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COMPUTING COORDINATES ON LIE GROUPS

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Abstract: We present a method for computing coordinates of the second kind on Lie groups and explain connections with representations. The approach is especially effective for Lie algebras admitting flags of subalgebras, including solvable (hence nilpotent) Lie algebras. A simple diagrammatic technique is developed that expresses the results directly in terms of the structure constants of the Lie algebra. We show how to do the calculations by symbolic computation with *MAPLE*.

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1. Introduction

Lie algebras and Lie groups are currently of special interest motivated by applications in control theory (Hazewinkel [16]), robotics (Murray, Li and Sastry [22])

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and computer science (Duchamp [4], Viennot [26]). The present authors have been interested in various aspects of computing with Lie algebras looking for systems of polynomials solving evolution equations such as arise in mathematical physics [10], connections with special functions [5], and explicit calculations of (certain) infinite-dimensional representations of Lie groups. [7] especially considers symbolic computations.

See Bourbaki [2] for general background on Lie algebras and groups.

This paper is organized as follows. Section 2 presents the necessary background of the methods that we have developed in this area. Section 3 indicates links with control theory and formal languages. In this section the emphasis is on Lie flags, that is, Lie algebras admitting flags of subalgebras. Details of the method are in Section 4. A diagrammatic technique is developed to find the expansion of the coordinates of the second kind to any order. For *MAPLE* worksheets showing symbolic computations, the reader is referred to the website: http://pfeinsil.math.siu.edu/lieflag/flagframe.html.

2. Background

Here we start with a summary of basic ideas as expounded in [8].

Our primary interest has been to develop effective means for computing representations of a Lie algebra (and the corresponding Lie group arising via exponentiation) on its universal enveloping algebra.

Basic to the approach is to consider the Poincaré-Birkhoff-Witt (PBW) basis for the universal enveloping algebra and to recognize the generating function for this basis as a typical group element factored into one-parameter subgroups, with the variables giving coordinates of the second kind. That is, consider a finite-dimensional Lie algebra $\mathcal{G}$ with basis $\{\xi_1, \ldots, \xi_d\}$, $d = \dim \mathcal{G}$, and ordered monomials

$$\xi^n = \xi_1^{n_1} \cdots \xi_d^{n_d}$$

all $n_i \geq 0$, comprising a PBW basis for its universal enveloping algebra $\mathcal{U}(\mathcal{G})$.

Then, form the exponential generating function with commuting variables $A = (A_1, \ldots, A_d)$:

$$g(A, \xi) = \sum \frac{A^n}{n!} \xi^n = e^{A_1 \xi_1} e^{A_2 \xi_2} \cdots e^{A_d \xi_d}$$

with the usual conventions $A^n = A_1^{n_1} \cdots A_d^{n_d}$, $n! = n_1! \cdots n_d!$. Now the action of left and right multiplication by basis elements ξ_i of the Lie algebra on the PBW basis of the enveloping algebra yields vector fields — ξ_i^* for right multiplication, $\xi_i^\ddagger$ for left multiplication:

$$\xi_i^* = \sum_\mu \pi_{j\mu}^*(A)\partial_\mu\,, \qquad \xi_i^\ddagger = \sum_\mu \pi_{j\mu}^\ddagger(A)\partial_\mu,$$

where partials refer to differentiating with respect to A-variables, $\partial_k = \partial/\partial A_k$. The coefficients of the vector fields form matrices of functions which are called pi-matrices. Theorem A below justifies (in one way at least) thinking of these as forming matrices.

Remark 2.1. In [7] we explain the technique of using matrices to find the dual representations avoiding use of the adjoint action. This is efficient if the Lie algebra is given in matrix terms. In the present paper, we are interested primarily in the case where the commutation relations are the basic data.

Now, taking a typical element $X \in \mathcal{G}$, $X = \sum_\lambda \alpha_\lambda \xi_\lambda$, with α_i coordinates of the first kind, consider the vector fields $X^* = \sum_\lambda \alpha_\lambda \xi_\lambda^*$ and $X^\ddagger = \sum_\lambda \alpha_\lambda \xi_\lambda^\ddagger$.

Notation 1. For convenience, write $\tilde{\xi}_i$, $\tilde{\pi}$, $\tilde{X}$ to denote either the left or right vector fields when the distinction is not essential.

We are interested in the evolution equation

$$\frac{\partial u}{\partial t} = \tilde{X}\, u$$

with $u(0) = f(A)$, with polynomial f. Going back to $\mathcal{G}$ and the exponential group, let us factor the exponential $\exp(tX)$ into one-parameter subgroups, emphasizing the dependence on t, suppressing the dependence on the α-variables considered as parameters:

$$g(t) = e^{tX} = e^{A_1(t)\xi_1} e^{A_2(t)\xi_2} \cdots e^{A_d(t)\xi_d}$$

the implicit dependence of A on α being in fact a change-of-coordinates map. Differentiating with respect to t yields

$$\dot{g} = Xg(t) = X^\ddagger g(t) = g(t)X = X^* g(t)$$

while, multiplication by ξ_i in each factor given by differentiating with respect to A_i,

$$\dot{g} = \left(\sum_\mu \dot{A}_\mu \partial_\mu \right) g.$$

Summarizing:

Proposition 2.2. *The characteristics for the flow generated by a vector* *field $\tilde{X}$,*

$$\frac{\partial u}{\partial t} = \tilde{X}\, u \tag{1}$$

are given by

$$\dot{A}_k = \sum_\lambda \alpha_\lambda \tilde{\pi}_{\lambda k}(A) \tag{2}$$

which in fact yield the coordinates of the second kind for the one-parameter subgroup generated by $\tilde{X}$.

Remark 2.3. See details of related work in Goodman [14] and Kawski [18]. The factorization into one-parameter subgroups was considered in Wei and Norman [27], [28].

The importance of equation (1) with X^* for polynomial initial conditions is illustrated by the *principal formula for the matrix elements* of the group acting on $\mathcal{U}(\mathcal{G})$ (see [8, p. 34]). Namely, define

$$g(A;\xi)\,\xi^n = \sum_m \left\langle {m \atop n} \right\rangle \xi^m. \tag{3}$$

Then the right dual representation yields

$$\left\langle {m \atop n} \right\rangle = (\xi_1^*)^{n_1} \cdots (\xi_d^*)^{n_d}\, A^m/m!. \tag{4}$$

Another way to look at this is to consider the group law in coordinates of the second kind,

$$g(B;\xi)\, g(A;\xi) = g(B \odot A;\xi). \tag{5}$$

With $g(A;\xi) = \exp(tX)$, $X = \sum \alpha_\mu \xi_\mu$ as above, we have

$$g(B \odot A;\xi) = e^{tX^*}\, g(B;\xi) \tag{6}$$

with $X^* = \sum\limits_{\lambda,\mu} \alpha_\lambda \pi^*_{\lambda\mu}(B)\partial/\partial B_\mu$. Comparing coefficients of ξ^m yields

$$e^{tX^*}\, B^m = (B \odot A)^m, \tag{7}$$

i.e.,

$$e^{tX^*}\, f(B) = f(B \odot A)$$

for polynomials f. Similarly, the left action yields

$$e^{tX^\ddagger}\, f(B) = f(A \odot B).$$

3. Present Approach

In this paper our goal is to show how to compute the variables A effectively as functions of α and data defining the Lie algebra, namely the structure constants. We look for expansion of $A(t)$ in powers of t, with the α-variables treated as parameters.

Remark 3.1. A link with control theory comes in studying the equations (2) for the characteristics: $\dot{A} = \alpha\pi(A)$. In control theory, the general case of time-dependent coefficients $\alpha(t)$ has been studied following Fliess [11] in the context of formal expansions (cf. Sussman [23]). This involves free Lie algebras, with good choices of bases such as the Hall basis playing an important rôle. The shuffle product provides an effective technique for doing calculations, see Berstel, Reutenauer [3], Melançon, Reutenauer [21], Schützenberger [24], and Viennot [26]. Also see Grayson, Grossmann [15] for the free nilpotent case. Time-dependent coefficients necessitate consideration of product integrals, of time-ordered products. Here algebraic methods have been developed by Agrachev, Gamkrelidze [1] and Kawski [19]. A recent summary from an algebraic viewpoint is Kawski, Sussman [20]. Shuffle algebras and Chen series from the point of view of quantum groups are indicated in Shnider, Sternberg [25].

Here we are interested in results starting with the data of a Lie algebra given according to commutation relations. This means one is given structure constants c_{ij}^k according to commutation rules $[\xi_i, \xi_j] = \sum c_{ij}^k \xi_k$, which may be neatly summarized in the *Kirillov matrix*, $K(x_1, \ldots, x_d)$, with entries linear in the x-variables:

$$K_{ij} = \sum_k c_{ij}^k x_k. \tag{8}$$

We present here a method for solving $\dot{A} = \alpha\pi(A)$ when the coefficients of the vector fields form a pi-matrix that arise from a Lie algebra as above. The novel feature of the present work is to show how to use the flag property of a Lie algebra effectively.

3.1. Lie Algebras with the Flag Property

Definition 3.2. A Lie algebra has the *flag property* if there is an increasing chain of *subalgebras* $\mathcal{B}_i$

$$\{0\} = \mathcal{B}_0 \subset \mathcal{B}_1 \subset \mathcal{B}_2 \subset \cdots \subset \mathcal{B}_d = \mathcal{G}, \tag{9}$$

each of codimension one in the next. Such a flag is an *increasing Lie flag.*

Note that this is a flag in the usual sense of $\mathcal{G}$ as a vector space, but is rather stringent as each $\mathcal{B}_i$ must be closed under Lie brackets. Note that with $\{\xi_1, \ldots, \xi_d\}$ a corresponding adapted basis for (9), i.e., $\{\xi_1, \ldots, \xi_i\}$ is a basis for $\mathcal{B}_i$, $1 \le i \le d$, reversing the order of the basis gives a *decreasing Lie flag.*

Every solvable, in particular every nilpotent, algebra has an increasing Lie flag consisting of ideals $\mathcal{B}_i$. In the nilpotent case, take the descending central series

$$\mathcal{G} = \mathcal{G}_0, \qquad \mathcal{G}_{j+1} = [\mathcal{G}, \mathcal{G}_j].$$

Refining this to a decreasing flag to satisfy the codimension one condition, choose an adapted basis. Reversing the order of any such basis yields an increasing flag. If we do this, let us say that the Lie flag and adapted basis are adapted to the central series. For a solvable Lie algebra in general, the Lie-Engel Theorem guarantees the existence of a flag of ideals (see Humphreys [17] for background and proofs).

The simple Lie algebra, sl(2), with basis E_+, E_-, , H and commutation relations $[E_+, E_-] = H$, $[H, E_\pm] = E_\pm$ admits the Lie flag with adapted basis $\{E_+, H, E_-\}$. Direct sums of sl(2) thus have the flag property.

3.2. Dual Representations and Flags

Given an increasing flag, with adapted basis $\{\xi_1, \ldots, \xi_d\}$, denote by $\check{\xi}_i^*$ the transpose of the matrix of ξ_i in the adjoint representation restricted to the subalgebra $\mathcal{B}_i$, i.e., columns $i+1$ through d of $\check{\xi}_i$ are zero'd out and then the matrix is transposed. In terms of the structure constants the entries of $\check{\xi}_i^*$ are

$$(\check{\xi}_i^*)_{jk} = c_{ij}^k \tag{10}$$

with the condition that $j, k \le i$, otherwise null. Dually, for a decreasing flag, we denote by $\check{\xi}_i^\ddagger$ the transposed matrix of the restriction of the adjoint action of ξ_i to the subalgebra $\mathcal{B}_i^d = \mathrm{span}\{\xi_i, \ldots, \xi_d\}$, i.e., the first i columns are zero'd out, then the matrix transposed. So the entries of $\check{\xi}_i^\ddagger$ are c_{ij}^k as in equation (10) except with the condition $j, k \ge i$ otherwise null.

We recall the main theorem from [8, p. 33] (see there for the proof).

Theorem A. *For the dual representations we have:*

1. *Given an increasing flag, the pi-matrix for the right dual is given by*

$$\pi^*(A) = \exp(A_d \check{\xi}_d^*) \exp(A_{d-1} \check{\xi}_{d-1}^*) \cdots \exp(A_1 \check{\xi}_1^*).$$

2. Given a decreasing flag, the pi-matrix for the left dual is given by

$$\pi^{\ddagger}(A) = \exp(-A_1\check{\xi}_1^{\ddagger})\exp(-A_2\check{\xi}_2^{\ddagger})\cdots\exp(-A_d\check{\xi}_d^{\ddagger}).$$

For a nilpotent Lie algebra, the matrices of the adjoint representation are nilpotent, so from Theorem A we note

Proposition 3.3. *For a Lie flag and basis adapted to the central series, the pi-matrices have polynomial entries.*

And the principal formula, equation (4), yields polynomial matrix elements for the action of the group on the enveloping algebra (cf. Kawski [18]). These are thus 'polynomial representations'.

4. Expansion of Coordinates of the Second Kind

We now present a method for expanding the A-variables in terms of the parameter t.

1. Let $\mathcal{G}$ be a Lie algebra with the flag property. With adapted basis $\{\xi_i\}_{1\le i\le d}$ and $\tilde{X} = \sum_\lambda \alpha_\lambda\tilde{\xi}_\lambda$ given by the corresponding dual representation, the equations (2) — $\dot{A} = \alpha\pi(A)$ — and the chain rule yield

$$\frac{d}{dt} = \sum_\mu \frac{\partial A_\mu}{\partial t}\frac{\partial}{\partial A_\mu} = \sum_{\lambda,\mu}\alpha_\lambda\pi_{\lambda\mu}\frac{\partial}{\partial A_\mu}.$$

Thus,

$$\left(\frac{d}{dt}\right)^n\bigg|_{t=0} A_i = \left(\sum_{\lambda,\mu}\alpha_\lambda\pi_{\lambda\mu}\frac{\partial}{\partial A_\mu}\right)^n A_i. \qquad (11)$$

Note that $t = 1$ recovers the coordinate change $\alpha \to A$.

2. Observe that $\tilde{X}A_i = \sum\alpha_\lambda\tilde{\pi}_{\lambda i}(A)$. So in applying $\tilde{X}$ n times in equation (11), the first application replaces A_i by $\sum\alpha_\lambda\tilde{\pi}_{\lambda i}(A)$. The point is that we are reduced to calculating derivatives of $\pi(A)$ at $A = (0,\dots,0)$.

In general, this would not be possible to do so 'generically', but from Theorem A, we have, say for an increasing flag,

$$\left(\frac{\partial}{\partial A_1}\right)^{m_1}\cdots\left(\frac{\partial}{\partial A_d}\right)^{m_d}\pi^*(A)\bigg|_{A=(0,\dots,0)} = \left(\check{\xi}_d^*\right)^{m_d}\cdots\left(\check{\xi}_1^*\right)^{m_1} \qquad (12)$$

— directly in terms of the structure constants.

3. In equation (11), write the i-th factor from the right as $\sum\alpha_{\lambda_i}\pi_{\lambda_i\mu_i}\partial_{\mu_i}$.

Note that the last application of $\tilde{X}$ when evaluated at 0 contracts to $\sum \alpha_{\mu_n}\partial_{\mu_n}$.

4. Consider the case of an increasing flag (as the decreasing flag is similar). The problem is to keep track of the results of applying Leibniz' rule as the derivatives are applied to factors to their right. We denote by a numeral k the summation index μ_k corresponding to differentiation with respect to A_{μ_k}. Fix n, the order of differentiation. Then for terms of the expanded right side of equation (11) we have a diagram consisting of a sequence of numerals from 0,2,3 to n and $n-1$ bars setting off n boxes. The sequence begins with a 0 and ends with a 2, e.g. for $n=5$: 0|5|0|43|2. The boxes are numbered from 1 to n from right to left. The rule is that each numeral from 2 to n is used exactly once and may go in any box with a strictly lower number. The 0's may be thought of as indicating empty boxes. Since numeral k has $k-1$ choices, there are $(n-1)!$ such diagrams.

The number of the box indicates which π is being differentiated.

5. It remains to convert a diagram into a term of the result. In every case, the α factors are

$$\alpha_{\lambda_1}\cdots\alpha_{\lambda_n}.$$

For a given diagram, consider the k-th box. Each non-zero numeral i in that box contributes a factor of $\check{\xi}^*_{\mu_i}$. If the box contains 0, just the identity matrix appears. The matrices are ordered according to decreasing subscripts, multiplied together, and the matrix entry $\lambda_k\mu_k$ taken. We use Wick symbols to indicate that the matrices are taken in decreasing order, e.g.,

$$:\check{\xi}^*_3\check{\xi}^*_5\check{\xi}^*_2: \;=\check{\xi}^*_5\check{\xi}^*_3\check{\xi}^*_2.$$

Remark 4.1. Note that this completes the expansion given to two terms in [8, p. 29]. The method used there, calculating powers of X directly, will work in general, but is unwieldy for high-order terms.

Example 4.2. For $n=3$, we have the diagrams 0|0|32 and 0|3|2 which translate to

$$0|0|32 \quad\to\quad \sum \alpha_{\lambda_1}\alpha_{\lambda_2}\alpha_{\lambda_3}\delta_{\lambda_3\mu_3}\delta_{\lambda_2\mu_2}\big(:\check{\xi}^*_{\mu_3}\check{\xi}^*_{\mu_2}:\big)_{\lambda_1\mu_1}$$

and

$$0|3|2 \quad\to\quad \sum \alpha_{\lambda_1}\alpha_{\lambda_2}\alpha_{\lambda_3}\delta_{\lambda_3\mu_3}\big(\check{\xi}^*_{\mu_3}\big)_{\lambda_2\mu_2}\big(\check{\xi}^*_{\mu_2}\big)_{\lambda_1\mu_1}$$

A short *MAPLE* procedure can be used to calculate the action of the vector field X^* (say) and then iterated to produce terms of any order. For some *MAPLE* worksheets illustrating Theorem A and the method of Section 4, see `http://pfeinsil.math.siu.edu/lieflag/flagframe.html`.

5. Conclusion

For Lie algebras with the flag property we have presented an effective method for computing coordinates of the second kind. Connections with polynomial representations have been given. This approach is implemented using symbolic computation (in *MAPLE*). Here we have developed a diagrammatic approach that yields explicit formulae in terms of structure constants.

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