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INTERVAL-VALUED H-FUZZY RELATIONS

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Abstract: We introduce the category **IVRel**(H) consisting of interval-valued H-fuzzy relational spaces and relation preserving mappings between them and we study structures of the category **IVRel**(H) in the viewpoint of the topological universe introduced by Nel. Thus we show that **IVRel**(H) satisfies all the conditions of a topological universe over **Set** except the terminal separator property and **IVRel**(H) is Cartesian closed over **Set**. Furthermore, we study some relations among **Rel**(H), **IRel**(H) and **IVRel**(H).

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1. Introduction

Nel [20] introduced the notion of a topological universe which implies a Cartesian closed and a concrete quasitopos [1]. Every topological universe satisfies all the properties of a topos except one condition on the subobject classifier. The notion of a topological universe has already been put to effective use in

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several areas of mathematics in [18], [19], [21]. In 1980, Cerruti [5] introduced the category of L-fuzzy relations and investigated some of its properties. After that time, Hur [11] introduced the category **Rel**(H) of the fuzzy relational spaces with a complete Heyting algebra H as a codomain and he studied the category **Rel**(H) in the sense of a topological universe. Moreover, by using the concept of intuitionistic fuzzy set introduced by Atanassov [2], Hur et al [13] investigated the category **IVRel**(H) consisting of intuitionistic H-fuzzy relational spaces and morphisms between them in a topological universe viewpoint.

In 1975, Zadeh [24] introduced the concept of an interval-valued fuzzy set as the generalization of fuzzy sets. After then, Gorzalczany [7] studied a method of inference in approximate reasoning based on interval-valued fuzzy sets, and Roy and Biswas [23] investigated interval-valued fuzzy relations. Recently, Hur et al [14] studied them in the sense of a lattice theory. In particular, Hur et al [15] introduced the category **IVSet**(H) of interval-valued H-fuzzy sets and investigated it in a topological universe viewpoint.

In this paper, we introduce the category **IVRel**(H) of interval-valued H-fuzzy relational spaces and study the category **IVRel**(H) in the sense of a topological universe. In particular, we show that **IVRel**(H) satisfies all the conditions of a topological universe over **Set** except the terminal separator property. And **IVRel**(H) is shown to be Cartesian closed over **Set**. Furthermore, we study some relations among **Rel**(H), **IRel**(H) and **IVRel**(H).

2. Preliminaries

In this section, we will introduce some basic definitions and well-known results which are needed in the next sections.

Definition 2.1. (see [9]) Let **A** be a concrete category and let Γ be a class.

(1) A *source in* **A** is a pair $(X, (f_\alpha)_\Gamma)$ (simply, (X, f_α) or $(f_\alpha)_\Gamma$), where X is an **A**-object and $(f_\alpha : X \to X_\alpha)_\Gamma$ is a family of **A**-morphisms each with domain X. In this case, X is called the *domain of the source* and the family $(X_\alpha)_\Gamma$ is called the *codomain of the source*

(2) A source (X, f_α) is called a *mono-source* providing that the f_α can be simutaneously cancelled from the left; i.e., providing that for any pair $Y \overset{r}{\underset{s}{\rightrightarrows}} X$, of morphisms such that $f_\alpha \circ r = f_\alpha \circ s$ for each $\alpha \in \Gamma$, it follows that $r = s$.

Dual Notions. Sink in **A** and epi-sink.

Definition 2.2. (see [17]) Let **A** be a concrete category and let $((Y_\alpha, \xi_\alpha))_\Gamma$ be a family of objects in **A** indexed by a class Γ. For any set X, let $(f_\alpha : X \to Y_\alpha)_\Gamma$ be a source of mappings indexed by Γ. An **A**-structure ξ on X is said to be *initial with respet to* $(X, (f_\alpha), ((Y_\alpha, \xi_\alpha)))$ providing that the following conditions hold:

(1) For each $\alpha \in Z$, $f_\alpha : (X, \xi) \to (Y_\alpha, \xi_\alpha)$ is an **A**-morphism.

(2) If (Z, ρ) is an **A**-object and $g : Z \to X$ is mapping such that for each $i \in Z$, the mapping $f_\alpha \circ g : (Z, \rho) \to (Y_\alpha, \xi_\alpha)$ is an **A**-morphism, then $g : (Z, \rho) \to (X, \xi)$ is an **A**-morphism. In this case, $(f_\alpha : (X, \xi) \to (Y_\alpha, \xi_\alpha))_\Gamma$ is called an *initial source in* **A**.

Dual Notions. Final structure and final sink.

Definition 2.3. (see [17]) A concrete category **A** is said to be *topological over* **Set** providing that for each set X, for any family $((Y_\alpha, \xi_\alpha))_\Gamma$ of **A**-objects, and for any source $(f_\alpha : X \to Y_\alpha)_\Gamma$ of mappings, there exists a unique **A**-structure ξ on X which is initial with respect to $(X, (f_\alpha), ((Y_\alpha, \xi_\alpha)))$.

Dual Notions. Cotopological category.

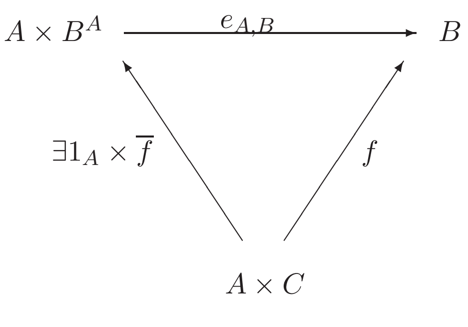
Result 2.A. (see [17, Theorem 1.5]) A concrete category **A** is topological if and only if **A** is cotopological.

Result 2.B. (see [17, Theorem 1.6]) Let **A** be a topological category over **Set**. Then **A** is complete and cocomplete.

Definition 2.4. (see [8]) A category **A** is called *Cartesian closed* providing that the following conditions hold:

(1) For any **A**-objects A and B, there exists a product $A \times B$ in **A**.

(2) Exponential exists in **A**, i.e., for any **A**-object A, the functor $A \times - : \mathbf{A} \to \mathbf{A}$ has a right adjoint, i.e., for any **A**-object B, there exists an **A**-object B^A and a **A**-morphism $e_{A,B} : A \times B^A \to B$(called the *evaluation*) such that for any **A**-object C and any **A**-morphism $f : A \times C \to B$, there exists a unique **A**-morphism $\overline{f} : C \to B^A$ such that the diagram



commutes.

Definition 2.5. (see [17]) Let $\mathbf{A}$ be a concrete category.

(1) The $\mathbf{A}$-*fibre* of a set X is the class of all $\mathbf{A}$-structures on X.

(2) $\mathbf{A}$ is called *properly fibred over* **Set** providing that the following conditions hold:

(i) (*Fibre-smallness*) For each set X, the $\mathbf{A}$-fibre of X is a set.

(ii) (*Terminal separator property*) For each singleton set X, the $\mathbf{A}$-fibre of X has precisely one element.

(iii) If ξ and η are $\mathbf{A}$-structures on a set X such that $1_X : (X, \xi) \to (X, \eta)$ and $1_X : (X, \eta) \to (X, \xi)$ are $\mathbf{A}$-morphisms, then $\xi = \eta$.

Definition 2.6. (see [20]) A category $\mathbf{A}$ is called a *topological universe over* **Set** providing that the following conditions hold:

(1) $\mathbf{A}$ is well-structured over **Set**, i.e.: (i) $\mathbf{A}$ is a concrete category; (ii) $\mathbf{A}$ has the fibre-smallness condition; (iii) $\mathbf{A}$ has the terminal separator property.

(2) $\mathbf{A}$ is cotopological over **Set**.

(3) Final episinks in $\mathbf{A}$ are preserved by pullbacks, i.e., for any final episink $(g_\lambda : X \to Y)_\Lambda$ and any $\mathbf{A}$-morphism $f : W \to Y$, the family $(e_\lambda : U_\lambda \to W)_\Lambda$, obtained by taking the pullback of f and g_λ for each λ, is again a final episink.

Definition 2.7. (see [22]) A category $\mathbf{A}$ is called a *topos* providing that the following conditions hold:

(1) There is a terminal object U in $\mathbf{A}$, i.e., for each $\mathbf{A}$-object A, there exists one and only one $\mathbf{A}$-morphism from A to U.

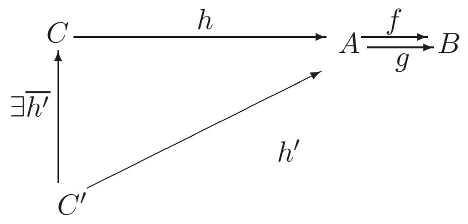
(2) $\mathbf{A}$ has equalizers i.e., for any $\mathbf{A}$-objects A and B and $\mathbf{A}$-morphisms

$$A \underset{g}{\overset{f}{\rightrightarrows}} B,$$

there exist an $\mathbf{A}$-object C and an $\mathbf{A}$-morphism $h : C \to A$ such that:

(a) $f \circ h = g \circ h$,

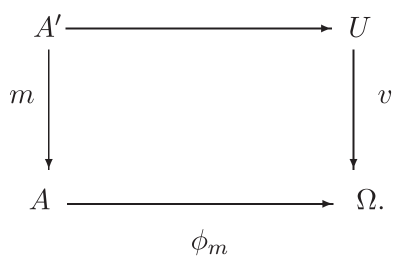
(b) for each $\mathbf{A}$-object C' and $\mathbf{A}$-morphism $h' : C' \to A$ with $f \circ h' = g \circ h'$, there exists a unique $\mathbf{A}$-morphism $\overline{h'} : C' \to C$ such that $h' = h \circ \overline{h'}$, i.e., the diagram



commutes.

(3) **A** is Cartesian closed.

(4) There is a subobject classifier in **A**, i.e., there is an **A**-object Ω and **A**-morphism $v : U \to \Omega$ such that for each **A**-monomorphism $m : A' \to A$, there exists a unique **A**-morphism $\phi_m : A \to \Omega$ such that the following diagram is a pullback:



Remark 2.8. Let **A** be any category with a subobject classifier. If f is any bimorphism in **A**, then f is an isomorphism in **A** (cf. [4]).

Definition 2.9. (see [5], [22]) A lattice H is called a *complete Heyting algebra*, if H satisfies the following conditions hold:

(1) H is a complete lattice.

(2) For any $a, b \in H$, the set $\{x \in H : x \wedge a \leq b\}$ has a greatest element denoted by $a \to b$ (called *pseudo-complement of* a *and* b), i.e., $x \wedge a \leq b$ if and only if $x \leq (a \to b)$.

In particular, for each $a \in H$, $N(a) = a \to 0$ is called the *negation* or the *pseudocomplement* of a.

Result 2.C. (see [5, Example 6 on p. 46]) Let H be a complete Heyting algebra and let $a, b \in H$. Then:

(1) If $a \leq b$, then $N(b) \leq N(a)$, i.e., $N : H \to H$ is an involutive order reversing operation in $(H, \leq)$.

(2) $a \leq NN(a)$.

(3) $N(a) = NNN(a)$.

(4) $N(a \vee b) = N(a) \wedge N(b)$ and $N(a \wedge b) = N(a) \wedge N(b)$.

Throughout this paper, we use H as a complete Heyting algebra with the least element 0 and the largest element 1.

3. The Category IVRel(H)

We introduce the category **IVRel**(H) consisting of interval-valued H-fuzzy relational spaces and relation preserving mappings between them, and show that it has similar structures as those of **IVRel**(H).

Let $D(H)$ be the set of all closed subintervals of H. The elements of $D(H)$ are generally denoted by capital letters M, N, $\cdots$, and note that $M = [M^L, M^U]$, where M^L and M^U are the lower and the upper end points respectively. Especially, we denote $\mathbf{0} = [0, 0]$,$\mathbf{1} = [1, 1]$, and $\mathbf{a} = [a, a]$ for every $a \in H$. We also note that

(i) $(\forall M, N \in D(H))(M = N \Leftrightarrow M^L = N^L, M^U = N^U)$.

(ii) $(\forall M, N \in D(H))(M \leq N \Leftrightarrow M^L \leq N^L, M^U \leq N^U)$.

For every $M \in D(H)$, the *complement* of M, denoted by M^c, is defined by $M^c = N(M) = [N(M^U), N(M^L)]$.

Definition 3.1. Let X be a nonempby set. Then a mapping $A = [A^L, A^H] : X \to D(H)$ is called an *interval-valued H-fuzzy set* (in short, *IVHFS*) in X, where A^L and A^U are H-fuzzy sets in X satisfying $A^L(x) \leq A^U(x)$ for each $x \in X$.

We will denote the set of all IVHFSs in X as $D(H)^X$

Definition 3.2. Let X be a nonempty set. Then a mapping $R = [R^L, R^U] : X \times X \to D(H)$ is called an *interval-valued H-fuzzy relation*(in shot, *IVHFR*) on X, where R^L and R^U are H-fuzzy relations on X satisfying $R^L(x, y) \leq R^U(x, y)$ for each $(x, y) \in X \times X$. The pair (X, R) is called an *interval-valued H-fuzzy relational space* (in short, *IVHFRS*).

Definition 3.3. Let (X, R_X) and (Y, R_Y) be an IVHFRSs. A mapping $f : X \to Y$ is called a *relation preserving mapping* if $R^L_X \leq R^L_Y \circ f^2$ and $R^U_X \leq R^U_Y \circ f^2$, where $f^2 = f \times f$.

The following is the immediate result of Definition 3.3.

Proposition 3.4. *Let (X, R_X), (Y, R_Y) and (Z, R_Z) be IVHFRSs.*

(1) The identity mapping $id_X : (X, R_X) \to (X, R_X)$ is a relation preserving mapping.

(2) If $f : (X, R_X) \to (Y, R_Y)$ and $g : (Y, R_Y) \to (Z, R_Z)$ are relation

preserving mappings, then $g \circ f : (X, R_X) \to (Z, R_Z)$ *is a relation preserving mapping.*

From Definitions 3.2 and 3.3, and Proposition 3.4, we can form a concrete category $\mathbf{IVRel}(H)$ consisting of all IVHFRSs and relation preserving mappings between them. Every $\mathbf{IVRel}(H)$-morphism will be called an $\mathbf{IVRel}(H)$-*mapping.*

Theorem 3.5. $\mathbf{IVRel}(H)$ *is topological over* **Set**.

Proof. Let X be any set and let $((X_\alpha, R_\alpha))_\Gamma$ be any family of IVHFRSs indexed by a class Γ. Let $(f_\alpha : X \to X_\alpha)_\Gamma$ be any source of mappings. We define the mapping $R = [R^L, R^U] : X \times X \to D(H)$ as follows: For each $(x, y) \in X \times X$,

$$R^L(x, y) = \bigwedge_{\alpha\in\Gamma} R^L_\alpha(f(x), f(y)) \text{ and } R^U(x, y) = \bigwedge_{\alpha\in\Gamma} R^U_\alpha(f(x), f(y)).$$

Then, by the definition of R, $R^L \leq R^U$. Thus $(X, R) \in \mathbf{IVRel}(H)$. Moreover, $f_\alpha : (X, R) \to (X_\alpha, R_\alpha)$ is an $\mathbf{IVRel}(H)$-mapping for each $\alpha \in \Gamma$.

For any $(Y, R_Y) \in \mathbf{IVRel}(H)$, let $g : Y \to X$ be any mapping for which $f_\alpha \circ g : (Y, R_Y) \to (X_\alpha, R_\alpha)$ is an $\mathbf{IVRel}(H)$-mapping for each $\alpha \in \Gamma$. Then we can easily check that $g : (Y, R_Y) \to (X, R)$ is an $\mathbf{IVRel}(H)$-mapping. Hence R is the initial structure on X with respect to $(X, (f_\alpha), ((X_\alpha, R_\alpha)))$. This completes the proof. □

Example 3.5. (1) *Inverse image of an* IVHFR. Let X be a set, let (Y, R_Y) be an IVHFRS and let $f : X \to Y$ be any mapping. Then there exists the initial IVHFR R on X for which $f : (X, R) \to (Y, R_Y)$ is an $\mathbf{IVRel}(H)$-mapping. In this case, R is called the *inverse image* of R_Y under f. In particular, if $X \subset Y$ and $f : X \to Y$ is the canonical mapping, then (X, R) is called an *interval-valued H-fuzzy relational subspace* of (Y, R_Y), where $R = [R^L, R^U]$ is the inverse image of R_Y under f. In fact, $R^L = R^L_Y|_{X\times X}$ and $R^U = R^U_Y|_{X\times X}$.

(2) *Interval-valued fuzzy product structure.* Let $((X_\alpha, R_\alpha))_\Gamma$ be any family of IVHFRSs and let $X = \prod X_\alpha$ be the product set of $(X_\alpha)_\Gamma$. Then there exists the initial IVHFR R on X for which each projection $\pi_\alpha : (X, R) \to (X_\alpha, R_\alpha)$ is an $\mathbf{IVRel}(H)$-mapping. In this case, R is called the *product* of $(R_\alpha)_\Gamma$ and denoted by $R = \prod R_\alpha$ and $(\prod X_\alpha, \prod R_\alpha)$ is called the *interval-valued H-fuzzy product relational space* of $((X_\alpha, R_\alpha))_\Gamma$. In fact, $(\Pi R)^L = \bigwedge_\Gamma R^L_\alpha \circ \pi^2_\alpha$ and $R^L_\alpha = \bigwedge_\Gamma R^U_\alpha \circ \pi^2_\alpha$.

In particular, if $H = \{1, 2\}$, then

$$(R_1 \times R_2)^L((x_1, y_1), (x_2, y_2)) = R^L_1(x_1, x_2) \wedge R^L_2(y_1, y_2)$$

and
$$(R_1 \times R_2)^U((x_1, y_1), (x_2, y_2)) = R_1^U(x_1, x_2) \wedge R_2^U(y_1, y_2)$$
for any
$$(x_1, y_1), (x_2, y_2) \in X_1 \times X_2.$$

Corollary 3.5. **IRel**(H) *is complete and cocomplete. Moreover, by definition, it is easy to show that* **IRel**(H) *is well-powered and co-well-powered.*

From Result 2.A and Theorem 3.5, it is clear that **IVRel**(H) is cotopological. However, we show directly that **IVRel**(H) is cotopological.

Theorem 3.6. **IVRel**(H) *is cotopological over* **Set**.

Proof. Let X be any set and let $((X_\alpha, R_\alpha))_\Gamma$ be any family of IVHFRSs indexed by a class Γ. Let $(f_\alpha : X_\alpha \to X)_\Gamma$ be any sink of mappings. We define the mapping $R = [R^L, R^U] \to D(H)$ as follows: For each $(x, y) \in X \times X$,
$$R^L(x, y) = \bigvee_\Gamma \bigvee_{(x_\alpha, \alpha) \in f_\alpha^{-1^2}(x,y)} R_\alpha^L(x_\alpha, y_\alpha)$$
and
$$R^U(x, y) = \bigvee_\Gamma \bigvee_{(x_\alpha, y_\alpha) \in f_\alpha^{-1^2}(x,y)} R_\alpha^L(x_\alpha, y_\alpha),$$
where $f_\alpha^{-1^2} = f_\alpha^{-1} \times f_\alpha^{-1}$. Then clearly $(X, R) \in$ **IVRel**(H). Moreover, $f_\alpha : (X_\alpha, R_\alpha) \to (X, R)$ is a **IVRel**(H)-mapping for each $\alpha \in \Gamma$.

For any $(Y, R_Y) \in$ **IVRel**(H), let $g : X \to Y$ be any mapping for which $g \circ f_\alpha : (X_\alpha, R_\alpha) \to (Y, R_Y)$ is an **IVRel**(H)-mapping for each $\alpha \in \Gamma$. Then we can easily check that $g : (X, R) \to (Y, R_Y)$ is an **IVRel**(H)-mapping. Hence R is the final structure on X with respect to $(((X_\alpha, R_\alpha)), (f_\alpha), X)$. This completes the proof. □

Example 3.6. (1) *Interval-valued H-fuzzy quotient relation.* Let $(X, R) \in$ **IVRel**(H), let $\sim$ be an equivalence relation on X and let $\varphi : X \to X/\sim$ be the canonical mapping. Then there exists the final interval-valued H-fuzzy relation $R_{X/\sim} = [R_{X/\sim}^L, R_{X/\sim}^U]$ on $X/\sim$ for which $\varphi : (X, R) \to (X/\sim, R_{X/\sim})$ is an **IVRel**(H)-mapping. In this case, $R_{X/\sim}$ is called the *interval-valued H-fuzzy quotient relation* of X by R.

(2) *Sum of interval-valued H-fuzzy relations.* Let $((X_\alpha, R_\alpha))_\Gamma$ be a family of IVHFRSs, let be X the sum of $(X_\alpha)_\Gamma$ and let $j_\alpha : X_\alpha \to X$ be the canonical (injection) mapping for each $\alpha \in \Gamma$. Then there exists the final IVHFR R on X. In fact, for each $((x_\alpha, \alpha), (y_\beta, \beta)) \in X \times X$, $R^L((x_\alpha, \alpha), (y_\beta, \beta)) = \bigvee_\Gamma \mu_{R_\alpha}(x, y)$ and $R^U((x_\alpha, \alpha), (y_\beta, \beta)) = \bigvee_\Gamma R^U(x, y)$. In this case, R is called the *sum* of

$(R_\alpha)_\Gamma$ and (X, R) is called the *sum* of $((X_\alpha, R_\alpha))_\Gamma$.

Theorem 3.7. *Final episinks in* $\mathbf{IVRel}(H)$ *are preserved by pullbacks.*

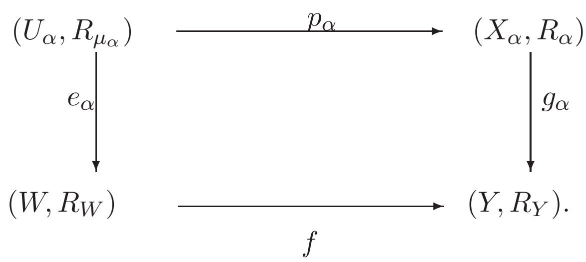
Proof. Let $(g_\alpha : (X_\alpha, R_\alpha) \to (Y, R_Y))_\Gamma$ be any final episink in $\mathbf{IVRel}(H)$ and let $f : (W, R_W) \to (Y, R_Y)$ be any $\mathbf{IVRel}(H)$-mapping. For each $\alpha \in \Gamma$, let $U_\alpha = \{(w, x_\alpha) \in W \times X_\alpha : f(w) = g_\alpha(x_\alpha)\}$ and let us define the mapping $R_{U_\alpha} = [R^L_{U_\alpha}, R^U_{U_\alpha}]$:$U_\alpha \times U_\alpha \to D(H)$ as follows: For each $((w, x_\alpha), (w', x'_\alpha)) \in U_\alpha \times U_\alpha$,

$$R_{U_\alpha}((w, x_\alpha), (w', x'_\alpha)) = R^L_W(w, w') \wedge R^U_\alpha(x_\alpha, x'_\alpha)$$

and

$$R^U_{U_\alpha}((w, x_\alpha), (w', x'_\alpha)) = R^U_W(w, w') \wedge R^U_\alpha(x_\alpha, x'_\alpha).$$

Let $e_\alpha : U_\alpha \to W$ and $p_\alpha : U_\alpha \to X_\alpha$ denote the usual projections of U_α. Then clearly $(U_\alpha, R_{U_\alpha}) \in \mathbf{IVRel}(H)$ for each $\alpha \in \Gamma$. Moreover, $e_\alpha : (U_\alpha, R_{U_\alpha}) \to (W, R_W)$ and $p_\alpha : (U_\alpha, R_{U_\alpha}) \to (X_\alpha, R_\alpha)$ are $\mathbf{IVRel}(H)$-mappings for each $\alpha \in \Gamma$. And the following diagram is a pullback square in $\mathbf{IVRel}(H)$:



By the process of the proof of Theorem 3.5 in [11], we can see that $(e_\alpha : (U_\alpha, RU_\alpha) \to (W, R_W)_\Gamma$ is a final episink in IVRel(H) and R_W is the unique final IVHFR on W with respect to $(e_\alpha)_\Gamma$. This completes the proof. □

For any singleton set $\{a\}$, since the IVHFR R on $\{a\}$ is not unique, the category $\mathbf{IVRel}(H)$ is not properly fibred over $\mathbf{Set}$. Hence, by Theorems 3.6 and 3.7, we obtain the following result.

Theorem 3.8. $\mathbf{IVRel}(H)$ *satisfies all the conditions of a topological universe over* $\mathbf{Set}$ *except the terminal separator property.*

Theorem 3.9. $\mathbf{IVRel}(H)$ *is Cartesian closed over* $\mathbf{Set}$.

Proof. It is clear that $\mathbf{IVRel}(H)$ has products by Corollary 3.5. We will show that $\mathbf{IVRel}(H)$ has exponential objects.

For any IVHFRSs $\mathbf{X} = (X, R_X)$ and $\mathbf{Y} = (Y, R_Y)$, let Y^X be the set of all mappings from X into Y. We define the mapping $R = [R^L, R^U] : Y^X \times Y^X \to$

$D(H)$ as follows: For each $(f,g) \in Y^X \times Y^X$,

$$R^L(f,g) = \bigwedge\{h \in H : R^L_X(x,y) \wedge h \leq R^L_Y(f(x),g(y))$$

for each $(x,y) \in X \times X\}$ and

$$R^U(f,g) = \bigwedge\{h \in H : R^U_X(x,y) \wedge h \leq R^U_Y(f(x),g(y))$$

for each $(x,y) \in X \times X\}$.

Then clearly $(Y^X, R) \in \mathbf{IVRel}(H)$. Let $\mathbf{Y^X} = (Y^X, R)$. Then, by the definition of R,

$$R^L_X(x,y) \wedge R^L(f,g) \leq R^L_Y(f(x),g(y))$$

and

$$R^U_X(x,y) \wedge R^U(f,g) \leq R^U_Y(f(x),g(y))$$

for each $(f,g) \in Y^X$ and $(x,y) \in X \times X$.

Define $e_{X,Y} : X \times Y^X \to Y$ by $e_{X,Y}(x,f) = f(x)$ for each $(x,f) \in X \times Y^X$. Let $((x,f),(y,g)) \in (X \times Y^X) \times (X \times Y^X)$. Then, by the process of the proof of Theorem 2.7 in [11], $e_{X,Y} : \mathbf{X} \times \mathbf{Y^X} \to \mathbf{Y}$ is an $\mathbf{IVRel}(H)$-mapping.

For any $\mathbf{Z} = (Z, R_Z) \in \mathbf{IVRel}(H)$, let $h : \mathbf{X} \times \mathbf{Z} \to \mathbf{Y}$ be an $\mathbf{IVRel}(H)$-mapping. We define $\overline{h} : Z \to Y^X$ by $[\overline{h}(z)](x) = h(x,z)$ for each $z \in Z$ and each $x \in X$. Let $z, z' \in Z$ and let $x, x' \in X$. Then, by the process of the proof of Theorem 2.7 in [11], $\overline{h} : \mathbf{Z} \to \mathbf{Y^X}$ is an $\mathbf{IVRel}(H)$-mapping. Moreover, $\overline{h}$ is the unique $\mathbf{IVRel}(H)$-mapping such that $e_{X,Y} \circ (1_X \times \overline{h}) = h$. This completes the proof. □

Remark 3.10. $\mathbf{IVRel}(H)$ has no subobject classifier. Hence $\mathbf{IVRel}(H)$ is not topos.

Example 3.11. Let $H = \{0,1\}$ be the two points chain and let $X = \{a\}$. Let R_1 and R_2 be the IVHFRs on X given by $R_1(a,a) = \mathbf{0}$ and $R_2(a,a) = \mathbf{1}$. Let $1_X : (X, R_1) \to (X, R_2)$ be the identity mapping. Then clearly, 1_X is both a monomorphism and an epimorphism in $\mathbf{IVRel}(H)$. But, 1_X is not an isomorphism in $\mathbf{IVRel}(H)$. Hence $\mathbf{IVRel}(H)$ has no subobject classifier(See [4]).

4. The Relations between Rel(*H*), IRel(*H*) and IVRel(*H*)

Definition 4.1. (see [11]) The concrete category $\mathbf{Rel}(H)$ is defined by: Objects are (X, R), called *H-fuzzy relational space* on X, where X is any set and R is a mapping from $X \times X$ to H. A morphism $f : (x, R_X) \to (Y, R_Y)$ is a mapping from X to Y satisfying $R_X(x,y) \leq R_Y(f(x), f(y))$, i.e., $R_X(x,y) \leq (R_Y \circ f^2)(x,y)$ for each $(x,y) \in X \times X$. Every **Rel**(H)-morphism is called a **Rel**(H)-*mapping*.

Definition 4.2. (see [13]) The concrete category **IRel**(H) is defined by: Objects are $(X,R) = (X, \mu_R, \nu_R)$, called an *intuitionistic H-fuzzy relational space* on X, where X is any set and $\mu_R, \nu_R \in H^{X\times X}$ satisfying $\mu_R(x,y) \leq N(\nu_R(x,y))$ for each $(x,y) \in X \times X$, A morphism $f : (X, R_X) \to (Y, R_Y)$ is a mapping satisfying $\mu_{R_X} \leq \mu_{R_Y} \circ f^2$ and $\nu_{R_X} \geq \nu_{R_Y} \circ f^2$, where $f^2 = f \times f$. Every **IRel**(H)-morphism is called an ***IRel**(H)-mapping.*

Lemma 4.3. *Define* $G_1, G_2 : \mathbf{IRel}(H) \to \mathbf{Rel}(H)$ *by*

$$G_1(X,R) = (X, R^L),\ G_2(X,R) = (X, R^U) \text{ and } G_1(f) = G_2(f) = f.$$

Then G_1 *and* G_2 *are functors.*

Proof. Clearly $G_1(X,R) = (X,R^L) \in \mathbf{Rel}(H)$ for each $(X,R) \in \mathbf{IVRel}(H)$. Let (X,R_X), $(Y,R_Y) \in \mathbf{IVRel}(H)$ and let $f : (X,R_X) \to (Y,R_Y)$ be an **IVRel**(H)-mapping. Then $R_X^L \leq R_Y^L \circ f^2$ and $R_X^U \leq R_Y^U \circ f^2$. Thus $G_1(f) = f : (X, R_X^L) \to (Y, R_Y)$ and $G_2(f) : (X, R_X^U) \to (Y, R_Y^U)$ are **Rel**(H)-mappings. Hence G_1 and G_2 are functors. □

Lemma 4.4. *Define* $F : \mathbf{Rel}(H) \to \mathbf{IVRel}(H)$ *by* $F_1(X,R) = (X,[R,R])$ *and* $F(f) = f$. *Then* F *is a functor.*

Proof. It is obvious. □

Theorem 4.5. *The functor* $F : \mathbf{Rel}(H) \to \mathbf{IVRel}(H)$ *is a left adjoint of the functors* G_1 *and* G_2.

Proof. For each $(X,R) \in \mathbf{Rel}(H)$, $1_X : (X,R) \to G_1F(X,R) = (X,R)$ is a **Rel**(H)-mapping. Let $(Y,R_Y) \in \mathbf{IVRel}(H)$ and let $f : (X,R) \to G_1(Y,R_Y)$ be an **IVRel**(H)-mapping. We will show that $f : F(X,R) = (X,[R,R]) \to (Y,R_Y)$ is an **IVRel**(H)-mapping. Since $f : (X,R) = G_1(Y,R_Y) \to (Y,R_Y^L)$ is a **Rel**(H)-mapping, $R \leq R_Y^L \circ f^2$. Since $R_Y^L \leq R_Y^U$, $R \leq R_Y^U \circ f^2$. So $f : F(X,R) \to (Y,R_Y)$ is an **IRRel**(H)-mapping. Hence 1_X is a G_1-universal map for (X,R) in **Rel**(H). Similarly, we can see that 1_X is a G_2-universal map for (X,R) in **Rel**(H). This completes the proof. □

For each $(X,R) \in \mathbf{Rel}(H)$, $F(X,R) = (X,[R,R])$ is called an *interval-valued H-fuzzy relation in* X *induced* by (X,R). Let us denote the category consistiong of all induced interval-valued H-fuzzy relations and **IVRel**(H)-mappings as $\mathbf{IVRel}^*(H)$. Then it is clear $\mathbf{IVRel}^*(H)$ is a full subcategory of **IVRel**(H).

Theorem 4.6. *Two categories* **Rel**(H) *and* $\mathbf{IVRel}^*(\mathbf{H})$ *are isomorphic.*

Proof. It is clear that $F : \mathbf{Rel}(H) \to \mathbf{IVRel}^*(H)$ is a functor by Lemma 4.4. Consider the restriction $G_1 : \mathbf{IRel}^*(H) \to \mathbf{Rel}(H)$ of the functor G_1 in Lemma 4.3. Let $(X, R) \in \mathbf{Rel}(H)$. Then, by Lemma 4.4, $F(X, R) = (X, [R, R])$. Thus $G_1F(X, R) = G_1(X, [R, R]) = (X, R)$. So $G_1 \circ F = 1_{\mathbf{Rel}(H)}$. Now let $(X, [R, R]) \in \mathbf{IRel}^*(H)$. Then, by Lemma 4.3, $G_1(X, [R, R]) = (X, R)$. Thus $FG_1(X, [R, R]) = (X, [R, R])$. So $F \circ G_1 = 1_{\mathbf{IRel}^*(H)}$. Hence $F : \mathbf{Rel}(H) \to \mathbf{IRel}^*(H)$ is an isomorphism. This completes the proof. □

Lemma 4.7. *We define $G : \boldsymbol{IVRel}(H) \to \boldsymbol{IRel}(H)$ as follows:*

$$G(X, R) = (X, R^L, N(R^U)) \quad \forall (X, R) \in Ob(\boldsymbol{IVRel}(H))$$

and

$$G(f) = f \quad \forall f \in Mor(\boldsymbol{IVRel}(H)).$$

Then G is a functor.

Proof. Let $(X, R) \in Ob(\mathbf{IVRel}(H))$. Then, by Result 2.C(2),

$$R^L \leq R^U \leq NN(R^U).$$

Thus $G(X, R) = (X, R, N(R^U)) \in \mathbf{IRel}(H)$. Let $f : (X, R_X) \to (Y, R_Y)$ be any **IVRel**(H)-mapping. Then

$$R^L_X \leq R^L_Y \circ f^2 \text{ and } R^U_X \leq R^U_Y \circ f^2.$$

Let $(x, y) \in X \times X$. Then $R^U_X(x, y) \leq R^U_Y(f(x), f(y))$. Thus $N(R^U_X(x, y)) \geq N(R^U_Y(f(x), f(y)))$. So $[N(R^U_X)](x, y) \geq [N(R^U_Y)] \circ f^2(x, y)$, i.e., $N(R^U_X) \geq N(R^U_Y) \circ f^2$. Hence $G(f) = f : G(X, R_X) \to G(Y, R_Y)$ is an **IRel**(H)-mapping. G is a functor. □

Lemma 4.8. *We define $K : \boldsymbol{IRel}(H) \to \boldsymbol{IVRel}(H)$ as follows:*

$$K(X, \mu_R, \nu_R) = (X, [\mu_R, N(\nu_R)]) \quad \forall (X, \mu_R, \nu_R) \in Ob(\boldsymbol{IRel}(H))$$

and

$$K(f) = f \quad \forall\ f \circ Mor(\boldsymbol{IRel}(H)).$$

Then K is a functor.

Proof. Let $(X, \mu_R, \nu_R) \in \mathrm{Ob}(\mathbf{IRel}(H))$. Then clearly $\mu_R \leq \mathrm{N}(\nu_R)$. Thus $(X, \mu_R, \nu_R) \in \mathbf{IRel}(H)$. Let $f : (X, \mu_R, \nu_R) \to (Y, \mu_R, \nu_R)$ be as $\mathbf{IRel}(H)$-mapping. Then

$$\mu_{R_X} \leq \mu_{R_Y} \circ f^2 \text{ and } \nu_{R_X} \geq \nu_{R_Y} \circ f^2.$$

We can easily see that $\mathrm{N}(\nu_{R_X}) \leq \mathrm{N}(\nu_{R_Y}) \circ f^2$. Thus

$$K(f) : K(X, \mu_{R_X}, \nu_{R_X}) \to (Y, \mu_{R_Y}, \nu_{R_Y})$$

is an $\mathbf{IVRel}(H)$-mapping. Hence K is a functor. □

Theorem 4.9. *Two categories **IRel**(H) and **IVRel**(H) are isomorphic.*

Proof. By Lemmas 4.7 and 4.8, we can easily show that

$$G \circ K = 1_{\mathrm{IRel}(H)} \text{ and } K \circ G = 1_{\mathrm{IVRel}(H)}. \qquad \square$$

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