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SOME SUFFICIENT CONDITIONS FOR INTEGRAL OPERATORS DEFINED BY HYPERGEOMETRIC FUNCTIONS

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Abstract: In the present paper, we consider some sufficient conditions for generalized integral operators defined by hypergeometric functions to be starlike, close-to-convex and uniformly convex functions defined in the open unit disk.

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1. Introduction and Preliminaries

Let A denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad z \in U, \tag{1.1}$$

which are analytic in the open unit disk $U = \{z : z \in C \text{ and } |z| < 1\}$. Let S denote the subclass of A consisting of all functions which are univalent in U. We also denoted by $S^*(\alpha)$, $C(\alpha)$ and $K(\alpha)$, the familiar subclasses of A consisting of functions which are, respectively, starlike, convex, close-to-convex of order α $(0 \leq \alpha < 1)$ in U and uniformly convex function. Thus, we have (see, for details, [5], [8]; see also [9], [14])

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$$S^*(\alpha) := \left\{ f : f \in A \text{ and } \Re\left(\frac{zf'(z)}{f(z)}\right) > \alpha, \quad (z \in U\,;\, 0 \le \alpha < 1) \right\},$$

$$C(\alpha) := \left\{ f : f \in A \text{ and } \Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > \alpha, \ (z \in U\,;\, 0 \le \alpha < 1) \right\},$$

$$K(\alpha) := \left\{ f : f \in A \text{ and } \Re\left(\frac{f'(z)}{g'(z)}\right) > \alpha, \ (z \in U\,;\, 0 \le \alpha < 1;\ g \in C) \right\},$$

and

$$UCV := \left\{ f : f \in A \text{ and } \Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > \left| \frac{zf''(z)}{f'(z)} \right|, \ (z \in U) \right\}.$$

The Hadamard product of two functions $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ and $g(z) = z + \sum_{n=2}^{\infty} a_n z^n$ in A is given by $(f * g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n$.

For complex parameters $\alpha_1, \alpha_2, ..., \alpha_q$ and $\beta_1, \beta_2, ..., \beta_s (\beta_j \in C \backslash z_0^-; z_0^- = 0, -1, -2, ..., \ j = 1, ..., s)$, we define the generalized hypergeometric function ${}_qf_s(\alpha_1, ..., \alpha_q; \beta_1, ..., \beta_s; z)$ by

$${}_qf_s(\alpha_1, ..., \alpha_q; \beta_1, ..., \beta_s; z) = \sum_{n=0}^{\infty} \frac{(\alpha_1)_n \cdots (\alpha_q)_n}{(\beta_1)_n \cdots (\beta_s)_n} \frac{z^n}{n!}$$

$$(q \le s + 1; q, s \in N_0 = N \cup \{0\}\,;\ z \in U),$$

where N denotes the set of positive integral and $(x)_k$ is the Pochhamer symbol defined in terms of Gamma function Γ by

$$(x)_k = \frac{\Gamma(x+k)}{\Gamma(x)} = \begin{cases} 1, & \text{if } k = 0, \\ x(x+1)(x+2)....(x+k-1), & \text{if } k \in N = \{1, 2, ..., n\}. \end{cases}$$

Corresponding to a function $h(\alpha_1, ..., \alpha_q; \beta_1, ..., \beta_s; z) = z_q f_s(\alpha_1, ..., \alpha_q; \beta_1, ..., \beta_s; z)$. The Dziok-Srivastava operator [6] (see also [4], [12]) $H_{q,s}(\alpha_1, ..., \alpha_q; \beta_1, ..., \beta_s)$ is defined by the Hadamard product:

$$\begin{aligned} H_{q,s}(\alpha_1, ..., \alpha_q; \beta_1, ..., \beta_s) f(z) &:= h(\alpha_1, ..., \alpha_q, \beta_1, ..., \beta_s; z) * f(z) \\ &= z + \sum_{n=0}^{\infty} \frac{(\alpha_1)_{n-1} \cdots (\alpha_q)_{n-1}}{(\beta_1)_{n-1} \cdots (\beta_s)_{n-1}} \frac{a_n z^n}{(n-1)!}. \end{aligned} \tag{1.2}$$

For brevity, we write

$$H_{q,s} f(z) := H_{q,s}(\alpha_1, ..., \alpha_q; \beta_1, ..., \beta_s) f(z).$$

Lemma 1.1. (see [15]) *If* $f \in A$ *satisfies*

$$\Re\left\{1+\frac{zf''(z)}{f'(z)}\right\}<\frac{3}{2} \quad (z\in U),$$

then $f \in S^*$.

Lemma 1.2. (see [10]) *If* $f \in A$ *satisfies*

$$\left|1+\frac{zf''(z)}{f'(z)}\right|<2 \quad (z\in U),$$

then $f \in S^*$.

Lemma 1.3. (see [13]) *If* $f \in A$ *satisfies*

$$\left|\frac{zf''(z)}{f'(z)}\right|<\frac{1}{2} \quad (z\in U),$$

then $f \in UCV$.

Lemma 1.4. (see [11]) *If* $f \in A$ *satisfies*

$$\Re\left(1+\frac{zf''(z)}{f'(z)}\right)>\frac{1+3\alpha}{2(1+\alpha)} \quad (z\in U;\ 0\le\alpha<1),$$

then

$$\Re\left\{f'(z)\right\}>\frac{1+\alpha}{2} \qquad (z\in U;\ 0\le\alpha<1),$$

or equivalently,

$$f\in C\left(\frac{1+\alpha}{2}\right) \quad (0\le\alpha<1).$$

Lemma 1.5. (see [11]) *If* $f \in A$ *satisfies*

$$\Re\left(1+\frac{zf''(z)}{f'(z)}\right)<\frac{3+2\alpha}{2+\alpha} \quad (z\in U;\ 0\le\alpha<1),$$

then

$$\Re\left\{f'(z)\right\}>-\alpha \quad (z\in U;\ 0\le\alpha<1)\ .$$

By using Dziok-Srivastava operator given by (1.2), we introduce the following integral operator:

Definition 1.6. Let $q, s, m \in N_0 = N\cup\{0\}$, $\gamma_i \in C$, $0 \le \delta_i \le 1$, $i=\{1,2,...,m\}$. Define the integral operator $G_{\gamma_1\gamma_2,...,\gamma_m} : A^m \to A$,

$$G_{\gamma_1\gamma_2,...,\gamma_m}(z)=\int_0^z\left[(1-\delta_1)\frac{H_{q,s}f_1(t)}{t}+\delta_1\left(H_{q,s}f_1(t)\right)'\right]^{\gamma_1}$$

$$... \left[(1-\delta_m) \frac{H_{q,s}f_m(t)}{t} + \delta_m \left(H_{q,s}f_m(t)\right)' \right]^{\gamma_m} dt, \quad z \in U, \quad (1.3)$$

where $f_i \in A$, $i = \{1, 2, ..., m\}$ and $H_{q,s}$ is the Dziok-Srivastava operator.

Remarks. (i) For $q = 2$, $s = 1$, $\alpha_1 = \beta_1$, $\alpha_2 = 1$, $m = 1$, $\gamma_1 = 1$, $\gamma_2, ..., \gamma_m = 0$,

$$H_{q,s}f(z) = f(z) \in A.$$

(ii) For $q = 2$, $s = 1$, $\alpha_1 = \beta_1$, $\alpha_2 = 1$, $\delta_i = 0$, $i = \{1, 2, ..., m\}$, $m \in N \cup \{0\}$ we obtain the integral operator

$$F(z) = \int_0^z \left(\frac{f_1(t)}{t}\right)^{\gamma_1} .... \left(\frac{f_m(t)}{t}\right)^{\gamma_m} dt$$

introduced and studied by Breaz and Breaz in [1]. In [2] Breaz and Guney considered this integral operator and obtained their properties on the classes $S^*_\alpha(b)$, $C_\alpha(b)$ of starlike and convex functions of complex order b, and type α introduced and studied dy the author [7].

(iii) For $q = 2$, $s = 1$, $\alpha_1 = \beta_1$, $\alpha_2 = 1$, $\gamma_i > 0$, $\delta_i = 0$, $i = \{1, 2, ..., m\}$, $m \in N \cup \{0\}$, we obtain the integral operator

$$F_{\alpha_1,...,\alpha_m}(z) = \int_0^z \left(f_1'(t)\right)^{\gamma_1} .... \left(f_m'(t)\right)^{\gamma_m} dt,$$

recently introduced and studied by Breaz, S. Owa, and N. Breaz in [3].

2. Sufficient Condition for the Integral Operator $G_{\gamma_1\gamma_2,...,\gamma_m}$ when $\delta_i = 0$ for $i = \{1, 2, ..., m\}$

Theorem 2.1. *Let $\gamma_i > 0$, $\delta_i = 0$, $i = \{1, 2, ..., m\}$. If $f_i \in A$ for $i = \{1, 2, ..., m\}$ satisfies*

$$\Re\left[\frac{z\left(H_{q,s}f_i(z)\right)'}{H_{q,s}f_i(z)}\right] < 1 + \frac{1}{2\sum_{i=1}^{m}\gamma_i}, \quad (2.1)$$

then $G_{\gamma_1\gamma_2,...,\gamma_m}(z) \in S^*$.

Proof. From (2.1), we have

$$G'_{\gamma_1\gamma_2,...,\gamma_m}(z) = \left(\frac{H_{q,s}f_1(z)}{z}\right)^{\gamma_1} .... \left(\frac{H_{q,s}f_m(z)}{z}\right)^{\gamma_m} \quad (2.2)$$

and

$$\log G'_{\gamma_1\gamma_2,...,\gamma_m}(z) = \gamma_1 \log\left(\frac{H_{q,s}f_1(z)}{z}\right) + ... + \gamma_m \log\left(\frac{H_{q,s}f_m(z)}{z}\right). \quad (2.3)$$

By differentiating (2.2), we have

$$\frac{G''_{\gamma_1\gamma_2,...,\gamma_m}(z)}{G'_{\gamma_1\gamma_2,...,\gamma_m}(z)} = \gamma_1\left[\frac{(H_{q,s}f_1(z))'}{H_{q,s}f_1(z)} - \frac{1}{z}\right] + ...\gamma_m\left[\frac{(H_{q,s}f_m(z))'}{H_{q,s}f_m(z)} - \frac{1}{z}\right].$$

By multiplying the rotation (2.2) with z, we obtain

$$\frac{zG''_{\gamma_1\gamma_2,...,\gamma_m}(z)}{G'_{\gamma_1\gamma_2,...,\gamma_m}(z)} = \sum_{i=1}^{m}\gamma_i\left[\frac{z(H_{q,s}f_i(z))'}{H_{q,s}f_i(z)} - 1\right],$$

or, equivalently

$$1 + \frac{zG''_{\gamma_1\gamma_2,...,\gamma_m}(z)}{G'_{\gamma_1\gamma_2,...,\gamma_m}(z)} = \sum_{i=1}^{m}\gamma_i\left[\frac{z(H_{q,s}f_i(z))'}{H_{q,s}f_i(z)}\right] + 1 - \sum_{i=1}^{m}\gamma_i. \quad (2.4)$$

Taking the real part of both terms of (2.4), we have

$$\Re\left(1 + \frac{zG''_{\gamma_1\gamma_2,...,\gamma_m}(z)}{G'_{\gamma_1\gamma_2,...,\gamma_m}(z)}\right) = \sum_{i=1}^{m}\gamma_i\Re\left[\frac{z(H_{q,s}f_i(z))'}{H_{q,s}f_i(z)}\right] + 1 - \sum_{i=1}^{m}\gamma_i. \quad (2.5)$$

Using (2.5) and (2.2), we obtain

$$\Re\left(1 + \frac{zG''_{\gamma_1\gamma_2,...,\gamma_m}(z)}{G'_{\gamma_1\gamma_2,...,\gamma_m}(z)}\right) < \frac{3}{2}.$$

Hence by Lemma 1.1, we get $G_{\gamma_1\gamma_2,...,\gamma_m} \in S^*$. This completes the proof. □

Taking $m = 1$ in Theorem 2.3, we have

Corollary 2.2. *Let* $\gamma_1 > 0$ *If* $f_1(z) \in A$ *and*

$$\Re\left[\frac{z(H_{q,s}f_1(z))'}{H_{q,s}f_1(z)}\right] < 1 + \frac{1}{2\gamma_1},$$

then $G_{\gamma_1} \in S^*$.

Theorem 2.3. *Let* $\gamma_i > 0$, $\delta_i = 0$, $i = \{1, 2, ..., m\}$. *If* $f_i \in A$ *for* $i = \{1, 2, ..., m\}$ *satisfies*

$$\left|\frac{z(H_{q,s}f_i(z))'}{H_{q,s}f_i(z)} - 1\right| < \frac{1}{\sum_{i=1}^{m}\gamma_i}, \quad (2.6)$$

then $G_{\gamma_1\gamma_2,...,\gamma_m} \in S^*$.

Proof. From (2.5), we have

$$1+\frac{zG''_{\alpha_1\alpha_2,\dots,\alpha_m}(z)}{G'_{\alpha_1\alpha_2,\dots,\alpha_m}(z)}=\sum_{i=1}^{m}\gamma_i\left[\frac{z\left(H_{q,s}f_i(z)\right)'}{H_{q,s}f_i(z)}-1\right]+1, \tag{2.7}$$

and hence

$$\left|1+\frac{zG''_{\alpha_1\alpha_2,\dots,\alpha_m}(z)}{G'_{\alpha_1\alpha_2,\dots,\alpha_m}(z)}\right|\le\sum_{i=1}^{m}|\gamma_i|\left|\frac{z\left(H_{q,s}f_i(z)\right)'}{H_{q,s}f_i(z)}-1\right|+1. \tag{2.8}$$

Using (2.6), (2.7) and applying Lemma 1.2 we get $G_{\gamma_1\gamma_2,\dots,\gamma_m}\in S^*$. □

Taking $m=1$ in Theorem 2.5, we have

Corollary 2.4. *Let $\gamma_1>0$. If $f_1(z)\in A$ and*

$$\left|\frac{z\left(H_{q,s}f_1(z)\right)'}{H_{q,s}f_1(z)}-1\right|<\frac{1}{\gamma_1},$$

then $G_{\gamma_1}\in S^*$.

Applying Lemma 1.3 and using (2.8), we easily get

Theorem 2.5. *Let $\gamma_i>0$, $\delta_i=0$, $i=\{1,2,...,m\}$. If $f_i\in A$ for $i=\{1,2,...,m\}$ satisfies*

$$\left|\frac{z\left(H_{q,s}f_i(z)\right)'}{H_{q,s}f_i(z)}-1\right|<\frac{1}{2\sum\limits_{i=1}^{m}\gamma_i}, \tag{2.9}$$

then $G_{\gamma_1\gamma_2,\dots,\gamma_m}\in UCV$.

Taking $m=1$ in Theorem 2.7, we have

Corollary 2.6. *Let $\gamma_1>0$. If $f_1(z)\in A$ and*

$$\left|\frac{z\left(H_{q,s}f_1(z)\right)'}{H_{q,s}f_1(z)}-1\right|<\frac{1}{2\gamma_1},$$

then $G_{\gamma_1}\in UCV$.

Theorem 2.7. *Let $\gamma_i>0$, $\delta_i=0$, $i=\{1,2,...,m\}$. If $f_i\in A$ for $i=\{1,2,...,m\}$ satisfies*

$$\Re\left[\frac{z\left(H_{q,s}f_i(z)\right)'}{H_{q,s}f_i(z)}\right]>\frac{-1+\alpha+2(1+\alpha)\sum\limits_{i=1}^{m}\gamma_i}{2(1+\alpha)\sum\limits_{i=1}^{m}\gamma_i}, \tag{2.10}$$

then $G_{\gamma_1\gamma_2,\dots,\gamma_m}\in C\left(\frac{1+\alpha}{2}\right)$, where $0\le\alpha<1$.

Proof. As in the proof of Theorem 2.3. From (2.6) and (2.11), we obtain

$$\Re\left(1+\frac{zG''_{\alpha_1\alpha_2,...,\alpha_m}(z)}{G'_{\alpha_1\alpha_2,...,\alpha_m}(z)}\right)>\frac{1+3\alpha}{2(1+\alpha)}.$$

Hence by Lemma 1.4 we get $G_{\gamma_1\gamma_2,...,\gamma_m}\in C\left(\frac{1+\alpha}{2}\right),\ (0\le\alpha<1)$. □

Taking $m=1$ in Theorem 2.9, we have

Corollary 2.8. *Let* $\gamma_1>0$ *If* $f_1(z)\in A$ *and*

$$\Re\left[\frac{z\left(H_{q,s}f_1(z)\right)'}{H_{q,s}f_1(z)}\right]>\frac{-1+\alpha+2(1+\alpha)\gamma_1}{2(1+\alpha)\gamma_1},$$

then $G_{\gamma_1}\in C\left(\frac{1+\alpha}{2}\right)$, *where* $0\le\alpha<1$.

Using Lemma 1.5 and applying similar proof as in Theorem 2.9, we obtain

Theorem 2.9. *Let* $\gamma_i>0$, $\delta_i=0$, $i=\{1,2,...,m\}$. *If* $f_i\in A$ *for* $i=\{1,2,...,m\}$ *satisfies*

$$\Re\left[\frac{z\left(H_{q,s}f_i(z)\right)'}{H_{q,s}f_i(z)}\right]<\frac{1+\alpha+2(1+\alpha)\sum\limits_{i=1}^{m}\gamma_i}{2(1+\alpha)\sum\limits_{i=1}^{m}\gamma_i},$$

then

$$\Re\left\{G'_{\gamma_1\gamma_2,...,\gamma_m}(z)\right\}>-\alpha\quad(z\in U;0\le\alpha<1).$$

Taking $m=1$ in Theorem 2.11, we have

Corollary 2.10. *Let* $\gamma_1>0$ *If* $f_1(z)\in A$ *and*

$$\Re\left[\frac{z\left(H_{q,s}f_1(z)\right)'}{H_{q,s}f_1(z)}\right]<\frac{1+\alpha+2(1+\alpha)\gamma_1}{2(1+\alpha)\gamma_1},$$

then $\Re\left\{G'_{\gamma_1}(z)\right\}>-\alpha,\quad(z\in U;0\le\alpha<1)$

3. Sufficient Condition for the Integral Operator $G_{\gamma_1\gamma_2,...,\gamma_m}$ when $\delta_i=1$ for $i=\{1,2,...,m\}$

Applying Lemma 1.1, we drive to

Theorem 3.1. *Let* $\gamma_i>0$, $\delta_i=1$, $i=\{1,2,...,m\}$. *If* $f_i\in A$ *for*

$i = \{1, 2, ..., m\}$ *satisfies*

$$\Re\left[1 + \frac{z\left(H_{q,s}f_1\left(z\right)\right)''}{\left(H_{q,s}f_1\left(z\right)\right)'}\right] < 1 + \frac{1}{2\sum\limits_{i=1}^{m}\gamma_i}, \tag{3.1}$$

then $G_{\gamma_1\gamma_2,...,\gamma_m} \in S^*$.

Proof. As in the proof of Theorem 2.3, we have

$$\frac{G''_{\gamma_1\gamma_2,...,\gamma_m}\left(z\right)}{G'_{\gamma_1\gamma_2,...,\gamma_m}\left(z\right)} = \gamma_1\frac{\left(H_{q,s}f_1\left(z\right)\right)''}{\left(H_{q,s}f_1\left(z\right)\right)'} + ... + \gamma_m\frac{\left(H_{q,s}f_m\left(z\right)\right)''}{\left(H_{q,s}f_m\left(z\right)\right)'}, \tag{3.2}$$

or, equivalently,

$$\frac{G''_{\gamma_1\gamma_2,...,\gamma_m}\left(z\right)}{G'_{\gamma_1\gamma_2,...,\gamma_m}\left(z\right)} = \sum_{i=1}^{m}\gamma_i\left[\frac{z\left(H_{q,s}f_i\left(z\right)\right)''}{\left(H_{q,s}f_i\left(z\right)\right)'}\right]. \tag{3.3}$$

From (3.3), we have

$$\Re\left(1 + \frac{zG''_{\gamma_1,\gamma_2,...,\gamma_m}\left(z\right)}{G'_{\gamma_1,\gamma_2,...,\gamma_m}\left(z\right)}\right) = \sum_{i=1}^{m}\gamma_i\Re\left[1 + \frac{z\left(H_{q,s}f_i\left(z\right)\right)''}{\left(H_{q,s}f_i\left(z\right)\right)'}\right] + 1 - \sum_{i=1}^{m}\gamma_i. \tag{3.4}$$

Using (3.4) and (3.1), we obtain

$$\Re\left(1 + \frac{zG''_{\gamma_1,\gamma_2,...,\gamma_m}\left(z\right)}{G'_{\gamma_1,\gamma_2,...,\gamma_m}\left(z\right)}\right) < \frac{3}{2}.$$

Hence by Lemma 1.1, we get $G_{\gamma_1\gamma_2,...,\gamma_m} \in S^*$. □

Using (4.3), (4.4) and Lemmas 1.2, 1.3, 1.4, and 1.5, we have respectively,

Theorem 3.2. *Let* $\gamma_i > 0$, $\delta_i = 1$, $i = \{1, 2, ..., m\}$. *If* $f_i \in A$ *for* $i = \{1, 2, ..., m\}$ *satisfies*

$$\left|1 + \frac{z\left(H_{q,s}f_i\left(z\right)\right)''}{\left(H_{q,s}f_i\left(z\right)\right)'}\right| < \frac{1}{\sum\limits_{i=1}^{m}\gamma_i} + 1,$$

then $G_{\gamma_1\gamma_2,...,\gamma_m} \in S^*$.

Theorem 3.3. *Let* $\gamma_i > 0$, $\delta_i = 1$, $i = \{1, 2, ..., m\}$. *If* $f_i \in A$ *for* $i = \{1, 2, ..., m\}$ *satisfies*

$$\left|\frac{z\left(H_{q,s}f_i\left(z\right)\right)''}{\left(H_{q,s}f_i\left(z\right)\right)'}\right| < \frac{1}{2\sum\limits_{i=1}^{m}\gamma_i},$$

then $G_{\gamma_1\gamma_2,...,\gamma_m} \in UCV$.

Theorem 3.4. *Let* $\gamma_i > 0$, $\delta_i = 1$, $i = \{1, 2, ..., m\}$. *If* $f_i \in A$ *for*

$i = \{1, 2, ..., m\}$ *satisfies*

$$\Re\left[1 + \frac{z\left(H_{q,s}f_1(z)\right)''}{\left(H_{q,s}f_1(z)\right)'}\right] > \frac{-1 + \alpha + 2(1+\alpha)\sum\limits_{i=1}^{m}\gamma_i}{2(1+\alpha)\sum\limits_{i=1}^{m}\gamma_i},$$

then $G_{\gamma_1\gamma_2,...,\gamma_m} \in C\left(\frac{1+\alpha}{2}\right)$, *where* $0 \le \alpha < 1$.

Theorem 3.5. *Let* $\gamma_i > 0$, $\delta_i = 1$, $i = \{1, 2, ..., m\}$. *If* $f_i \in A$ *for* $i = \{1, 2, ..., m\}$ *satisfies*

$$\Re\left[1 + \frac{z\left(H_{q,s}f_1(z)\right)''}{\left(H_{q,s}f_1(z)\right)'}\right] < \frac{1 + \alpha + (2+\alpha)\sum\limits_{i=1}^{m}\gamma_i}{(2+\alpha)\sum\limits_{i=1}^{m}\gamma_i},$$

then $\Re\left\{G'_{\gamma_1,\gamma_2,...,\gamma_m}(z)\right\} > -\alpha \ (z \in U; 0 \le \alpha < 1)$

Taking $m = 1$ in Theorems 3.1- 3.5, we have respectively

Corollary 3.6. *Let* $\gamma_1 > 0$ *If* $f_1(z) \in A$ *and*

$$\Re\left[1 + \frac{z\left(H_{q,s}f_1(z)\right)''}{\left(H_{q,s}f_1(z)\right)'}\right] < 1 + \frac{1}{2\gamma_1},$$

then $G_{\gamma_1} \in S^*$.

Corollary 3.7. *Let* $\gamma_1 > 0$ *If* $f_1(z) \in A$ *and*

$$\left|1 + \frac{z\left(H_{q,s}f_1(z)\right)''}{\left(H_{q,s}f_1(z)\right)'}\right| < \frac{1}{\gamma_1} + 1,$$

then $G_{\gamma_1} \in S^*$.

Corollary 3.8. *Let* $\gamma_1 > 0$ *If* $f_1(z) \in A$ *and*

$$\left|\frac{z\left(H_{q,s}f_1(z)\right)''}{\left(H_{q,s}f_1(z)\right)'}\right| < \frac{1}{2\gamma_1},$$

then $G_{\gamma_1} \in UCV$.

Corollary 3.9. *Let* $\gamma_1 > 0$ *If* $f_1(z) \in A$ *and*

$$\Re\left[1 + \frac{z\left(H_{q,s}f_1(z)\right)''}{\left(H_{q,s}f_1(z)\right)'}\right] > \frac{-1 + \alpha + 2(1+\alpha)\gamma_1}{2(1+\alpha)\gamma_1},$$

then $G_{\gamma_1} \in C\left(\frac{1+\alpha}{2}\right)$, *where* $0 \le \alpha < 1$.

Corollary 3.10. *Let* $\gamma_1 > 0$ *If* $f_1(z) \in A$ *and*

$$\Re\left[1 + \frac{z\left(H_{q,s}f_1(z)\right)''}{\left(H_{q,s}f_1(z)\right)'}\right] < \frac{1 + \alpha + (2+\alpha)\gamma_1}{(2+\alpha)\gamma_1},$$

then $\Re\left\{G'_{\gamma_1}(z)\right\} > -\alpha\ (z \in U;\ 0 \le \alpha < 1)$.

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References

[1] D. Breaz, N. Breaz, Two integral operator, *Studia Universitatis Babes-Bolyai, Mathematica*, Cluj-Napoca, **3** (2003), 13-21.

[2] D. Breaz, H. Guney, The integral operator on the classes $S^*_\alpha(b)$ and $C_\alpha(b)$ *J. Math. Ineq.*, **2**, No. 1 (2008), 97-100.

[3] D. Breaz, S. Owa, N. Breaz, A new integral univalent operator, *Acta Univ. Apulensis Math. Inform.*, No. 16 (2008), 11-16.

[4] B.C. Carlson, D.B. Shaffer, Starlike and prestarlike hypergeometric functions, *SIAM. J. Math. Anal.*, **15** (2002), 737-745.

[5] P.L. Duren, *Univalent Functions*, Grundlehren der Mathematischen Wissenschaften, **259**, Springer-Verlag, New York (1983).

[6] J. Dziok, H.M. Srivastava, Classes of analytic functions associated with the generalized hypergeometric function, *Appl. Math. Comput.*, **103** (1999), 1-13.

[7] B.A. Frasin, Family of analytic functions of complex order, *Acta. Math. Acad. Paed. Ny.*, **22** (2006), 179-191.

[8] A.W. Goodman, On uniformly convex functions, *Ann. Polon. Math.*, **65** (1991), 87-92.

[9] W. Ma, D. Minda, Uniformly convex functions, *Ann. Polon. Math.*, **57** (1992), 165-175.

[10] S.S. Miller, P.T. Mocanu, On some class of first order differential subordinations, *Michigan Math. J.*, **32** (1985), 185-195.

[11] S. Owa, M. Nunokawa, H. Saitoh, H.M. Srivastava, Close-to-convexity, starlikeness and convexity of certain analytic functions, *Appl. Math. Letters*, **15** (2002), 63-69.

[12] S. Owa, H.M. Srivastava, Univalent and starlike generalized hypergeometric functions, *Univ. Victoria Report*, No. DM.336-IR, 1-27.

[13] V. Ravichandran, On uniformly convex functions, *Ganita*, **53**, No. 2 (2002), 117-124.

[14] F. Ronning, Uniformly convex functions and a corresponding class of starlike functions, *Proc. Amer. Math. Sci.*, **118** (1993), 189-196.

[15] R. Singh, S. Singh, Some sufficient conditions for univalent and starlikeness, *Collq. Math.*, **47** (1982), 309-31.