t) = \frac{1}{m}\left[\cosh t + \sum_{k=1}^{m-1} \exp\left(t\cos\frac{k\pi}{m}\right)\cos\left(t\sin\frac{k\pi}{m}\right)\right] \tag{11}$$

for $2 \le m \in \mathbb{N}$, and

$$A_{2m+1}(t) = \frac{1}{2m+1}\left[e^t + 2\sum_{k=1}^{m} \exp\left(t\cos\frac{2k\pi}{2m+1}\right)\cos\left(t\sin\frac{2k\pi}{2m+1}\right)\right] \tag{12}$$

for $m \in \mathbb{N}$ and $t \in \mathbb{R}_0$.

Moreover, $A_1(t) = e^t$ and $A_2(t) = \cosh t = \frac{1}{2}(e^t + e^{-t})$ for $t \in \mathbb{R}_0$.

Applying (11) and (12) (or (6)), we easily derive the following

Lemma 2. *Let $r \in \mathbb{N}$ be fixed. Then*

$$\lim_{t\to\infty} \frac{A_r(t)}{e^t} = \frac{1}{r}\,.$$

Moreover, there exists a positive constant M_r^* *depending only on* r *such that*

$$1 \leq \frac{e^t}{A_r(t)} \leq M_r^* \quad \textit{for} \quad t \in \mathbb{R}_0. \tag{13}$$

2.3. Now we give some elementary properties of the operators $S_{n;r,q}$.

By (7), (6), (3) and (13) is obvious the following

Corollary 1. *For* $q > 0$, $r \in \mathbb{N}$ *and every non-negative function* $f \in C_q$ *there holds the inequality*

$$S_{n;r,q}(f;x) \leq M_r^* S_n(f;q;x) \quad \textit{for} \quad x \in \mathbb{R}_0, n \in \mathbb{N},$$

where $M_r^* = \text{const.} > 0$ *is given in (13).*

Applying Corollary 1 and the formulas for $S_n\left((t-x)^k;q;x\right)$, $k = 1,2,4$, and $S_n\left((t-x)^2/v_q(t);q;x\right)$ given in the paper [8], we immediately obtain

Corollary 2. *Let* $q > 0$ *and* $r \in \mathbb{N}$ *be fixed numbers. Then there exists* $M_{q,r} = \text{const.} > 0$ *depending only on* q *and* r *such that*

$$|S_{n;r,q}(t-x;x)| = \frac{e^{nx}}{A_r(nx)} |S_n(t-x;q;x)| \leq M_{q,r} \frac{x}{n+q}, \tag{14}$$

and analogously

$$S_{n;r,q}\left((t-x)^2;x\right) \leq M_{q,r} \frac{x^2+x}{n+q}, \tag{15}$$

$$S_{n;r,q}\left((t-x)^4;x\right) \leq M_{q,r} \frac{(x^2+x)^2}{(n+q)^2}, \tag{16}$$

$$v_q(x) S_{n;r,q}\left((t-x)^2/v_q(t);x\right) \leq M_{q,r} \frac{x^2+x}{n+q}, \tag{17}$$

for $x \in \mathbb{R}_0$ *and* $n \in \mathbb{N}$.

Applying (8), (9) and Lemma 2, we can easily obtain the following

Lemma 3. *For fixed* $q > 0$ *and* $r \in \mathbb{N}$ *there holds:*

$$\lim_{n\to\infty} n S_{n;r,q}(t-x;x) = -qx \quad \textit{and} \quad \lim_{n\to\infty} n S_{n;r,q}\left((t-x)^2;x\right) = x,$$

at every $x \in \mathbb{R}_0$.

Lemma 4. *Let* $q > 0$ *and* $r \in \mathbb{N}$ *be fixed and let* $M_r^* = \text{const.} > 0$ *be given by (13). Then for* $S_{n;r,q} \in \Omega_q$ *we have*

$$\|S_{n;r,q}(f)\|_q \leq M_r^* \|f\|_q \quad \textit{for} \quad f \in C_q \;\; \textit{and} \;\; n \in \mathbb{N}. \tag{18}$$

The formulas (7) and (6) and the inequality (18) show that $S_{n;r,q}$, $n \in \mathbb{N}$, *is a positive linear operator acting from the space* C_q *to* C_q.

Proof. From (7), (6), (2)-(4) and Corollary 1 we get

$$v_q(x)\,|S_{n;r,q}(f;x)| \le \|f\|_q v_q(x) S_{n;r,q}\left(1/v_q(t);x\right)$$
$$\le M_r^* \|f\|_q \left\|S_n\left(1/v_q;q\right)\right\|_q \le M_r^*\|f\|_q \quad \text{for} \quad x\in\mathbb{R}_0, n\in\mathbb{N},$$

which by (2) yields the inequality (18). □

3. Theorems

3.1. First we shall give two theorems on the degree of approximation of $f\in C_q$ by $S_{n;r,q}(f)$. We shall use the modulus of continuity $\omega_1(f;C_q;\cdot)$ and the modulus of smoothness $\omega_2(f;C_q;\cdot)$ of a function $f\in C_q$, $q>0$, i.e.

$$\omega_k(f;C_q;t) := \sup_{0\le h\le t}\left\|\Delta_h^k f(\cdot)\right\|_q \quad \text{for} \quad t\ge 0, k=1,2, \tag{19}$$

where $\Delta_h^1 f(x)=f(x+h)-f(x)$ and $\Delta_h^2 f(x) = f(x)-2f(x+h)+f(x+2h)$ for $x,h\in\mathbb{R}_0$ ([4], [11]). Moreover, let

$$\varphi(x) := (x^2+1)^{-1} \quad \text{for} \quad x\in\mathbb{R}_0, \tag{20}$$

$$\delta_{n,q} := (n+q)^{-1/2} \quad \text{for} \quad n\in\mathbb{N}, q>0, \tag{21}$$

and let C_q^2, $q>0$, be the set of all functions $f\in C_q$ which the derivatives f' and f'' belong to C_q also.

Theorem 1. *Let $q>0$ and $r\in\mathbb{N}$ be fixed numbers. Then there exists $M_{q,r}=\text{const.}>0$ depending only on q and r such that for every $f\in C_q^2$ the following inequality holds:*

$$\|(S_{n;r,q}(f)-f)\,\varphi\|_q \le M_{q,r}\frac{1}{n+q}\left(\|f'\|_q+\|f''\|_q\right) \quad n\in\mathbb{N}. \tag{22}$$

Proof. Choosing $f\in C_q^2$ and $x\in\mathbb{R}_0$, we have

$$f(t) = f(x)+f'(x)(t-x)+\int_x^t (t-u)f''(u)\,du$$

and by (2)

$$\left|\int_x^t (t-u)f''(u)\,du\right| \le \|f''\|_q\left(\frac{1}{v_q(t)}+\frac{1}{v_q(x)}\right)(t-x)^2,$$

for $t\in\mathbb{R}_0$. Using now the operator $S_{n;r,q}$ and (8), we get

$$S_{n;r,q}(f(t);x) = f(x)+f'(x)S_{n;r,q}(t-x;x)$$
$$+S_{n;r,q}\Big(\int_x^t (t-u)f''(u)du;x\Big), \quad n\in\mathbb{N}.$$

From the above and (14), (15) and (17) we deduce that

$$\begin{aligned} v_q(x)\,|S_{n;r,q}(f;x)-f(x)| &\le \|f'\|_q\,|S_{n;r,q}(t-x;x)| \\ &+ v_q(x)S_{n;r,q}\Big(\Big|\int_x^t (t-u)f''(u)\,du\Big|;x\Big) \le \|f'\|_q\,|S_{n;r,q}(t-x;x)| \\ &+ \|f''\|_q\left\{v_q(x)S_{n;r,q}\left((t-x)^2/v_q(t);x\right)+S_{n;r,q}\left((t-x)^2;x\right)\right\} \\ &\le M_{q,r}\frac{x^2+x}{n+q}\left(\|f'\|_q+\|f''\|_q\right), \quad n\in\mathbb{N}. \end{aligned}$$

Now using (20) and (2), we obtain (22). □

Theorem 2. *Suppose that $q>0$ and $r\in\mathbb{N}$ are fixed and φ and $\delta_{n,q}$ are defined by (20) and (21). Then there exists $M_{q,r}=\text{const.}>0$ depending only on q and r such that:*

$$\|(S_{n;r,q}(f)-f)\,\varphi\|_q \le M_{q,r}\left\{\delta_{n,q}\omega_1\left(f;C_q;\delta_{n,q}\right)+\omega_2\left(f;C_q;\delta_{n,q}\right)\right\}, \tag{23}$$

for every $f\in C_q$ and $n\in\mathbb{N}$, where $\omega_k(f;C_q;\cdot)$ is defined by (19).

Proof. Analogously to [2] we use the Steklov function f_h for $f\in C_q$:

$$f_h(x) := \frac{4}{h^2}\int_0^{h/2}\int_0^{h/2}\left[2f(x+s+t)-f(x+2(s+t))\right]ds\,dt,\ x\ge 0, h>0.$$

It is known ([2]) that $f_h\in C_q^2$ if $f\in C_q$ and

$$\|f_h-f\|_q \le \omega_2(f;C_q;h), \tag{24}$$

$$\|f_h'\|_q \le 5e^{hq}h^{-1}\omega_1(f;C_q;h), \tag{25}$$

and

$$\|f_h''\|_q \le 9h^{-2}\omega_2(f;C_q;h), \quad \text{for} \quad h>0. \tag{26}$$

By the linearity of the operator $S_{n;r,q}: C_q\to C_q$ and (20) we can write

$$\begin{aligned} \|[S_{n;r,q}(f)-f]\,\varphi\|_q &\le \|[S_{n;r,q}(f-f_h)]\,\varphi\|_q \\ &+ \|[S_{n;r,q}(f_h)-f_h]\,\varphi\|_q + \|f_h-f\|_q := Z_1+Z_2+Z_3. \end{aligned}$$

Next, by (20), (18) and (24), we have

$$Z_1 \le \|S_{n;r,q}(f-f_h)\|_q \le M_r^*\|f-f_h\|_q \le M_r^*\omega_1(f;C_q;h).$$

Applying Theorem 1 for $f_h\in C_q^2$ and next (25) and (26), we get

$$\begin{aligned} Z_2 &\le M_{q,r}\frac{1}{n+q}\left(\|f_h'\|_q+\|f_h''\|_q\right) \\ &\le M_{q,r}\frac{1}{n+q}\Big(e^{hq}h^{-1}\omega_1(f;C_q;h)+h^{-2}\omega_2(f;C_q;h)\Big). \end{aligned}$$

Consequently,

$$\|[S_{n;r,q}(f)-f]\,\varphi\|_q \le M_{q;r}\left\{e^{hq}h^{-1}(n+q)^{-1}\omega_1(f;C_q;h)\right.$$
$$\left.+\left((n+q)^{-1}h^{-2}+1\right)\omega_2(f;C_q;h)\right\} \quad \text{for} \quad n\in\mathbb{N}, h>0.$$

Choosing now $h=(n+q)^{-1/2}\equiv\delta_{n,q}$ for fixed $n\in\mathbb{N}$ and $q>0$, we obtain the desired estimation (23). □

The property $\lim_{t\to 0^+}\omega_k(f;C_q;t)=0$ for $f\in C_q$ and $k=1,2$, and Theorem 2 imply the following

Corollary 3. *For fixed $q>0$, $r\in\mathbb{N}$ and every $f\in C_q$ we have*

$$\lim_{n\to\infty} S_{n;r,q}(f;x)=f(x) \quad \textit{at every} \quad x\in\mathbb{R}_0.$$

This convergence is uniform on every interval $[a,b]$, $a\ge 0$.

3.2. Applying Corollary 3 and Lemma 3, we shall prove the Voronovskaya type theorem for the operators $S_{n;r,q}$.

Theorem 3. *Suppose that $f\in C_q^2$ with a fixed $q>0$ and $r\in\mathbb{N}$. Then*

$$\lim_{n\to\infty} n\left(S_{n;r,q}(f;x)-f(x)\right)=-qxf'(x)+\frac{x}{2}f''(x) \text{ at every } x\in\mathbb{R}_0. \tag{27}$$

Proof. The statement (27) is obvious for $x=0$ by (10).

Let now $x>0$ be fixed. By the Taylor formula for $f\in C_q^2$ we have

$$f(t)=f(x)+f'(x)(t-x)+\frac{1}{2}f''(x)(t-x)^2+\psi(t,x)(t-x)^2,$$

for $t\in\mathbb{R}_0$, where $\psi(t)\equiv\psi(t,x)$ is a function belonging to C_q and $\psi(x)=0$. From the above (7) and (8) we deduce that

$$S_{n;r,q}(f;x)=f(x)+f'(x)S_{n;r,q}(t-x;x)+\frac{1}{2}f''(x)S_{n;r,q}\left((t-x)^2;x\right)$$
$$+S_{n;r,q}\left(\psi(t)(t-x)^2;x\right), \quad \text{for} \quad n\in\mathbb{N}.$$

Using now Lemma 3, we get

$$\lim_{n\to\infty} n\left(S_{n;r,q}(f;x)-f(x)\right)=-qxf'(x)+\frac{x}{2}f''(x) \tag{28}$$
$$+\lim_{n\to\infty} nS_{n;r,q}\left(\psi(t)(t-x)^2;x\right).$$

Next by the Hölder inequality, we have

$$n\left|S_{n;r,q}\left(\psi(t)(t-x)^2;x\right)\right| \tag{29}$$
$$\le\left(S_{n;r,q}\left(\psi^2(t);x\right)\right)^{1/2}\left(n^2S_{n;r,q}\left((t-x)^4;x\right)\right)^{1/2}$$

and, by Corollary 3,

$$\lim_{n\to\infty} S_{n;r,q}\left(\psi^2(t);x\right) = \psi^2(x) = 0. \tag{30}$$

From (29), (30) and (16) it follows that

$$\lim_{n\to\infty} nS_{n;r,q}\left(\psi(x)(t-x)^2;x\right) = 0,$$

which used to (28) yields the statement (27). □

Remark. Theorems 1-3 show that the approximation properties of the Szász-Mirakyan operators $S_{n;r,q} \in \Omega_q$ are independent on $r \in \mathbb{N}$ but these are dependent only on $n \in \mathbb{N}$ and $q > 0$.

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