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INTERPOLATION OF FUZZY DATA BY USING $E(3)$ CUBIC SPLINES

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Abstract: In this paper, we will consider the interpolation of fuzzy data by fuzzy-valued $E(3)$ splines. Numerical examples will be presented to illustrate the differences between of using $E(3)$ spline and other interpolations that have been studied before.

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1. Introduction

The following problem was first posed by L.A. Zadeh, see for example [12].

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Suppose that we have $n+1$ distinct real numbers x_0, $x_1, \dots, x_n$ and for each of these numbers a fuzzy value in R, rather than a crisp value, is given. Zadeh asked the question whether it is possible to construct some kind of smooth function on $\mathbb{R}$ to fit with the collection of fuzzy data at these $n+1$ points.

Lagrange interpolation for fuzzy data was first investigated by Lowen [12]. Later, Kaleva [9], avoided the well-known computational troubles associated with crisp Lagrange interpolation by using linear spline and not-a-knot cubic spline approximations. If the fuzzy data are not convex, then a technical difficulty arises and in this case the Bernestein approximation can be constructed, see for example Diamond and Ramer [6]. The interpolation of fuzzy data by using spline functions of odd degree was considered in [2] with natural splines, in [1] with complete splines and in [3] with fuzzy splines. Constructing consistent fuzzy surfaces from fuzzy data in sense of Lagrange polynomials, linear splines and not-a-knot cubic splines were described in [11].

In [4], Behforooz compared the $E(3)$ cubic spline with the not-a-knot cubic spline and the natural cubic spline and he showed that the $E(3)$ cubic spline is more accurate than these two cubic splines, and also, it has superconvergence properties which the other two cubic splines do not have these properties. These superconvergence properties of the $E(3)$ cubic spline can be used in different fields for better approximation and this fact has motivated us to use $E(3)$ cubic spline for construction $E(3)$ cubic fuzzy spline. In Section 3, we will introduce $E(3)$ cubic fuzzy spline to interpolate the fuzzy data. Finally, in Section 4, some numerical examples will be presented to compare our results with other studies.

2. Preliminaries

In this section we recall some fundamental results of fuzzy numbers and fuzzy interpolations.

Definition 1. A fuzzy number is a mapping $u : \mathbb{R} \to I = [0, 1]$ with the following properties (see [10]):

(i) u is an upper semi-continuous function on $\mathbb{R}$.

(ii) $u(x) = 0$ outside of some interval $[c, d] \subset \mathbb{R}$.

(iii) There exist real numbers a, b such that $c \le a \le b \le d$, and

(1) $u(x)$ is a monotonic increasing function on $[c, a]$,

(2) $u(x)$ is a monotonic decreasing function on $[b,\ d]$,

(3) $u(x) = 1$, for all x in $[a,\, b]$.

Definition 2. A fuzzy number $u = (m, l, r)_{LR}$ of type $L - R$ is a function from the reals into the interval $[0, 1]$ satisfying

$$u(x) = \begin{cases} R(\frac{x-\alpha}{\beta}) & ,\ a \le x \le a + \beta, \\ L(\frac{\alpha - x}{\alpha}) & ,\ a - \alpha \le x \le a, \\ 0, & \text{otherwise,} \end{cases}$$

where L and R are decreasing and continuous functions from $[0, 1]$ to $[0, 1]$ satisfying $L(0) = R(0) = 1$ and $L(1) = R(1) = 0$.

The set of all fuzzy numbers is denoted by $\mathcal{F}$. A popular type of fuzzy number is the set of triangular fuzzy number $u = (c,\, \alpha, \beta)$ defined by

$$u(x) = \begin{cases} \dfrac{x - c + \alpha}{\alpha}, & c - \alpha \le x \le c, \\ \dfrac{c + \beta - x}{\beta}, & c \le x \le c + \beta, \\ 0, & \text{otherwise,} \end{cases}$$

where $\alpha > 0$ and $\beta > 0$. Note that the triangular fuzzy numbers are special cases of $L - L$ fuzzy numbers, see [4].

Definition 3. If $u \in \mathcal{F}$ then the $\alpha-$level set of u is denoted by $[u]^\alpha$ and defined by $[u]^\alpha = \{x \in \mathbb{R} |\, u(x)\ \ge \alpha\}$, where $0 < \alpha \le 1$. Also, $[u]^0$ is called the support of u and it is given by $[u]^0 = \overline{\cup_{\alpha \in (0,1]} [u]^\alpha}$. It follows that the level sets of u are closed and bounded intervals in $\mathbb{R}$.

It is well-known that the addition and multiplication operations of real numbers can be extended to $\mathcal{F}$. In other words, for any $0 < \alpha \le 1$, $\lambda \in \mathbb{R}$ and $u,\, v \in \mathcal{F}$, we have:

$$[u + v]^\alpha = [u]^\alpha + [v]^\alpha \quad \text{and} \quad [\lambda u]^\alpha = \lambda [u]^\alpha.$$

Consider $n + 1$ distinct real numbers $x_0 \le x_1 \le x_2 \le ... \le x_n$. For each x_{i} we associate a fuzzy number $u_i \in \mathcal{F}$. To solve Zadeh's problem, we must find a continuous function $F : \mathbb{R} \to \mathcal{F}$ such that $F(x_i) = u_i$; for $i = 0, 1, \ldots, n$.

Let $P_{y_0, y_1 \ldots, y_n}(x)$ be the Lagrange interpolation polynomial of degree n which interpolates the data $(x_i,\, y_i)$; $i = 0, 1, \ldots, n$. According to the extension principle [4], we can write the membership function $F(x)$ for each $x \in \mathbb{R}$ as follows:

$$\mu_{F(x)}(t) = \begin{cases} \sup_{\substack{y_0, y_1, \ldots, y_n \\ t = P_{y_0 \cdots y_n}(x)}} \min_{i=0,1,\ldots,n} \mu_{u_i}(y_i), & \text{if } P^{-1}_{y_0 \cdots y_n}(t) \ne \emptyset, \\ 0, & \text{otherwise,} \end{cases}$$

where μ_{u_i} is the membership function of u_i.

For each $\alpha \in$(0,1] and $i = 0, 1, \ldots, n$, let $J_i^\alpha = [u_i]^\alpha = \mu_{u_i}^{-1}[\alpha, 1]$, and $F^\alpha(x)$

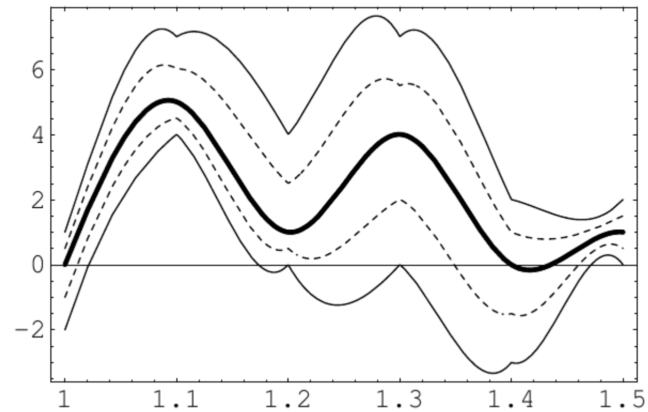

Figure 1: The solid line represents the support and the dashed line represents 0.5-level set and the thick line represents 1-level set of *natural* cubic spline

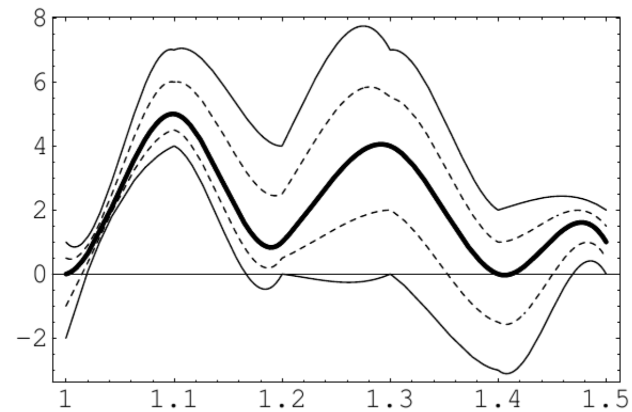

Figure 2: The solid line represents the support and the dashed line represents 0.5-level set and the thick line represents 1-level set of *complete* cubic spline

be the $\alpha-$level sets of u_i and $F(x)$, respectively. Hence,

$$F^{\ \alpha}(x) = \{t \in \mathbb{R} | \mu_{F(x)}(t) \geq \alpha\}$$
$$= \{t \in \mathbb{R} \,|\, \exists\, y_0, y_1, ..., y_n : \mu_{u_i}(y_i) \geq \alpha,\ i = 0, 1, \ldots, n$$

and $P_{y_0,y_1\ldots,y_n}(x) = t\} = \{t \in \mathbb{R} \,|\, \exists\, \overline{y} \in \prod_{i=0}^{n} J_i^{\alpha}\ : P_{y_0,y_1\ldots,y_n}(x) = t\}$,

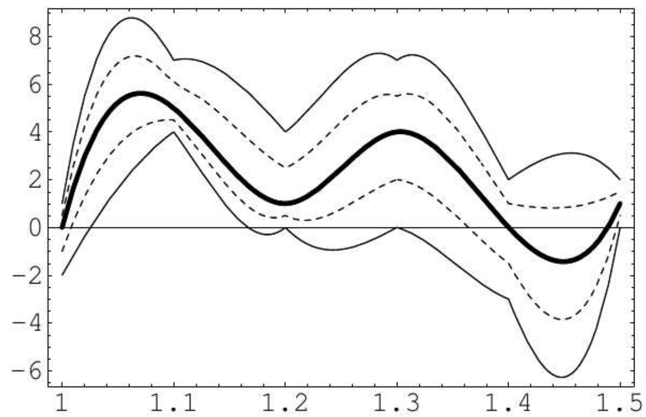

Figure 3: The solid line represents the support and the dashed line represents 0.5-level set and the thick line represents 1-level set of not-a-knot cubic spline

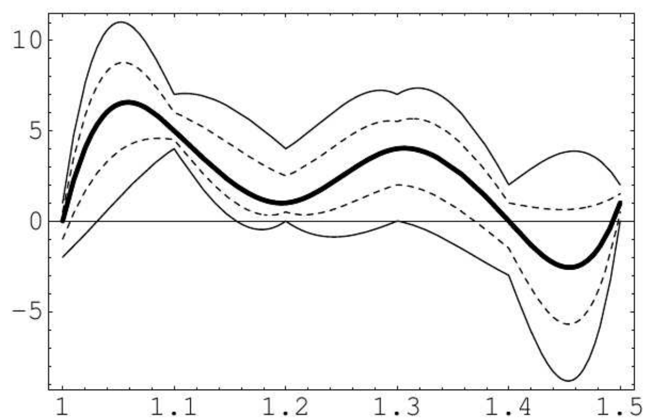

Figure 4: The solid line represents the support and the dashed line represents 0.5-level set and the thick line represents 1-level set of $E(3)$ cubic spline

where $\overline{y} = (y_0,\, y_1, ..., y_n) \in \mathbb{R}^{n+1}$. Now, we have

$$\mu_{F(x)}(t) \ = \sup\{\alpha \in (0,1] \,|\, \exists\ \overline{y} \in \prod_{i=0}^{n} J_i^{\alpha}\ : P_{y_0,y_1...,y_n}(x) = t\},$$

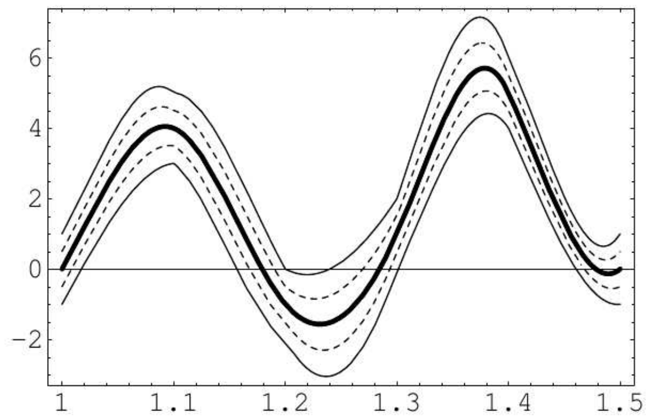

Figure 5: The solid line represents the support and the dashed line represents 0.5-level set and the thick line represents 1-level set of *natural* cubic spline

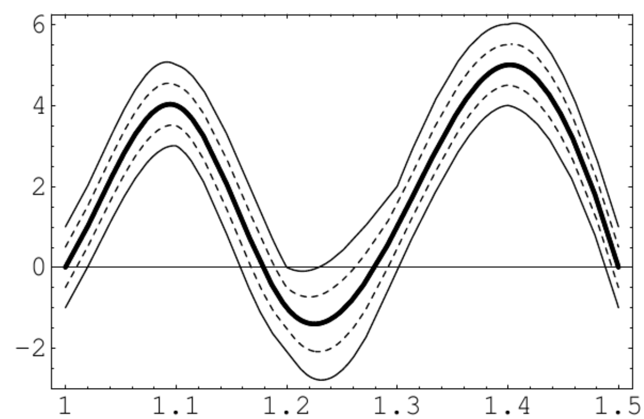

Figure 6: The solid line represents the support and the dashed line represents 0.5-level set and the thick line represents 1-level set of *complete* cubic spline

where, as mentioned by Lowen in [12], the supremum is attained and hence from Nguyen [10], we have

$$F^{\,\alpha}(x) = \{y \in \mathbb{R} \,|\, y = P_{y_0,y_1\ldots,y_n}(x),\, y_i \in J_i^{\alpha}\}.$$

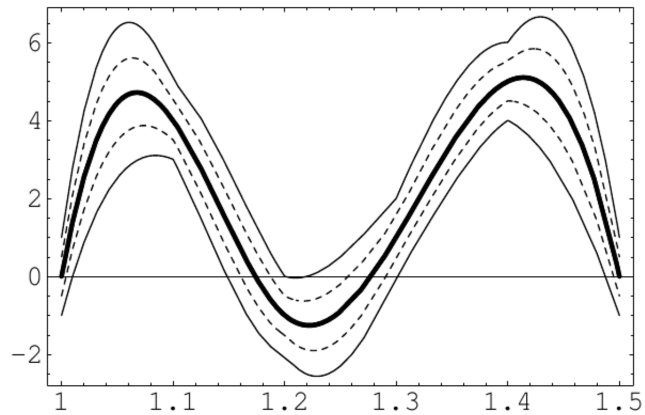

Figure 7: The solid line represents the support and the dashed line represents 0.5-level set and the thick line represents 1-level set of not-a-knot cubic spline

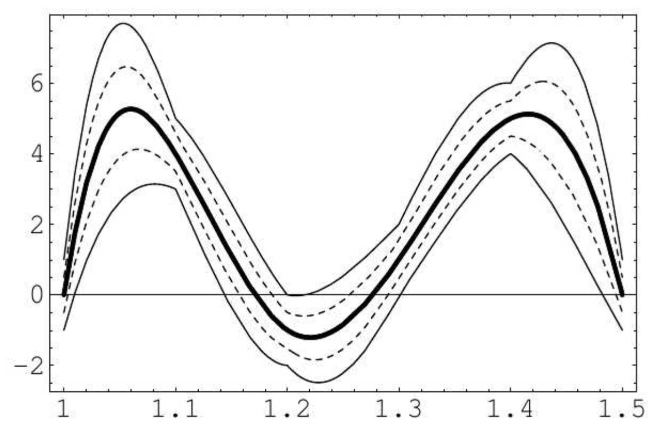

Figure 8: The solid line represents the support and the dashed line represents 0.5-level set and the thick line represents 1-level set of $E(3)$ cubic spline

But, from Lagrange interpolation formula, we have

$$F^{\alpha}(x) = \sum_{i=0}^{n} L_i(x) J_i^{\alpha},$$

where $L_i(x)$ represents the Lagrange polynomials.

3. $E(3)$ Cubic Fuzzy Splines

In this section we introduce a set of special cubic spline functions called "$E(3)$ Cubic Fuzzy Splines".

Definition 4. For a given data $\{(x_i, y_i)\}_{i=0}^{n}$ with equally spaced points $x_i = x_0 + ih$; $i = 0, 1, \cdots, n$, an $E(3)$ cubic spline with knots x_i is a piecewise polynomial function $s : [x_0,\, x_n] \to \mathbb{R}$, that possesses the following conditions:

a) $s(x_i) = y_i$,

b) s belongs $C^2[x_0,\, x_n]$,

c) $s(x)$ is a polynomial of degree 3 for $x \in [x_i, x_{i+1})$; $i = 0, 1, \ldots, n-1$,

d) $m_0 + 3m_1 = \frac{1}{6h}\{-17y_0 + 9y_1 + 9y_2 - y_3\}$,

e) $m_n + 3m_{n-1} = -\frac{1}{6h}\{-17y_n + 9y_{n-1} + 9y_{n-2} - y_{n-3}\}$,

where $m_i = s^{(1)}(x_i)$. If all of $n+1$ parameters m_i are known, then at any point $x \in [x_{i-1}, x_i]$; $i = 1, 2, \cdots, n$, the value of $s(x)$ can be obtained by using the two points Hermite interpolation polynomial formula, for more details see [4] and [5]. We denote the family of these splines by $\mathcal{S}_3(x_0, x_n)$. If the base splines s belong to $S_3(x_0, x_n)$ are such that $s_i(x_j) = 1$ for $i = j$ and $s_i(x_j) = 0$ for $i \neq j$, then similar to Lagrange interpolation polynomial, the fuzzy spline

$$S_{y_0, y_1 \ldots, y_n}(x) = \sum_{i=0}^{n} s_i(x)\, y_i$$

interpolates $(x_i,\, y_i)$; $i = 0, 1, \ldots, n$. Hence from Section 2, we have

$$F^{\,\alpha}(x) = \{t \in \mathbb{R} \mid \exists\, \overline{y} \in \prod_{i=0}^{n} J_i^{\alpha} \,:\, S_{y_0, y_1 \ldots, y_n}(x) = t\} = \sum_{i=1}^{n} s_i(x)\, J_i^{\alpha},$$

and

$$F(x) = \sum_{i=0}^{n} s_i(x) u_i.$$

Hence if all u_i are $L-L$ fuzzy numbers, then $F(x)$ is an $L-L$ fuzzy number for all $x \in [x_0,\, x_n]$.

4. Numerical Examples

Let $J_i^{\alpha} = [a_i^{\alpha},\, b_i^{\alpha}]$. Then the upper end point of $F^{\,\alpha}(x)$ is the solution of the following problem:

Maximize $S_{y_0 y_1 \ldots y_n}$ subject to $a_i^{\alpha} \leq y_i \leq b_i^{\alpha}$; $i = 0, 1, \ldots, n$, where the optimal solution is

$$y_i = \begin{cases} b_i^\alpha, & \text{if } s_i(x) \geq 0, \\ a_i^\alpha, & \text{if } s_i(x) < 0. \end{cases}$$

Similarly the lower end point of $F^\alpha(x)$ can be obtained. Hence if $u_i = (m_i, l_i, r_i)$ and $F(x) = (m(x), l(x), r(x))$, then we will have

$$m(x) = \sum_{i=0}^{n} s_i(x) m_i, \quad l(x) = \sum_{s_i(x)\geq 0} s_i(x) l_i - \sum_{s_i(x)<0} s_i(x) r_i;$$

$$r(x) = \sum_{s_i(x)\geq 0} s_i(x) r_i - \sum_{s_i(x)<0} s_i(x) l_i,$$

which are the same results in Kaleva [9].

Example 1. Suppose we have the data (x_i, u_i)

x_i	1	1.1	1.2	1.3	1.4	1.5
m_i	0	5	1	4	0	1
l_i	2	1	0	4	3	1
r_i	1	2	3	3	2	1

and using cubic spline. Figures 1, 2, 3 and 4 show the zero, 0.5 and one level sets for *natural*, *complete*, not-a-knot and $E(3)$ cubic spline.

Example 2. Here we have $u_i = y_i + A$; $i = 0, 1, ..., n$ and $A = (0, 1, 1)$ and cubic spline, where

x_i	1	1.1	1.2	1.3	1.4	1.5
y_i	0	4	-1	1	5	0

Figures 5, 6, 7 and 8 show the zero, 0.5 and one level sets.

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