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TOPOLOGICAL EXTENSIONS IN THE REALM OF B-CONVERGENCE

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Abstract: In 1964 Doitchinov introduced the notion of supertopological spaces in order to construct a unified theory of topological, proximity and uniform spaces. As an application he showed that the compactly determined Hausdorff-extensions of a given topological space X are closely related with a class of supertopologies on X, which he called b-supertopologies. In 1973 Herrlich introduced nearness spaces, which generalize both symmetrical topological spaces and uniform spaces. Bentley showed that those nearness spaces that can be extended to a topological space have an elegant internal characterization, namely that every nearness collection is grill-determined, more precisely, is the subset of some bunch. The concepts mentioned above both are subsumed by the so-called supernearness spaces introduced by myself in 2002. Consequently, their corresponding grill-defined supernear operators are describing these extensions in a common manner. The topological construct **PUCONV** of preuniform convergence spaces introduced by Preußin 1993 plays an important role in the study of strong topological universes, in which "convergence structures" are available, among them the filtermerotopies in the sense of Katetov, generalized convergence spaces, and uniform convergence structures such as quasiuniformities and various generalizations as well. Consequently, natural function spaces exist in such categories (i.e., they are Cartesian closed), quotients are stable under products, and in addition such categories are extensional. Here we only point out that supertopologies and their generalized set-convergences

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introduced by Wyler in 1988 as well as grill-defined supernear operators cannot be subsumed by **PUCONV**. This motivated establishing our broader concept of b-convergence (see IJPAM 2008). Within this concept we will now examine how topological extensions can be described.

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1. Introduction

In this paper we will present a new type of convergence, which generalizes the "classical" ones of set-convergence in the sense of Wyler [10] and of preuniform convergence in the sense of Preuß [9] by bringing them both together.

As a basic concept we consider "uniform filters" converging to bounded subsets, thus defining by suitable axioms the so-called b-convergences. Morphisms between the corresponding spaces are then defined in an obvious way, i.e., they are bounded maps which preserve uniform filters and are called *b-continuous functions*.

Moreover, we have that "topological extensions" are closely related to corresponding b-convergences.

Well-known "topological extensions" in the literature are the Smirnov-compactification of an Efremovic proximity space, or the T_1-extension related to a Lodato proximity space, or, more generally, the "Herrlich-Bentley"-extension of a so-called "bunch-determined" nearness space. All these constructions on a nearness structure may be viewed as special cases of a more *general* theory of topological extensions and their related b-convergences.

2. Basic Concepts

As usual, PX denotes the power set of a set X, and we use $\mathcal{B}^X \subseteq PX$ to denote a collection of *bounded* subsets of X, also known as B-sets. Moreover, $\mathbf{FIL}(X \times X)$ denotes the set of all *uniform* filters on X.

Definition 2.1. We call a pair $(\mathcal{B}^X, \tau)$ consisting of a B-set $\mathcal{B}^X$ and a function $\tau : \mathcal{B}^X \longrightarrow P(\mathbf{FIL}(X \times X))$ a *b-convergence space* and τ a *b-convergence* (on $\mathcal{B}^X$), if the following axioms are satisfied:

(bc1) $B' \subseteq B \in \mathcal{B}^X$ implies $B' \in \mathcal{B}^X$;

(bc2) $\emptyset \in \mathcal{B}^X$;

(bc3) $x \in X$ implies $\{x\} \in \mathcal{B}^X$;

(bc4) $x \in X$ implies $\dot{x} \times \dot{x} \in \tau(\{x\})$;

(bc5) $\tau(\emptyset) = \{P(X \times X)\}$;

(bc6) $B \in \mathcal{B}^X$, $\mathfrak{U} \in \tau(B)$ and $\mathfrak{U} \subseteq \mathfrak{V} \in \mathbf{FIL}(X \times X)$ imply $\mathfrak{V} \in \tau(B)$.

(Here $\dot{x}$ denotes the filter generated by the set $\{x\}$.) If $\mathfrak{U} \in \tau(B)$ for some $B \in \mathcal{B}^X$, we say the uniform filter $\mathfrak{U}$ *b-converges* to B.

Given two b-convergence spaces $(\mathcal{B}^X, \tau_X)$ and $(\mathcal{B}^Y, \tau_Y)$, a function $f : X \longrightarrow Y$ is called *b-continuous* iff it is *bounded*, which means

(c1) $\{ f[B] \; B \in \mathcal{B}^X \} \subseteq \mathcal{B}^Y$,

and in addition we have that f *preserves uniform filters* in the sense that

(c2) $B \in \mathcal{B}^X$ and $\mathfrak{U} \in \tau_X(B)$ imply $(f \times f)(\mathfrak{U}) \in \tau_Y(f[B])$, where

$$(f \times f)(\mathfrak{U}) := \{ V \subseteq Y \times Y \; (f \times f)^{-1}[V] \in \mathfrak{U} \}$$

with $(f \times f)^{-1}[V] := \{ (x_1, x_2) \in X \times X \; (f(x_1), f(x_2)) \in V \}$.

Moreover, we denote the corresponding category by **b-CONV**, and mention here its interesting property of being *topological*.

(Corresponding examples can be found in the literature provided in the references.)

Definition 2.2. **EXT** denotes the category whose objects are triples $(e, \mathcal{B}^X, Y)$ — called *extensions* — where $X = (X, cl_X)$ and (Y, cl_Y) are topological spaces (given by closure operators), $\mathcal{B}^X$ is a B-set on X, and $e : X \longrightarrow Y$ is a function satisfying the following conditions:

(E_1) $A \in PX$ implies $cl_X(A) = e^{-1}[cl_Y(e[A])]$;

(E_2) $cl_Y(e[X]) = Y$, which means that the image of X is *dense* in Y;

(E_3) $(e, \mathcal{B}^X, Y)$ is T_1*-relative*, i.e., $x \in X$ and $y \in cl_Y(\{e(x)\})$ imply $y = e(x)$;

(E_4) $(e, \mathcal{B}^X, Y)$ is *strict*, i.e., $\{ cl_Y(e[A]) \; A \subseteq X \}$ is a *base* for the *closed* subsets of Y.

Morphisms in **EXT** have the form $(f, g) : (e, \mathcal{B}^X, Y) \longrightarrow (e', \mathcal{B}^{X'}, Y')$, where $f : X \longrightarrow X'$, $g : Y \longrightarrow Y'$ are *continuous* maps such that f is also *bounded* and the following diagram commutes:

$$\begin{array}{ccc} X & \xrightarrow{e} & Y \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ X' & \xrightarrow[e']{} & Y' \end{array}$$

If $(f, g) : (e, \mathcal{B}^X, Y) \longrightarrow (e', \mathcal{B}^{X'}, Y')$ and $(f', g') : (e', \mathcal{B}^{X'}, Y') \longrightarrow (e'', \mathcal{B}^{X''}, Y'')$ are **EXT**- morphisms, then they can be *composed* according to the rule

$$(f', g') \diamond (f, g) = (f' \circ f, g' \circ g) : (e, \mathcal{B}^X, Y) \longrightarrow (e'', \mathcal{B}^{X''}, Y''),$$

where "$\circ$" denotes the composition of maps.

3. Extensions and Related b-Convergences

Theorem 3.1. *We obtain a functor* $F : \mathbf{EXT} \longrightarrow \mathbf{b} - \mathbf{CONV}$ *by setting*

(a) $F(e, \mathcal{B}^X, Y) := (\mathcal{B}^X, \tau_{\mathbf{EXT}})$ *with* $\tau_{\mathbf{EXT}}(\emptyset) := \{P(X \times X)\}$ *and* $\tau_{\mathbf{EXT}}(B) := \{\, \mathfrak{U} \in \mathbf{FIL}(X \times X)\ \exists \mathfrak{F} \in \mathbf{FIL}(X).\ (\mathfrak{F} \times \mathfrak{F} \subseteq \mathfrak{U}$ *and* $\bigcap \{\, cl_Y(e[F])\ F \in \mathbf{sec}\, \mathfrak{F} \cup \{B\}\,\} \neq \emptyset)\,\}$ *in case* $B \neq \emptyset$.

(b) $F(f, g) := f$ *for an* **EXT***-morphism* $(f, g) : E := (e, \mathcal{B}^X, Y) \longrightarrow E' := (e', \mathcal{B}^{X'}, Y')$.

Recall that $\mathbf{sec}\, \mathfrak{G} := \{\, T \subseteq X\ \forall G \in \mathfrak{G}.\ G \cap T \neq \emptyset \,\}$ for a set-system $\mathfrak{G} \subseteq PX$.

Remark 3.2. We also note that each b-convergence τ defines an *underlying closure* operator cl_τ by setting

$$cl_\tau(A) := \{\, x \in X\ \exists \mathfrak{G} \in \mathbf{GRL}(X).\ (A \in \mathfrak{G} \text{ and } \mathbf{sec}\, \mathfrak{G} \times \mathbf{sec}\, \mathfrak{G} \in \tau(\{x\}))\,\}$$

Moreover, this closure cl_τ already is a *pretopology*. Here $\mathbf{GRL}(X)$ denotes the set of all grills on X, where $\mathfrak{G} \subseteq PX$ is called a *grill* on a set X (G. Choquet), provided that:

(G1) $\emptyset \notin \mathfrak{G}$;

(G2) $G_1 \cup G_2 \in \mathfrak{G}$ iff $G_1 \in \mathfrak{G}$ and $G_2 \in \mathfrak{G}$.

Focussing on the b-convergence $\tau_{\mathbf{EXT}}$ of the $F(e, \mathcal{B}^X, Y)$, we obtain $cl_X = cl_{\tau_{\mathbf{EXT}}}$, since by Axiom (E_3) the extension $(e, \mathcal{B}^X, Y)$ is T_1-relative. In addition, $\tau_{\mathbf{EXT}}$ turns out to have all the properties defined below:

Definition 3.3. A b-convergence τ (on $\mathcal{B}^X$) is called:

(i) *isotone* iff $\emptyset \neq B_1 \subseteq B_2 \in \mathcal{B}^X$ implies $\tau(B_1) \subseteq \tau(B_2)$;

(ii) *linked* iff $B_1, B_2 \in \mathcal{B}^X \setminus \{\emptyset\}$ and $B_1 \cup B_2 \in \mathcal{B}^X$ imply $\tau(B_1 \cup B_2) \subseteq \tau(B_1) \cup \tau(B_2)$;

(iii) *closed* iff $B \in \mathcal{B}^X \setminus \{\emptyset\}$ and $cl_\tau(B) \in \mathcal{B}^X$ imply $\tau(cl_\tau(B)) \subseteq \tau(B)$;

(iv) *singular* iff $x \in X$ implies $e_\tau(x) \times e_\tau(x) \in \tau(\{x\})$, where $e_\tau(x) := \mathbf{sec}\{\, T \subseteq X \; x \in cl_\tau(T) \,\}$;

(v) *clipped* iff $B \in \mathcal{B}^X \setminus \{\emptyset\}$ and $\mathfrak{U} \in \tau(B)$ imply the existence of a B-clip $\mathcal{C}$ in τ with $\mathcal{C} \times \mathcal{C} \subseteq \mathfrak{U}$, where $\mathcal{C} \in \mathbf{FIL}(X)$ is called a *B-clip* in τ iff :

(a) $B \in \mathbf{sec}\,\mathcal{C}$;

(b) $\mathcal{C} \times \mathcal{C} \in \tau(B)$;

(c) $F \subseteq X$ and $cl_\tau(F) \in \mathbf{sec}\,\mathcal{C}$ imply $F \in \text{sec}\ \mathcal{C}$;

(vi) *compressed* iff $B \in \mathcal{B}^X \setminus \{\emptyset\}$, $\mathcal{C} \in \mathbf{FIL}(X)$ and $\mathcal{C} \times \mathcal{C} \in \tau(B)$ imply $\mathcal{C} \times \mathcal{C} \in \bigcap\{\, \tau(F) \; F \in \mathbf{sec}\,\mathcal{C} \cap \mathcal{B}^X \,\}$.

Lemma 3.4. *For a clipped b-convergence τ, cl_τ is a topological closure operator.* □

Corollary 3.5. *$cl_{\tau_{\mathbf{EXT}}}$ is always a topological closure operator.* □

Definition 3.6. We call a b-convergence τ *extendible*, if τ satisfies the conditions (i) through (vi) above, and denote by **EbCONV** the corresponding full subcategory of **b-CONV**.

4. b-Convergences and Related Extensions

Lemma 4.1. *Let $(\mathcal{B}^X, \tau)$ be an extendible b-convergence space. We put*

$$\hat{X} := \{\, \mathcal{C} \subseteq PX \; \mathcal{C} \text{ is a } B\text{-clip for some } B \in \mathcal{B}^X \setminus \{\emptyset\} \,\}$$

and for each $\hat{A} \subseteq \hat{X}$ we set

$$cl_{\hat{X}}(\hat{A}) := \{\, \mathcal{C} \in \hat{X} \; \mathcal{C} \subseteq \delta\hat{A} \,\}\ ,$$

where $\delta\hat{A} := \{\, F \subseteq X \; \exists \mathcal{C} \in \hat{A}.\ F \in \mathcal{C} \,\}$. Then $cl_{\hat{X}}$ is a topological closure operator on $\hat{X}$.

Theorem 4.2. *For extendible b-convergence spaces $(\mathcal{B}^X, \tau_X)$, $(\mathcal{B}^Y, \tau_Y)$ let $f : X \longrightarrow Y$ be a b-continuous map. Define a function $\hat{f} : \hat{X} \longrightarrow \hat{Y}$ by setting for each $\mathcal{C} \in \hat{X}$*

$$\hat{f}(\mathcal{C}) := \{\, D \subseteq Y \; f^{-1}[cl_{\tau_Y}(D)] \in \mathbf{sec}\,\mathcal{C} \,\}.$$

Then the following statements are valid:

(i) $\hat{f}$ is a continuous map from $(\hat{X}, cl_{\hat{X}})$ to $(\hat{Y}, cl_{\hat{Y}})$.

(ii) The composites $\hat{f} \circ e_{\tau_X}$ *and* $e_{\tau_Y} \circ f$ *coincide, where* $e_{\tau_X} : X \longrightarrow \hat{X}$ *denotes the function that assigns the* $\{x\}$*-clip* $e_{\tau_X}(x)$ *to each* $x \in X$.

(iii) $f(\mathbf{sec}\,\mathcal{C}) \subseteq \hat{f}(\mathcal{C})$ *for each* $\mathcal{C} \in \hat{X}$*, where in general* $f\,dirD2 := \{\, f[F]\, f \in dirD2 \,\}$ *for* $dirD2 \subseteq PX$.

(iv) $\bigcup e_{\tau_X}[B] := \bigcup \{\, e_{\tau_X}(x)\; x \in B \,\} = \mathbf{sec}\{\, F \subseteq X\; B \subseteq cl_{\tau_X}(F) \,\}$ *for every* $B \subseteq X$.

Theorem 4.3. *We obtain a functor* $G : EbCONV \longrightarrow \mathbf{EXT}$ *by setting*

(a) $G(\mathcal{B}^X, \tau_X) := (e_{\tau_X}, \mathcal{B}^X, \hat{X})$ *for any extendible b-convergence space* $(\mathcal{B}^X, \tau_X)$ *with* $X := (X, cl_{\tau_X})$ *and* $\hat{X} := (\hat{X}, cl_{\hat{X}})$.

(b) $g(f) := (f, \hat{f})$ *for any b-continuous map* $f : (\mathcal{B}^X, \tau_X) \longrightarrow (\mathcal{B}^Y, \tau_Y)$.

Theorem 4.4. *Let* $F : \mathbf{EXT} \longrightarrow EbCONV$ *and* $G : EbCONV \longrightarrow \mathbf{EXT}$ *be the functors defined above. For each object* $(\mathcal{B}^X, \tau)$ *of EbCONV let* $t_{(\mathcal{B}^X,\tau)}$ *denote the identity map* $t_{(\mathcal{B}^X,\tau)} := \boldsymbol{id}_X : F(G(\mathcal{B}^X, \tau)) \longrightarrow (\mathcal{B}^X, \tau)$. *Then* t *is a natural equivalence from* $F \circ G$ *to the identity functor on the category EbCONV.*

Remark 4.5. $t_{(\mathcal{B}^X,\tau)} := \boldsymbol{id}_X : F(G(\mathcal{B}^X, \tau)) \longrightarrow (\mathcal{B}^X, \tau)$ is a b-continuous map in "both directions", and the following diagram commutes for each b-continuous map $f : (\mathcal{B}^X, \tau_X) \longrightarrow (\mathcal{B}^Y, \tau_Y)$:

$$\begin{array}{ccc} F(G(\mathcal{B}^X, \tau_X)) & \xrightarrow{\boldsymbol{id}_X} & (\mathcal{B}^X, \tau_X) \\ {\scriptstyle F(G(f))}\downarrow & & \downarrow{\scriptstyle f} \\ F(G(\mathcal{B}^Y, \tau_Y)) & \xrightarrow[\boldsymbol{id}_Y]{} & (\mathcal{B}^Y, \tau_Y) \end{array}$$

Remark 4.6. If $\mathcal{B}^X$ coincides with PX, then extendible b-convergence is leading to bunch-determined nearness spaces and vice versa.

Remark 4.7. Note also, if (X, cl_X) is a T_1-space, the corresponding function that assigns the $\{x\}$-clip $e_{\tau_{\mathbf{EXT}}}(x)$ to each $x \in X$ is injective and hence a topological embedding.

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