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SOME SUBCATEGORIES OF THE CATEGORY $\mathbf{IVRel_R}(H)$

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Abstract: We introduce the subcategory $\mathbf{IVRel_R}(H)$ of $\mathbf{IVRel}(H)$ consisting of interval-valued H-fuzzy reflexive relational space on sets and we study structures of $\mathbf{IVRel_R}(H)$ in a viewpoint of the topological universe introduce by Nel. We show that $\mathbf{IVRel_R}(H)$ is a topological universe over **Set**. Moreover, we show that exponential objects in $\mathbf{IVRel_R}(H)$ are quite different from those in $\mathbf{IVRel}(H)$. Also we introduce the subcategories $\mathbf{IVRel_{PR}}(H)$, $\mathbf{IVRel_P}(\mathbf{H})$ and $\mathbf{IVRel_E}(H)$ of $\mathbf{IVRel_R}(H)$ and investigate their structures in the sense of a topological universe.

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1. Introduction

In 1975, Zadeh [19] introduced the concept of an interval-valued fuzzy set as the generalization of fuzzy sets. After that time, Gorzalczany [5] studied a method of inference in approximate reasoning based on interval-valued fuzzy sets, and

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Roy and Biswas [18] investigater interval-valued fuzzy relations. Recently, Hur et al [10] studied them in the sense of a lattice theory. In particular, Hur et al [11], [12] studied categorical structures of the category $\mathbf{IVSet}(H)$ consisting of interval-valued H-fuzzy sets and the category $\mathbf{IVRel}(H)$ consisting of interval-valued H-fuzzy relational spaces in a viewpoint of topological universe, implying a Cartesian closed and a quasitops [1], defined by Nel [15].

In this paper, we investigate categorical structures of the subcategory $\mathbf{IVRel_R}(H)$ of $\mathbf{IVRel}(H)$ consisting of interval-valued H-fuzzy reflexive relational spaces on sets in a viewpoint of a topological universe. In particular, it is very interesting that exponential objects in $\mathbf{IVRel_R}(H)$ are shown to be quite different from those in $\mathbf{IVRel}(H)$(see [12]). Also we are introduce the subcategories $\mathbf{IVRel_{PR}}(H)$, $\mathbf{IVRel_P}(H)$ and $\mathbf{IVRel_E}(H)$ of $\mathbf{IVRel_R}(H)$ and study their structures in the sense of a topological universe. Moreover, we show that $\mathbf{IVRel_E}(H)$ has exponential objects.

2. Preliminaries

We will introduce some well-known definitions and results which are needed in the later sections.

Definition 2.1. (see [7]) A category $\mathbf{A}$ is said to be *well-powered* if each $\mathbf{A}$-object has a representative class of subobjects that is a set.

Dual Notion. Co-(well-powered) (i.e., each object has a representative class of quotient objects which is a set).

Definition 2.2. (see [14]) Let $\mathbf{A}$ be a concrete category.

(1) The $\mathbf{A}$-*fibre* of a set X is the class of all $\mathbf{A}$-structures on X.

(2) $\mathbf{A}$ is called *properly fibred over* **Set** provided that the following conditions hold:

(i) (*Fibre-smallness*) For each set X, the $\mathbf{A}$-fibre of X is a set.

(ii) (*Terminal separator property*) For each singleton set X, the $\mathbf{A}$-fibre of X has precisely one element.

(iii) If ξ and η are $\mathbf{A}$-structures on a set X such that $1_X : (X, \xi) \to (X, \eta)$ and $1_X : (X, \eta) \to (X, \xi)$ are $\mathbf{A}$-morphisms, then $\xi = \eta$.

Result 2.A. (see [14], Theorem 2.4; [10], Propositions 36.10 and 36.11) *Let* $\mathbf{A}$ *be a well-powered and co-(well-powered) topological category and let* $\mathbf{B}$ *be a subcategory of* $\mathbf{A}$. *Then the following are equivalent:*

(1) $\mathbf{B}$ *is epireflective in* $\mathbf{A}$.

(2) $\mathbf{B}$ *is closed under the formation of initial monosources.*

(3) $\mathbf{B}$ *is closed under the formation of products and pullbacks in* $\mathbf{A}$.

Result 2.B. (see [14], Theorem 2.5) *Let* $\mathbf{A}$ *be a well-powered and co-(well-powered) topological category and let* $\mathbf{B}$ *be a subcategory of* $\mathbf{A}$. *Then the following are equivalent:*

(1) $\mathbf{B}$ *is bireflective in* $\mathbf{A}$.

(2) $\mathbf{B}$ *is closed under the formation of initial sources.*

Result 2.C. (see [14], Theorem 2.6) *If* $\mathbf{A}$ *is a (property fibred, resp.) topological category and* $\mathbf{B}$ *is a bireflective subcategory of* $\mathbf{A}$, *then* $\mathbf{B}$ *is also a (property fibred, resp.) topological category. Moreover, every source in* $\mathbf{B}$ *which is initial in* $\mathbf{A}$ *is initial in* $\mathbf{B}$.

Definition 2.3. (see [6]) A category $\mathbf{A}$ is called *Cartesian closed* providing that the following conditions hold:

(1) For any $\mathbf{A}$-objects A and B, there exists a product $A \times B$ in $\mathbf{A}$.

(2) Exponential exist in $\mathbf{A}$, i.e., for any $\mathbf{A}$-object A, the functor $A \times - : \mathbf{A} \to \mathbf{A}$ has a right adjoint, i.e., for any $\mathbf{A}$-object B, there exists an $\mathbf{A}$-object B^A and a $\mathbf{A}$-morphism $e_{A,B} : A \times B^A \to B$ (called the *evaluation*) such that for any $\mathbf{A}$-object C and any $\mathbf{A}$-morphism $f : A \times C \to B$, there exists a unique $\mathbf{A}$-morphism $\overline{f} : C \to B^A$ such that the diagram

$$\begin{array}{ccc} A \times B^A & \xrightarrow{\;e_{A,B}\;} & B \\ {\scriptstyle \exists 1_A \times \overline{f}} \nwarrow & & \nearrow {\scriptstyle f} \\ & A \times C & \end{array}$$

commutes.

Definition 2.4. (see [15]) A category $\mathbf{A}$ is called a *topological universe over* $\mathbf{Set}$ providing that the following conditions hold:

(1) $\mathbf{A}$ is well-structured over $\mathbf{Set}$, i.e.: (i) $\mathbf{A}$ is a concrete category; (ii) $\mathbf{A}$ has the fibre-smallness condition; (iii) $\mathbf{A}$ has the terminal separator property.

(2) $\mathbf{A}$ is cotopological over $\mathbf{Set}$.

(3) Final episinks in $\mathbf{A}$ are preserved by pullbacks, i.e., for any final episink

$(g_\lambda : X \to Y)_\Lambda$ and any **A**-morphism $f : W \to Y$, the family $(e_\lambda : U_\lambda \to W)_\Lambda$, obtained by taking the pullback of f and g_λ for each λ, is again a final episink.

Definition 2.5. (see [17]) A category **A** is called a *topos* providing that the following conditions hold:

(1) There is a terminal object U in **A**, i.e., for each **A**-object A, there exists one and only one **A**-morphism from A to U.

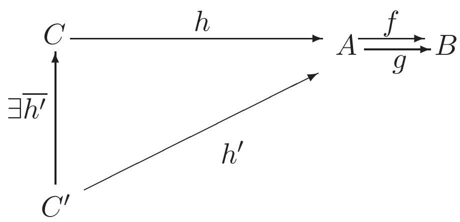
(2) **A** has equalizers, i.e., for any **A**-objects A and B and **A**-morphisms

$$A \underset{g}{\overset{f}{\rightrightarrows}} B,$$

there exist an **A**-object C and an **A**-morphism $h : C \to A$ such that:

(a) $f \circ h = g \circ h$,

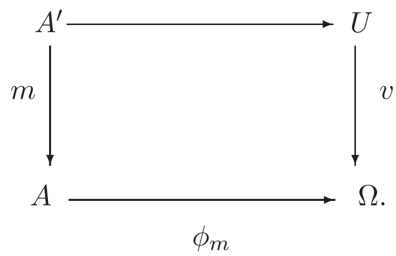
(b) for each **A**-object C' and **A**-morphism $h' : C' \to A$ with $f \circ h' = g \circ h'$, there exists a unique **A**-morphism $\overline{h'} : C' \to C$ such that $h' = h \circ \overline{h'}$, i.e., the diagram



commutes.

(3) **A** is Cartesian closed.

(4) There is a subobject classifier in **A**, i.e., there is an **A**-object Ω and **A**-morphism $v : U \to \Omega$ such that for each **A**-monomorphism $m : A' \to A$, there exists a unique **A**-morphism $\phi_m : A \to \Omega$ such that the following diagram is a pullback:



Remark 2.6. Let **A** be any category with a subobject classifier. If is any bimorphism in **A**, then f is an isomorphism in **A** (cf. [3]).

Definition 2.7. (see [15]) A category **A** is called a *topological universe* *over* **Set** provided that the following conditions hold:

(1) **A** is well-structured over **Set**, i.e.: (i) **A** is a concrete category; (ii) **A** has the fibre-smallness condition; (iii) **A** has the terminal separator property.

(2) **A** is cotopological over **Set**.

(3) Final episinks in **A** are preserved by pullbacks, i.e., for any final episink $(g_\lambda : X \to Y)_\Lambda$ and any **A**-morphism $f : W \to Y$, the family $(e_\lambda : U_\lambda \to W)_\Lambda$, obtained by taking the pullback of f and g_λ for each λ, is again a final episink.

Definition 2.8. (see [2], [13]) A lattice H is called a *complete Heyting algebra*, if H satisfies the following conditions:

(1) H is a complete lattice.

(2) For any $a, b \in H$, the set $\{x \in H : x \wedge a \leq b\}$ has a greatest element denoted by $a \to b$ (called *pseudo-complement of* a *and* b), i.e., $x \wedge a \leq b$ if and only if $x \leq (a \to b)$.

In particular, for each $a \in H, N(a) = a \to 0$ is called the *negation* or the *pseudocomplement* of a.

Result 2.D. (see [4], Example 6 on p. 46) *Let H be a complete Heyting algebra and let $a, b \in H$. Then:*

(1) If $a \leq b$, then $N(b) \leq N(a)$, i.e., $N : H \to H$ is an involutive order reversing operation in $(H, \leq)$.

(2) $a \leq NN(a)$.

(3) $N(a) = NNN(a)$.

(4) $N(a \vee b) = N(a) \wedge N(b)$ and $N(a \wedge b) = N(a) \wedge N(b)$.

Throughout this paper, we use H as a complete Heyting algebra with the least element 0 and the largest element 1.

3. The Category $\mathbf{IVRel_R(H)}$

In this section, we obtain a subcategory $\mathbf{IRel_R}(H)$ of $\mathbf{IRel}(H)$ which is a topological universe over **Set**. It is very interesting that exponential objects in $\mathbf{IVRel_R}(H)$ are shown to be quite different from those in $\mathbf{IRel}(H)$ constructed in [12].

Let $D(H)$ be the set of all closed subintervals of H. The elements of $D(H)$ are generally denoted by capital letters $M, N, \cdots$, and note that $M = [M^L, M^U]$, where M^L and M^U are the lower and the upper end points respectively. Especially, we denote $\mathbf{0} = [0, 0], \mathbf{1} = [1, 1]$, and $\mathbf{a} = [a, a]$ for every $a \in H$.

We also note that:

(i) $(\forall M, N \in D(H))(M = N \Leftrightarrow M^L = N^L, M^U = N^U)$.

(ii) $(\forall M, N \in D(H))(M \leq N \Leftrightarrow M^L \leq N^L, M^U \leq N^U)$.

For every $M \in D(H)$, the *complement* of M, denoted by M^c, is defined by $M^c = N(M) = [N(M^U), N(M^L)]$.

Definition 3.1. (see [11]) Let X be a given nonempby set. Then a mapping $A : X \to D(H)$ is called an *interval-valued H-fuzzy set* (in short, *IVHFS*) on X, where we denote $A = [A^L, A^U]$, and $A^L, A^H : X \to H$are mappings satisfying $A^L(x) \leq A^U(x)$ for each $x \in X$. We will denote the set of all IVHFSs on a set X as $D(H)^X$.

Definition 3.2. (see [12]) The concrete category $\mathbf{IVRel_R}(H)$ is defined by: Objects are (X, R), called an *interval-valued H-fuzzy relational space* on X, where X us any set and $R \in D(H)^{X \times X}$. A morphism $f : (X, R_X) \to (Y, R_Y)$ is a mapping from X to Y satisfying $R_X^L \leq R_Y^U \circ f^2$, where $f^2 = f \times f$. Every $\mathbf{IVRel_R}(H)$ mapping is called an $\mathbf{IVRel_R}(H)$-mapping.

Definition 3.3. An interval-valued H-fuzzy relation R on a set X is said to be *reflexive* if and $R(x, x) = \mathbf{1}$ $\forall x \in X$.

The class of all interval-valued H-fuzzy reflexive relational spaces and $\mathbf{IVRel}(H)$-mappings between them forms a subcategory of $\mathbf{IVRel}(H)$ and denoted by $\mathbf{IVRel_R}(H)$. It is clear that $\mathbf{IVRel_R}(H)$ is a full and isomorphism-closed subcategory of $\mathbf{IVRel}(H)$.

We can easily obtain the following.

Proposition 3.4. $\mathbf{IVRel_R}(H)$ *is properly fibred over* **Set**.

Lemma 3.5. $\mathbf{IVRel_R}(H)$ *is closed under the formation of initial sources in* $\mathbf{IVRel}(H)$.

Proof. Let $(f_\alpha : (X, R) \to (X_\alpha, R_\alpha))_\Gamma$ be any initial source in $\mathbf{IVRel}(H)$ such that $(X_\alpha, R_\alpha) \in \mathbf{IVRel_R}(H)$ for each $\alpha \in \Gamma$. Let $x \in X$. Since R_α is reflexive for each $\alpha \in \Gamma$, $R_\alpha^L \circ f_\alpha^2(x, x) = 1$ and $R_\alpha^U \circ f_\alpha^2(x, x) = 1$. Thus $R^L) = \bigwedge_\Gamma R_\alpha^L \circ f_\alpha^2(x, x) = 1$ and $R^U = \bigwedge_\Gamma R_\alpha^U \circ f_\alpha^2(x, x) = 1$. So R is reflexive. Hence $(X, R) \in \mathbf{IRel_R}(H)$. This completes the proof. □

From Result 2.B, Result 2.C and Lemma 3.5, we obtain the following result.

Theorem 3.6. *(1)* $\mathbf{IVRel_R}(H)$ *is a bireflective subcategory of* $\mathbf{IVRel}(H)$.

(2) $\mathbf{IVRel_R}(H)$ *is topological over* **Set**.

We show that $\mathbf{IVRel_R}(H)$ is cotopological over **Set**, directly.

Theorem 3.7. $\mathbf{IVRel_R}(H)$ *has final structures over* **Set**.

Proof. Let X be any set and let $(X_\alpha, R_\alpha)_\Gamma$ any family of interval-valued H-fuzzy reflexive relational spaces indexed by a class Γ. Let $(f_\alpha : X_\alpha \to X)_\Gamma$ be any sink of mappings. We define the mapping $R = [R^L, R^U] : X \times X \to D(H)$ R^L as follows: For each $(x, y) \in X \times X$,

$$R^L(x,y) = \bigvee_\Gamma \bigvee_{(x_\alpha,y_\alpha)\in f_\alpha^{-1^2}(x,y)} R_\alpha^U(x_\alpha, y_\alpha) \qquad \text{if } (x,y) \in (X \times X - \Delta_X)$$
$$= 1 \qquad \text{if } (x,y) \in \Delta_X$$

and

$$R^U(x,y) = \bigvee_\Gamma \bigvee_{(x_\alpha,y_\alpha)\in f_\alpha^{-1^2}(x,y)} R_\alpha^U(x_\alpha, y_\alpha) \qquad \text{if } (x,y) \in (X \times X - \Delta_X)$$
$$= 1 \qquad \text{if } (x,y) \in \Delta_X,$$

where $\Delta_X = \{(x,x) : x \in X\}$ and $f_\alpha^{-1^2} = f_\alpha^{-1} \times f_\alpha^{-1}$. Then clearly $(X, R) \in \mathbf{IVRel_R}(H)$. Moreover, we can easily check that $(f_\alpha : (X_\alpha, R_\alpha) \to (X, R))_\Gamma$ is a final sink in $\mathbf{IVRel_R}(H)$. □

Theorem 3.8. *Final episinks in* $\mathbf{IVRel_R}(H)$ *are preserved by pullbacks.*

Proof. Let $(g_\alpha : (X_\alpha, R_\alpha) \to (Y, R_Y))_\Gamma$ be any final episink in $\mathbf{IVRel_R}(H)$ and let $f : (W, R_W) \to (Y, R_Y)$ be any $\mathbf{IVRel}(H)$-mapping, where $(W, R_W) \in \mathbf{IVRel_R}(H)$. For each $\alpha \in \Gamma$, let us take $U_\alpha, R_{U_\alpha}, e_\alpha$ and p_α as in the process of the proof of Theorem 3.7 in [12]. By Theorem 3.6(1) and Result 2.A, $\mathbf{IVRel_R}(H)$ is closed under the formation of pullbaks in $\mathbf{IVRel}(H)$. Thus it is enough to show that $(e_\alpha : (U_\alpha, R_{U_\alpha}) \to (W, R_W))_\Gamma$ is final in $\mathbf{IVRel_R}(H)$.

Suppose R is the final interval-valued H-fuzzy relation on W with respect to $(e_\alpha)_\Gamma$. By the process of the proof of Theorem 2.6 in [9], we can easily see that $R_W = R$. □

Hence, by Proposition 3.4, Theorems 3.6(2) and 3.8, we obtain the following result.

Theorem 3.9. $\mathbf{IVRel_R}(H)$ *is a topological universe over* **Set**. *Hence* $\mathbf{IVRel_R}(H)$ *is a concrete quasitopos in the sense of E.J. Dubuc [4].*

Theorem 3.10. $\mathbf{IVRel_R}(H)$ *has exponential objects. Hence* $\mathbf{IVRel_R}(H)$ *is Cartesian closed over* **Set**.

Proof. For any $\mathbf{X} = (X, R_X), \mathbf{Y} = (Y, R_Y) \in \mathbf{IVRel_R}(H)$, let $Y^X = \hom_{\mathbf{IVRel_R}(H)}(X, Y)$. We define the mapping $R = [R^L, R^U] : Y^X \times Y^X \to D(H)$ as follows: For each $(f, g) \in Y^X \times Y^X$,

$$R^L(f,g) = 1 \qquad \text{if } D(f,g) = \emptyset,$$
$$= \bigwedge_{(x,y)\in D(f,g)} R_Y^U(f(x), g(y)) \qquad \text{if } D(f,g) \neq \emptyset$$

and

$$R^U(f,g) = 1 \qquad \text{if } E(f,g) = \emptyset,$$

$$= \bigwedge_{(x,y)\in E(f,g)} R_Y^U(f(x), g(y)) \qquad \text{if } E(f,g) \neq \emptyset,$$

where $D(f,g) = \{(x,y) \in X \times X : R_X^L(x,y) > R_Y^U(f(x), g(y))\}$ and $E(f,g) = \{(x,y) \in X \times X : R_X^U(x,y) > R_Y^U(f(x), g(y))\}$. Then it is clear that $E(f,g) \neq \emptyset$ if and only if $D(f,g) \neq \emptyset$ for each $(f,g) \in Y^X \times Y^X$. Thus $(Y^X, R) \in$ **IVRel**(H). Since $f : \mathbf{X} \to \mathbf{Y}$ is an **IVRel**(H)-mapping, $D(f,f) = \emptyset = E(f,f)$. So $(Y^X, R) \in \mathbf{IVRel_R}(H)$. Let $\mathbf{Y^X} = (Y^X, R)$. Now we define a mapping $e_{X,Y} : X \times Y^X \to Y$ by $e_{X,Y}(a,f) = f(a)$ for each $(a,f) \in X \times Y^X$. Let $((a,f),(b,g)) \in (X \times Y^X) \times (X \times Y^X)$. Then, by the process of the proof of Remark 2.8 in [9], $(R_X \times R_X)^L \leq R_Y^L \circ e_{X,Y}^2$. Suppose $E(f,g) = \emptyset$. Then:

$$\begin{aligned}(R_X \times R)^U((a,f),(b,g)) &= R_X^U(a,b) \wedge R^U(f,g)\\ &= R_X^U(a,b)\\ &\leq R_Y^U(f(a), g(b))\\ &= R_Y^U(e_{X,Y}(a,f), e_{X,Y}(b,g))\\ &= R_Y^U \circ e_{X,Y}^2((a,f),(b,g)).\end{aligned}$$

Suppose $E(f,g) \neq \emptyset$. Then:

$$\begin{aligned}(R_X \times R)^U((a,f),(b,g)) &= R_X^U(a,b) \wedge R^U(f,g)\\ &= R_X^U(a,b) \wedge [\textstyle\bigwedge_{(x,y)\in E(f,g)} R_Y^U(f(x), g(y))]\\ &\leq R_Y^U(f(a), g(b))\\ &= R_Y^U \circ e_{X,Y}^2((a,f),(b,g)).\end{aligned}$$

In all, $(R_X \times R)^U \leq R_Y^U \circ e_{X,Y}^2$. So $e_{X,Y} : \mathbf{X} \times \mathbf{Y^X} \to \mathbf{Y}$ is an **IVRel**(H)-mapping.

For any $\mathbf{Z} = (Z, R_Z) \in \mathbf{IVRel_R}(H)$, let $h : \mathbf{X} \times \mathbf{Z} \to \mathbf{Y^X}$ be any **IVRel**(H)-mapping. Define $\overline{h} : Z \to Y^X$ by $[\overline{h}(c)](a) = h(a,c)$ for each $c \in Z$ and each $a \in X$. Let $c \in Z$ and let $a, b \in X$. Then, by the process of the proof of Remark 2.8 in [9], we can see that, $R_X^L \leq R_Y^L \circ [\overline{h}(c)]^2$ and $R_X^U \leq R_Y^U \circ [\overline{h}(c)]^2$. Thus $\overline{h}(c) : \mathbf{X} \to \mathbf{Y}$ is an **IVRel**(H)-mapping for each $c \in Z$. So $\overline{h}$ is well-defined. Now let $c, c' \in Z$. Then, by the process of the proof of Remark 2.8 in [9], we also can easily see that $R_Z^L \leq R^L \circ \overline{h}^2$ and $R_Z^U \leq R^U \circ \overline{h}^2$. Hence $\overline{h}$ is an **IVRel**(H)-mapping. Moreover, $\overline{h}$ is unique and $e_{X,Y} \circ (1_X \times \overline{h}) = h$. This completes the proof. □

Remark 3.11. (1) In [16], Y. Noh obtained exponential objects in $\mathbf{Rel_R}(I)$, where $I = [0,1]$. In Theorem 3.10, we showed that the construction of an exponential object in $\mathbf{Rel_R}(I)$ is applicable to the case of $\mathbf{IVRel_R}(H)$.

(2) We note that exponential objects in $\mathbf{IVRel_R}(H)$ are quite different from those in **IVRel**(H) constructed in Theorem 3.9 in [12].

(3) $\mathbf{IVRel_R}(H)$ has no subobject classifier.

Example 3.12. Let $H = \{0, 1\}$ be the two points chain and let $X = \{a, b\}$. Let R_1 and R_2 be the interval-valued H-fuzzy reflexive relations on X given by:

$$R_1(a,a) = R_1(b,b) = \mathbf{1}, \quad R_1(a,b) = R_1(b,a) = \mathbf{0},$$

$$R_2(a,a) = R_2(b,b) = \mathbf{1}, \quad R_2(a,b) = R_2(b,a) = \mathbf{0}.$$

Let $id_X : (X, R_1) \to (X, R_2)$ be the identity mapping. Then clearly id_X is both a monomorphism and an epimorphism in $\mathbf{IVRel_R}(H)$. But id_X is not an isomorphism in $\mathbf{IVRel_R}(H)$. Hence $\mathbf{IVRel_R}(H)$ has no subobject classifier (see [3]).

4. Subcategory of $\mathbf{IVRel_R(H)}$

We introduce some subcategories of $\mathbf{IVRel_R}(H)$ which are topological universes over **Set**.

Definition 4.1. Let R be an interval-valued H-fuzzy relation on a set X. Then R is said to be:

(i) *symmetric* if $\mu_R(x,y) = \mu_R(y,x)$ and $\nu_R(x,y) = \nu_R(y,x)$ for each $x, y \in X$.

(ii) *transitive* if $R^L \circ R^L \leq R^L$ and $R^U \circ R^U \leq R^U$, where $R^L \circ R^L(x,y) = \bigvee_{z \in X}[R^L(x,z) \wedge R^L(z,y)]$ and $R^U \circ R^U(x,y) = \bigvee_{z \in X}[R^U(x,z) \wedge R^U(z,y)]$ for any $(x,y) \in X \times X$.

(iii) an *interval-valued H-fuzzy proximity relation* if it is reflexive and symmetric.

(iv) an *interval-valued H-fuzzy preorder relation* if it is reflexive and transitive.

(v) an *interval-valued H-fuzzy equivalence relation* if it is reflexive, symmetric and transitive.

Notation 4.1. (1) $\mathbf{IVRel_S}(H)$ denotes the full subcategory of $\mathbf{IVRel}(H)$ determined by all interval-valued H-fuzzy symmetric relational spaces.

(2) $\mathbf{IVRel_T}(H)$ denotes the full subcategory of $\mathbf{IVRel}(H)$ determined by all interval-valued H-fuzzy transitive relational spaces.

(3) $\mathbf{IVRel_{PR}}(H) = \mathbf{IVRel_R}(H) \cap \mathbf{IVRel_S}(H)$ denotes the full subcategory of $\mathbf{IVRel_R}(H)$ determined by all interval-valued H-fuzzy proximity relational spaces.

(4) $\mathbf{IVRel_P}(H) = \mathbf{IVRel_R}(H) \cap \mathbf{IVRel_T}(H)$ denotes the full subcategory of $\mathbf{IVRel_R}(H)$ determined by all interval-valued H-fuzzy preorder relational spaces.

(5) $\mathbf{IVRel_E}(H) = \mathbf{IVRel_R}(H) \cap \mathbf{IVRel_S}(H) \cap \mathbf{IVRel_T}(H)$ denotes the full subcategory of $\mathbf{IVRel_R}(H)$ determined by all interval-valued H-fuzzy equivalence relational spaces.

It is easy to show that the following result holds.

Proposition 4.2. *The category* $\mathbf{IVRel_{PR}}(H)$*(resp.* $\mathbf{IVRel_P}(H)$ *and* $\mathbf{IVRel_E}(H)$*) is properly fibred over* **Set**.

Lemma 4.3. $\mathbf{IVRel_{PR}}(H)$*(resp.* $\mathbf{IVRel_P}(H)$ *and* $\mathbf{IVRel_E}(H)$*) is closed under the formation of initial sources in* $\mathbf{IVRel_R}(H)$.

Proof. Let $(f_\alpha : (X, R) \to (X_\alpha, R_\alpha))_\Gamma$ be any initial source in $\mathbf{IVRel_R}(H)$, for each (X_α, R_α) belongs to $\mathbf{IVRel_{PR}}(H)$ (resp. $\mathbf{IVRel_P}(H)$ and $\mathbf{IVRel_E}(H)$). Then clearly, by the definition of R, R is reflexive and symmetric. Thus it is enough to show that R is transitive. By the process of the proof of Lemma 3.3 in [9], we can see that $R^L \circ R^L \leq R^L$ and $R^U \circ R^U \leq R^U$.

This completes the proof. □

From Result 2.B, Result 2.C and Lemma 4.3, we obtain the following result.

Theorem 4.4. $\mathbf{IVRel_{PR}}(H)$, $\mathbf{IVRel_P}(H)$ *and* $\mathbf{IVRel_E}(H)$ *are bireflective subcategories of* $\mathbf{IVRel_R}(H)$ *and hence topological categories over* **Set**.

Theorem 4.5. $\mathbf{IVRel_{PR}}(H)$, $\mathbf{IVRel_P}(H)$ *and* $\mathbf{IVRel_E}(H)$ *are closed under the formation of final structures in* $\mathbf{IVRel_R}(H)$ *and hence all of them are bicoreflective subcategories of* $\mathbf{IVRel_R}(H)$.

Proof. Let $(f_\alpha : (X_\alpha, R_\alpha) \to (X, R))_\Gamma$ be any final sink in $\mathbf{IVRel_R}(H)$ such that each (X_α, R_α) belongs to $\mathbf{IVRel_{PR}}(H)$ (resp. $\mathbf{IVRel_P}(H)$ and $\mathbf{IVRel_E}(H)$). By the definition of R, R is reflexive and symmetric. Thus, it is enough to show that R is transitive. By the process of the proof of Theorem 3.5 in [9], we can easily see that, $R^L \circ R^L \leq R^L$ and $R^U \circ R^U \leq R^U$. Hence R is an interval-valued H-fuzzy transitive relation on X. This completes the proof. □

By the similar argument as the process of the proof of Theorem 2.10 in [9], we can easily show the following result.

Lemma 4.6. $\mathbf{IVRel_S}(H)$ *(resp.* $\mathbf{IVRel_T}(H)$*) is closed under the formation of pullbacks in* $\mathbf{IVRel}(H)$.

From Theorems 3.8, and 2.5 and Lemma 4.6, we obtain the following result.

Theorem 4.7. *Final episinks in* $\mathbf{IVRel_{PR}}(H)$ *(resp.* $\mathbf{IVRel_P}(H)$ *and*

$\mathbf{IVRel_E}(H)$) *are preserved by pullbacks.*

By Theorems 4.4 and 2.7, we obtain the following result.

Theorem 4.8. $\mathbf{IVRel_{PR}}(H)$ *(resp.* $\mathbf{IVRel_P}(H)$ *and* $\mathbf{IVRel_E}(H)$*) is topological universe over* **Set**. *Hence each category is a concrete quasitopos in the sense of E.J. Dubuc [4].*

Lemma 4.9. $\mathbf{IVRel_{PR}}(H)$ *has exponential objects. Hence* $\mathbf{IVRel_{PR}}(H)$ *is Cartesian closed over* **Set**.

Proof. For any $\mathbf{X} = (X, R_X)$, $\mathbf{Y} = (Y, R_Y) \in \mathbf{IVRel_{PR}}(H)$, let R be the interval-valued H-fuzzy reflexive relation on $Y^X = \hom_{\mathbf{IVRel_R}(H)}(X, Y)$ defined in the process of the proof of Theorem 3.10. Let $f, g \in Y^X$. Since R_X and R_Y are symmetric, $D(f,g) = D(g,f)$ and $E(f,g) = E(g,f)$. Thus, by the definition of R, $\mu_R(f,g) = \mu_R(g,f)$ and $\nu_R(f,g) = \nu_R(g,f)$. Hence R is an interval-valued H-fuzzy proximity relation on Y^X. □

Theorem 4.10. $\mathbf{IVRel_E}(H)$ *is Cartesian closed over* **Set**.

Proof. For any $\mathbf{X} = (X, R_X)$, $\mathbf{Y} = (Y, R_Y) \in \mathbf{IVRel_E}(H)$, let R be the intuitionistic H-fuzzy reflexive relation on $Y^X = \hom_{\mathbf{IVRel_R}(H)}(X, Y)$ defined in the process of the proof of Theorem 3.10. Since $\mathbf{IVRel_E}(H)$ is a full isomorphism closed subcategory of $\mathbf{IVRel_R}(H)$ and R is symmetric by Lemma 4.9, it is sufficient to show that R is transitive, i.e., for any $f, h \in Y^X$

$$R^L(f,g) \wedge R^L(g,h) \leq R^L(f,h) \quad \text{for each} \quad g \in Y^X, \tag{$*$}$$

and

$$R^U(f,g) \wedge R^U(g,h) \leq R^U(f,h) \quad \text{for each} \quad g \in Y^X. \tag{$**$}$$

Let $f, g, h \in Y^X$. We consider the four cases:

(i) $D(f,g) = \emptyset, D(g,h) = \emptyset;\ E(f,g) = \emptyset, E(g,h) = \emptyset.$

(ii) $D(f,g) \neq \emptyset, D(g,h) = \emptyset;\ E(f,g) \neq \emptyset, E(g,h) = \emptyset.$

(iii) $D(f,g) = \emptyset, D(g,h) \neq \emptyset;\ E(f,g) = \emptyset, E(g,h) \neq \emptyset.$

(iv) $D(f,g) \neq \emptyset, D(g,h) \neq \emptyset;\ E(f,g) \neq \emptyset, E(g,h) \neq \emptyset.$

Case (i). Suppose $D(f,g) = \emptyset, D(g,h) = \emptyset;\ E(f,g) = \emptyset$ and $E(g,h) = \emptyset$. Let $(x,y) \in X \times X$. Since R_X is symmetric and $g \in Y^X$,

$$R^L(x,y) = R^L(y,x) \leq R^L_Y(g(y), g(x)).$$

Since $D(f,g) = \emptyset$ and R^L_Y is transitive,

$$R^L_X(x,y) \leq R^Y_L(f(x), g(y)) \wedge R^L_Y(g(y), g(x)) \leq R^L_Y(f(x), g(x)).$$

Since $D(g,h) = \emptyset$ and R^L_Y is transitive,

$$R^L_X(x,y) \leq R^L_Y(f(x), g(x)) \wedge R^L_Y(g(x), h(y)) \leq R^L_Y(f(x), h(y)).$$

Thus $D(f,h)=\emptyset$. By the definition of R, $R^L(f,g)\wedge R^L(g,h)=1=R^L(f,h)$. So (*) holds. Similarly, we can see that (**) holds.

Case (ii). Suppose $D(f,g)\neq\emptyset, D(g,h)=\emptyset$; $E(f,g)\neq\emptyset$ and $E(g,h)=\emptyset$. Then $D(f,h)\subset D(f,g)$ and $E(f,h)\subset E(f,g)$: Let $(a,b)\notin D(f,g)$. Then

$$R_X^L(a,b)\leq R_Y^L(f(a),g(b)).$$

Since $g\in Y^X$ and R_Y is symmetric,

$$R_X^L(a,b)\leq R_Y^L(g(a),g(b))=R_Y^L(g(b),g(a)).$$

Since R_Y^L is transitive,

$$R_X^L(a,b)\leq R_Y^L(f(a),g(b))\wedge R_Y^L(g(b),g(a))\leq R_Y^L(f(a),g(a)).$$

Since $D(g,h)=\emptyset$,

$$R_X^L(a,b)\leq R_Y^L(g(a),h(b)).$$

Since R_Y is transitive,

$$R_X^L(a,b)\leq R_Y^L(f(a),g(b))\wedge R_Y^L(g(a),h(b))\leq R_Y^L(f(a),h(b)).$$

Thus $(a,b)\notin D(f,h)$. So $D(f,h)\subset D(f,g)$. By the similar argument as the above proof, we can see that $E(f,h)\subset E(f,g)$.

Let $(x,z)\in D(f,g)$. Then $R_X^L(x,y)>R_Y^L(f(x),g(y))$. Since $D(g,h)=\emptyset$, $R_X^L(x,y)\leq R_Y^L(g(x),h(y))$. Since $g\in Y^X$ and R_Y is symmetric and transitive,

$$\begin{aligned}R_X^L(x,y)&\leq R_Y^L(g(x),g(y))\wedge R_Y^L(g(x),h(y))\\&=R_Y^L(g(y),g(x))\wedge R_Y^L(g(x),h(y))\\&\leq R_Y^L(g(y),h(y)).\end{aligned}$$

Thus $R_Y^L(f(x),g(y))<R_X^L(x,y)\leq R_Y^L(g(y),h(y))$. So

$$\begin{aligned}R_Y^L(f(x),g(y))&=R_Y^L(f(x),g(y))\wedge R_Y^L(g(y),h(y))\\&\leq R_Y^L(f(x),h(y))\text{ for each }(x,y)\in D(f,g).\end{aligned}$$

Since $D(f,h)\subset D(f,g)$,

$$\begin{aligned}R^L(f,g)\wedge R^L(g,h)&=[\textstyle\bigwedge_{(x,y)\in D(f,g)}R_Y^L(f(x),g(y))]\wedge 1\\&=\textstyle\bigwedge_{(x,y)\in D(f,g)}R_Y^L(f(x),g(y))\\&\leq\textstyle\bigwedge_{(x,y)\in D(f,g)}R_Y^L(f(x),h(y))\\&\leq\textstyle\bigwedge_{(x,y)\in D(f,h)}R_Y^L(f(x),h(y))\\&=R^L(f,h).\end{aligned}$$

So (*) holds. Similarly, we can see that (**) holds.

Case (iii). Suppose $D(f,g)=\emptyset, D(g,h)\neq\emptyset$; $E(f,g)=\emptyset$ and $E(g,h)\neq\emptyset$. Then it is dual of Case (ii).

Case (iv). Suppose $D(f,g) \neq \emptyset, D(g,h) \neq \emptyset$; $E(f,g) \neq \emptyset$ and $E(g,h) \neq \emptyset$. Then we can show that $D(f,h) \subset D(f,g) \cup D(g,h)$ and $E(f,h) \subset E(f,g) \cup E(g,h)$ by the similar argument as proof in Case (ii).

Suppose $(a,b) \in D(f,g) \cap D(g,h)$. Then, by the definition of R,

$$R^L(f,g) \wedge R^L(g,h) \leq R^L_Y(f(a),g(b))) \wedge R^L_Y(g(a),h(b)).$$

Suppose $(a,b) \in D(f,g) - D(g,h)$. Then,

$$R^L_Y(f(a),g(b)) < R^L_X(a,b) \leq R^L_Y(g(a),h(b)).$$

Thus, by the definition of R,

$$\begin{aligned} R^L(f,g) \wedge R^L(g,h) &= [\textstyle\bigwedge_{(x,y)\in D(f,g)} R^L_Y(f(x),g(y))] \\ &\leq R^L_Y(f(a),g(b)) \\ &= R^L_Y(f(a),g(b)) \wedge R^L_Y(g(a),h(b)). \end{aligned}$$

Suppose $(a,b) \in D(g,h) - D(f,g)$. Then, by the similar way,

$$R^L(f,g) \wedge R^L(g,h) \leq R^L_Y(f(a),g(b)) \wedge R^L_Y(g(a),h(b))$$

In all, for any $(a,b) \in D(f,g) \cup D(g,h)$.

$$R^L(f,g) \wedge R^L(g,h) \leq R^L_Y(f(x),g(y)) \wedge R^L_Y(g(x),h(y)).$$

Now let $(a,b) \in D(f,g) \cup D(g,h)$. Then:

$$R^L_Y(f(x),g(y)) < R^L_Y(x,y) \text{ or } R^L_Y(g(x),h(y)) < R^L_X(x,y).$$

Since $g \in Y^X$ and R_Y is symmetric and transitive,

$$\begin{aligned} & R^L_Y(f(x),g(x)) \wedge R^L_Y(g(x),h(y)) \\ &= R^L_Y(f(x),g(y)) \wedge R^L_Y(g(x),h(y)) \wedge R^L_X(x,y) \\ &\leq R^L_Y(f(x),g(x)) \wedge R^L_Y(g(x),h(y)) \wedge R^L_Y(g(x),g(y)) \\ &= R^L_Y(f(x),g(x)) \wedge R^L_Y(g(y),g(x)) \wedge R^L_Y(g(x),h(y)) \\ &\leq R^L_Y(f(x),g(y)) \wedge R^L_Y(g(y),h(y)) \\ &\leq R^L_Y(f(x),h(y)). \end{aligned}$$

Thus $$\begin{aligned} R^L(f,g) \wedge R^L(g,h) &\leq \textstyle\bigwedge_{(x,y)\in D(f,g)\cup D(g,h)} R^L_Y(f(x),h(y)) \\ &\leq \textstyle\bigwedge_{(x,y)\in D(f,h)} R^L_Y(f(x),h(y)) \\ &= R^L_Y(f,h). \end{aligned}$$

By the similar argument, we can see that:

$$R^U(f,g) \wedge R^U(g,h) \leq R^U(f,h).$$

In all, (*) and (**) hold. Hence R is transitive. This completes the proof. □

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