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RANKS FOR SINGULAR CURVES: THE NODAL DEGREE $n+1$ LINEARLY NORMAL SUBCURVE OF $\mathbb{P}^n$

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Abstract: Let $X \subset \mathbb{P}^n$ be an integral and non-degenerate curve. Here we introduce some invariants related to the X-ranks and study these invariants and the X-rank when X is a linearly normal curve of degree $n+1$ with a node.

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Let $X \subseteq \mathbb{P}^n$ be an integral and non-degenerate curve defined over an algebraically closed field $\mathbb{K}$. Let $f : C \to X$ be the normalization map. For any $P \in \mathbb{P}^n$ the X-rank $r_X(P)$ of P is the minimal cardinality of a finite set $S \subset X$ such that $P \in \langle S \rangle$, where $\langle \ \rangle$ denote the linear span. Set $d := \deg(X)$. For any integer $t \geq 1$ let $C^{(t)}$ denote the set of all effective divisors of C with degree t. For any hyperplane $H \subset \mathbb{P}^n$, let $f^*(H) \in C^{(d)}$ denote the associated effective divisor. Let $|\mathcal{O}_f(1)|$ denote the n-dimensional projective subspace of $C^{(d)}$ formed by the divisors $f^*(H)$, $H \in \mathbb{P}^{n*}$. Fix $P \in \mathbb{P}^n$ and $D \in C^{(t)}$. We say that $P \in \langle D \rangle$ if $f^*(H) - D$ is effective for every $H \in \mathbb{P}^{n*}$ containing P. Let $\beta_f(P)$ (resp. $\rho_f(P)$) denote the minimal integer t such that there is $D \in C^{(t)}$ (resp. a reduced $D \in C^{(t)}$) with $P \in \langle D \rangle$. Since $\langle E \rangle = \langle f(E) \rangle$ for any reduced $A \subset C$, we have $\rho_f(P) = r_X(P)$ for all $P \in \mathbb{P}^n$. However, a small modification of this invariant will give a new invariant (an element of $\mathbb{N}^{\oplus(s+1)}$, $s := \sharp(f^{-1}(\mathrm{Sing}(X)))$). For any integer $t \geq 1$ let $\sigma_t(X)$ denote the closure in $\mathbb{P}^n$ of the union of all $(t-1)$-

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dimensional linear subspaces of $\mathbb{P}^n$ spanned by t points of X. Hence $\sigma_1(X) = X$. Set $\sigma_0(X) := \emptyset$. We say that a point $P \in \sigma_t(X)\setminus\sigma_{t-1}(X)$ has border rank t and write $b_X(P) = t$. Set $\sigma_{t,x}(X) := \{P \in \mathbb{P}^n : b_X(P) = t, r_X(P) = t\}$. If X is a smooth curve (but we are not interested in this case) and $P \in \sigma_t(X)\setminus\sigma_{t-1}(X)$, then there is a degree t zero-dimensional scheme $Z \subset X$ such that $P \in \langle Z\rangle$ and $P \notin \langle Z'\rangle$ for any $Z' \subsetneqq Z$. Since $C^{(t)}$ is complete, for arbitrary X to any $P \in \sigma_t(X)\setminus\sigma_{t-1}(X)$ we may associate at least one element $D \in C^{(t)}$. It is not clear when $P \in \langle D\rangle$. Set $A_f := f^{-1}(\mathrm{Sing}(X))$ (seen as a reduced subscheme of C). Since X, C and f is fixed, A_f is a finite fixed subset of C. Since we are interested in the variable point $P \in \mathbb{P}^n$ it is not very expensive from the computational point of view to compute several derivatives at the points of the fix set A_f. For every effetive divisor $D \subset C$ let D' be the part supported by $C \setminus A_f$ and $D'' := D - D'$. Hence D'' is the part of D supported by A_f (but with arbitrary non-negative multiplicities). For any $P \in \mathbb{P}^n$ let $V_f(P)$ be the set of all pairs $(\deg(D'), \deg(D''))$, where $D \subset C$ is effective, D is reduced outside A_f and $P \in \langle D\rangle$. When $s := \sharp(A_f) \geq 2$ we may refine the set $V_f(P)$ in the following way. We fix an ordering $Q_1, \dots, Q_s$ of the points of A_f and instead of the pair $(\deg(D'), \deg(D''))$ we write the pair (whose second term is an s-ple of integers) $(\deg(D_1); a_1, \dots, a_s)$, where a_i is the degree of the connected component of D'' supported by Q_i.

We prove the following result (part (b) gives another examples of pairs (X, P) for which [5], Proposition 4.1, is sharp).

Theorem 1. *Let $X \subset \mathbb{P}^n$, $n \geq 3$, be an integral and non-degenerate curve such that $\deg(X) = n+1$ and X has an ordinary node, O. Set $\{Q_1, Q_2\} := f^{-1}(O)$. Let L_i, $i = 1, 2$, be the tangent line to the branch of X at O corresponding to Q_i. Let $T_OX \cong \mathbb{P}^2$ be the Zariski tangent space of X at O.*

(a) Fix any $P \in T_OX \setminus \{O\}$. Then $(0; 2, 2) \in V_f(P)$, $(0; 1, 2) \in V_f(P)$ if and only if $P \in L_2$ and $(0; 1, 2) \in V_f(P)$ if and only if $P \in L_1$.

(b) If $P \in L_1 \cup L_2$, then $r_X(P) = n$.

(c) If $n = 3$, then there are constructible subsets $\mathcal{A}$ and $\mathcal{B}$ of T_OX such that $T_OX \setminus (L_1 \cup L_2) = \mathcal{A} \cup \mathcal{B}$, $\mathcal{A} \neq \emptyset$, $\mathcal{B} \neq \emptyset$, $\mathcal{A} \cap \mathcal{B} = \emptyset$, $\dim(\mathcal{A}) = 2$, $\dim(\mathcal{B}) = 1$, $r_X(P) = 2$ for all $P \in \mathcal{A}$ and $r_X(P) = 3$ for all $P \in \mathcal{B}$.

(d) Assume $n \geq 7$ and that either char$(\mathbb{K}) = 0$ or char$(\mathbb{K}) \geq n+2$. Then $r_X(P) \leq n-1$ for all $P \in \mathbb{P}^n \setminus (L_1 \cup L_2)$. We have $r_X(P) = n-1$ for all $P \in T_AX \setminus \{A\}$ for some $A \in X_{reg}$ and for all $P \in T_OA \setminus (L_1 \cup L_2)$.

Remark 1. Assume $p := \mathrm{char}(\mathbb{K}) > 0$. Let $X \subset \mathbb{P}^n$, $n \geq 3$, be a non-degenerate strange curve with strange point $P \notin X$. Let q and s respectively the inseparable and the separable degree of the morphism of X induced by the linear projection from P. Thus q is a p-power. Assume $s = 1$. We have $r_X(P) = n + 1$ and all pairs (X, P) with $r_X(P) = n + 1$ arises in this way (see [1], Theorem 2). Fix any $O \in \mathrm{Sing}(X)$ and any $Q \in C$ such that $f(Q) = O$. For any $A \in X_{reg}$ the line $\langle \{P, A\} \rangle$ contains the effective divisor qA of A. Hence $P \notin |(q-1)(f^{-1}(A))|$. Hence $P \notin (q-1)Q'$. If $q > n$ it is easy to check the non existence of a divisor $D \in C^{(n)}$ such that $P \in \langle D \rangle$, at least when X has a unique singular point. For an example with arbitrary n and arbitrary q and a unique singular point, see [1], Example 1.

Linear algebra and the definition of seminormal point gives the following result.

Proposition 1. *Fix $O \in Sing(X)$ which is a seminormal point with multiplicity s. Fix an ordering $Q_1, \dots, Q_s$ of $f^{-1}(O)$. Let $L_1, \dots, L_s$ the corresponding lines of $\mathbb{P}^n$ such that $O \in L_i$ for all i and L_j is the the tangent line of the branch of X at O associated to Q_j. Fix an integer k such that $1 \le k \le \min\{n, s\}$ and $S \subseteq \{1, \dots, s\}$ such that $\sharp(S) = k$. Let $E_S \subseteq \mathbb{P}^n$ denote the k-dimensional linear subspace of $\mathbb{P}^n$ spanned by the line L_j with $j \in S$. Fix any $P \in E_S$. Then $(0; a_1, \dots, a_s) \in V_f(P)$, where $a_i = 1$ if $i \notin S$ and $a_i := 2$ if $i \in S$.*

Proof of Theorem 1. For any $B \in \mathbb{P}^m$, $m \ge 3$, let $\ell_B : \mathbb{P}^m \setminus \{B\} \to \mathbb{P}^{m-1}$ denote the linear projection from B. Obviously $C \cong \mathbb{P}^1$, $p_a(X) = 1$, O is the unique singular point of X and X is linearly normal. Proposition 1 gives the statements concerning $(0; 2, 2)$, $(0; 1, 2)$ and $(0; 2, 1)$. Part (a) easily follows.

(i) We have $r_X(P) \le n$ for all $P \in \mathbb{P}^n$ (if $\mathrm{char}(\mathbb{K}) = 0$ apply [5], Proposition 5.1; if $\mathrm{char}(\mathbb{K}) > 0$ apply [1], Theorem 2, and the fact that X is not strange because f is unramified (see [4], Proposition 3). First assume $P \in (L_1 \cup L_2) \setminus \{O\}$, say $P \in L_1$. The closure $Y \subset \mathbb{P}^{n-1}$ of the image of $X \setminus \{O\}$ by the linear projection ℓ_O from O is a rational normal curve. Since L_1 is tangent to a branch of X at O, it corresponds to a point $O_1 \in Y$. The closure $T \subset \mathbb{P}^{n-2}$ of the image of X by the linear projection from L_1 is the closure of the image of Y by the linear projection from O_1. Thus T is a rational normal curve of $\mathbb{P}^{n-2}$. Hence any $n - 1$ points of T are linearly independent. Since $P \in L_1$, we get $P \notin \langle S \rangle$ for any set $S \subset X \setminus \{O\}$ such that $\sharp(S) \le n - 1$. Now assume the existence of $S \subset X$ such that $O \in S$, S is linearly independent, $P \in \langle S \rangle$ and $\sharp(S) \le n - 1$. Set $S_1 := \ell_O(S \setminus \{O\}$. Notice that S' is a subset of Y with cardinality at most $n - 2$. Since $P \in \langle S \rangle$, we have $O_1 := \ell_O(P) \in \langle S_1 \rangle$. Since $(L_1 \cap X)_{red} = \{O\}$ and S is linearly independent, we have $S_1 \subset Y \setminus \{O_1\}$. Since any $n-1$ points of the rational normal curve Y are linearly independent, while $O_1 \in \langle S_1 \rangle$, we obtained a contradiction, proving part (b).

(iii) Now assume $n = 3$ and fix $P \in T_O X \setminus (L_1 \cup L_2)$. Since $P \notin L_1 \cup L_2$, the degree 4 plane curve $\ell_P(X)$ has a tacnode with arithmetic genus 2 at the point $\ell_P(O)$. Hence the genus formula for plane curves shows that there is another line $R \subset \mathbb{P}^3$ such that the scheme $R \cap X$ has degree ≥ 2, that this line is unique and $\deg(R \cap X) = 2$. Since $O \notin R$, $X \cap R$ is not reduced if and only if there is $A \in X_{reg}$ such that $R = T_A X$. The set X_{reg} is non-empty and of dimension 1. Every line $T_A X$ meets the plane $T_O X$ and only finitely many of them meets $L_1 \cup L_2$. Thus taking the points $(T_A X \cap (T_O X \setminus (L_1 \cup L_2)))$, $A \in X_{reg}$, we get a non-empty constructible subset $\mathcal{B}$ of $T_O X \setminus (L_1 \cup L_2)$ with pure dimension 1. If $X \cap D$ is reduced, then $r_X(P) = 2$. Hence to prove part (c) it is sufficient to take $\mathcal{A} := T_O X \setminus (L_1 \cup L_2 \cup \mathcal{B})$.

(iv) From now on we assume $n \geq 7$ and that either $\mathrm{char}(\mathbb{K}) = 0$ or $\mathrm{char}(\mathbb{K}) \geq n+2$. Fix $P \in \mathbb{P}^n \setminus (L_1 \cup L_2)$. We want to prove that prove $r_X(P) \leq n-1$. There are a smooth rational normal curve $D \subset \mathbb{P}^{n+1}$, $B_1, B_2 \in D$ such that $B_1 \neq B_2$ and $U \in \langle \{B_1, B_2\} \rangle \setminus \{B_1, B_2\}$ such that $X = \ell_U(D)$ and $P_i = \ell_U(B_i)$, $i = 1, 2$. Moreover, $L_i = \ell_U(T_{B_i} D)$, $i = 1, 2$, and $T_O X = \ell_U(\langle T_{B_1} \cup T_{B_2} D \rangle \setminus \{U\})$. Fix any $P' \in \mathbb{P}^{n+1} \setminus \{U\}$ such that $\ell_U(P') = P$. If there is $S \subset D$ such that $\sharp(S \cap \{B_1, B_2\}) \leq 1$ and $P' \in \langle S' \rangle$, then $P \in \langle \ell_U(S) \rangle$ and $\sharp(\ell_U(S)) = \sharp(S)$. If there is $S \subset D$ such that $P' \in \langle S \rangle$ and $\{B_1, B_2\} \subseteq S$, then $P \in \langle \ell_U(S \setminus B_1) \rangle$ and $\sharp(\ell_U(S \setminus B_1)) = \sharp(S) - 1$. Thus to prove the inequality $r_X(P) \leq n-1$, it would be sufficient to find P' as above and with $r_D(P') \leq n-1$. This condition is satisfied if and only if there is no unreduced set $Z_{P'} \subset D$ such that $\deg(Z_{P'}) \leq 3$ and $P' \in \langle Z \rangle$ (see [3] or [5], Theorem 4.1, if $\mathrm{char}(\mathbb{K}) = 0$, [2] if $\mathrm{char}(\mathbb{K}) \geq n+2$). Assume that $Z_{P''}$ exists for all $P'' \in \mathbb{P}^{n+1} \setminus \{U\}$ such that $\ell_U(P'') = P$, i.e. such that $P'' \in \langle \{P', U\} \rangle \setminus \{U\}$. Take P'' as above and different from P' we get $U \in \langle Z_{P',P''} \rangle$, where $Z_{P',P''}$ is the minimal positive divisor of D containing both $Z_{P'}$ and $Z_{P''}$. Notice that if $Z_{P'} = Z_{P''}$ for all P'', then $U \in Z_{P'}$. Since any degree $n+2$ effective divisor of D is linearly independent, $U \in \langle \{B_1, B_2\} \rangle$ and $n+2 \geq 5$, we get $\{B_1, B_2\} \subseteq (Z_{P'})_{red}$. Since $Z_{P'}$ is not reduced, we get that either $Z_{P'} = 2B_1 + B_2$ or $Z_{P'} = B_1 + 2B_2$. In the first case we get $P \in L_1$, while in the second case we get $P \in L_2$. Since $P \notin L_1 \cup L_2$ by assumption, we get $Z_{P'} \neq Z_{P''}$ and that at least one $Z_{P''}$ has the property that one of the points B_1, B_2, is not in the support of $Z_{P''}$. Since $\langle \{P', U\} \rangle \setminus \{U\}$ is an integral variety, while the set $\{B_1, B_2\}$ is finite, for general $(Q, Q', Q'') \in \langle (\{P', U\} \rangle \setminus \{U\})^3$ one of the points B_1, B_2 is not in the support of $Z_Q + Z_{Q'} + Z_{Q''}$. Since $U \in \langle Z_{Q,Q'} \rangle \cap \langle Z_{Q,Q''} \rangle \cap \langle Z_{Q',Q''} \rangle$, we get the existence of a linearly dependent subscheme of D with degree ≤ 9. Since

$n + 2 \geq 9$, we got a contradiction, proving the first part of (d). Now assume either $P \in T_O X \setminus (L_1 \cup L_2)$ or the existence of $A \in X_{reg}$ such that $P \in T_A X$. In both cases there is an unreduced zero-dimensional scheme $Z \subset X$ such that $\deg(Z) = 2$ and $P \in \langle Z \rangle$. Since any zero-dimensional subscheme of X with degree at most n is linearly independent, we get $r_X(P) \geq n - 1$. The first part of (d) gives $r_X(P) = n - 1$. □

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