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# A PLAUSIBILITY ARGUMENT FOR THE RIEMANN HYPOTHESIS USING A VARIANT OF THE DIRICHLET ETA FUNCTION

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**Abstract:** The Riemann zeta function is $\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \frac{1}{1^s} + \frac{1}{2^s} + \frac{1}{3^s} + \cdots$ and the hypothesis is equivalent to the statement that all zeros of the Dirichlet eta function $\eta(s) = \sum_{k=1}^{\infty} \frac{(-1)^{(k-1)}}{k^s} = (1-2^{1-s})\zeta(s)$ have a $Re(s) = \frac{1}{2}$. For the purpose of this study, $k$ will now be a complex number and $\eta(\{b\}) = \sum_{a=1}^{\infty} (-1)^{(a+bi)-1} \frac{1}{(a+bi)^{(c+di)}}$, where the independent variable becomes a selected set of $b$ values. Using the trigonometric form of the denominator, $\begin{bmatrix} \ln r_2 \\ \theta_2 \end{bmatrix} = \begin{bmatrix} c & -d \\ d & c \end{bmatrix} \begin{bmatrix} -\ln r_1 \\ \theta_1 \end{bmatrix}$, the eigenvalues and eigenvectors are $\lambda = c \pm di$ and $\begin{bmatrix} \ln r_1 \\ \mp i \ln r_1 \end{bmatrix}$, respectively. Since the eigenvalues are complex, $(r_2, \theta_2)$ is rotated from $(r_1, \theta_1)$ and we can set $r_2 = r_1$ and $\theta_2$ to be constant, chosen from $\pm\frac{\pi}{3}, \pm\frac{5\pi}{3}, \cdots$. This approach first demonstrates the viability of the critical strip, $0 \leq c \leq 1$, and then solving for a particular eigenvector yields $c = \frac{1}{2}$. Each value of $a$ determines values of $b$ and $d$ from the equations derived. In this plausibility argument for the Riemann Hypothesis, a process for selecting eigenvectors results in reducing the real part of the eta function to as close to zero as is arbitrarily desired. Additionally, the methods and results of this study are applied to the question of the Mass Gap Hypothesis in Appendix.

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## 1. Introduction

The Riemann Hypothesis is one of the most important unsolved math problems known to mankind. The Riemann Hypothesis states that the real value of the zeros of the Riemann zeta function will be $\frac{1}{2}$. If one is able to prove this hypothesis, it will unlock many secrets in the mathematical world. A proof of this hypothesis could put online financial security at risk because of its potential to decode prime number based ciphers. While it is assumed that the Riemann Hypothesis is true, having proved it true for the first ten trillion numbers or so, a formal proof has not been found. In this project, a plausibility argument has been created for this hypothesis using a variant of the Dirichlet eta function. The new function will use complex numbers and will be focused on the real parts. This number theory based study will also require the use of vectors, determinants, matrices, eigenvectors, eigenvalues, and linear transformations.

## 2. The Riemann Zeta Function and the Dirichlet Eta Function

The Riemann zeta function is

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \frac{1}{1^s} + \frac{1}{2^s} + \frac{1}{3^s} + \cdots . \tag{1}$$

The function is defined by the series on the half-plane of convergence and can be continued analytically to all complex $s \neq 1$, see [1].

"The Riemann Hypothesis is equivalent to the statement that all zeros of the Dirichlet eta function

$$\eta(s) = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k^s} = (1 - 2^{1-s})\zeta(s) \tag{2}$$

that fall in the critical strip $0 \leq Re(s) \leq 1$ lie on the critical line $Re(s) = \frac{1}{2}$" (see [2]).

## 3. A Variant Dirichlet Eta Function

Rockmore [3] "... in order to settle the Riemann hypothesis, perhaps we should try [to] find a way to view Riemann's zeta function as one of an infinitude of zeta functions, each with its own 'Riemann Hypothesis'. Perhaps a distantly related cousin might reveal a previously hidden aspect of the original Riemann zeta function, and thus shed some light on a path toward the resolution of the original Riemann hypothesis." As suggested, using a variant function is the approach taken in this study.

For the purpose of this study, $k$ will now be a complex number. Let $k = a+bi$ and now $s = c + di$. Then

$$\eta(\{b\}) = \sum_{a=1}^{\infty} (-1)^{(a+bi)-1} \frac{1}{(a+bi)^{(c+di)}}. \tag{3}$$

We will see that for each value of $a$ we select values of $b$ and $d$ and that the value of $\eta$ is a function of the values of $b$ that are selected; that is, $\{b\}$.

The eta function can be written in this form to show the magnitude and the phase angle of the vector it produces: $(a + bi)^{(c+di)} = r_2 e^{i\theta_2}$. Then

$$\begin{aligned} r_2 e^{i\theta_2} &= (a+bi)^{(c+di)} \\ &= (r_1 e^{i\theta_1})^{(c+di)}, \quad \text{where} \quad r_1 = \sqrt{a^2+b^2} \quad \text{and} \quad \theta_1 = \tan^{-1}\frac{b}{a} \\ &= r_1^{(c+di)} e^{i\theta_1 c} e^{-d\theta_1} \\ &= r_1^c e^{d \ln r_1 i} e^{i\theta_1 c} e^{-d\theta_1} \\ &= r_1^c e^{-d\theta_1} e^{i(\theta_1 c + d \ln r_1)}. \end{aligned}$$

Therefore,

$$r_2 = r_1^c e^{-d\theta_1}, \tag{4}$$

$$\theta_2 = \theta_1 c + d \ln r_1. \tag{5}$$

$\theta_2$ is a linear combination of the variables $\theta_1$ and $\ln r_1$.

So, we choose to take the logarithm of the $r_2$ equation, resulting in $r_2$ and $\theta_2$ becoming linear combinations of the state vector $\ln r_1$ and $\theta_1$. Rewriting yields

$$\ln r_2 = c \ln r_1 - d\theta_1, \tag{6}$$

$$\theta_2 = d \ln r_1 + c\theta_1. \tag{7}$$

Using linear algebra tools of matrices and vectors, this becomes

$$\begin{bmatrix} \ln r_2 \\ \theta_2 \end{bmatrix} = \begin{bmatrix} c & -d \\ d & c \end{bmatrix} \begin{bmatrix} \ln r_1 \\ \theta_1 \end{bmatrix}. \tag{8}$$

## 4. Eigenvalues and Eigenvectors

Now, we can solve for the eigenvalues.

$$\begin{vmatrix} c-\lambda & -d \\ d & c-\lambda \end{vmatrix} = 0, \qquad (c-\lambda)^2 + d^2 = 0, \tag{9}$$

$\lambda_1 - c - di$ and $\lambda_2 = c + di$ are the eigenvalues.

Using these, we can find the eigenvectors. For $\lambda_1$,

$$\begin{bmatrix} c & -d \\ d & c \end{bmatrix} \begin{bmatrix} \ln r_1 \\ \theta_1 \end{bmatrix} = (c - di) \begin{bmatrix} \ln r_1 \\ \theta_1 \end{bmatrix},$$

$$c \ln r_1 - d\theta_1 = c \ln r_1 - di \ln r_1, \qquad \theta = i \ln r_1,$$

and the eigenvector is

$$\vec{v}_1 = \begin{bmatrix} \ln r_1 \\ i \ln r_1 \end{bmatrix}. \tag{10}$$

Similarly, for $\lambda_2$, $\lambda_2 = c + di$ and

$$\vec{v_2} = \begin{bmatrix} \ln r_1 \\ -i \ln r_1 \end{bmatrix}. \tag{11}$$

## 5. Quantifying *c*

We can invoke linear algebra principles for the case of complex eigenvalues, namely, $\lambda_1 \vec{v_1} + \lambda_2 \vec{v_2} = \begin{bmatrix} Re[\ln r_2 e^{i\theta_2}]_1 + Re[\ln r_2 e^{i\theta_2}]_2 \\ Re[\theta_2]_1 + Re[\theta_2]_2 \end{bmatrix}$ where the subscripts 1 and 2 refer to the two eigenvectors. Then the left side of the equation becomes

$$\begin{aligned} &(c - di) \begin{bmatrix} \ln r_1 \\ i \ln r_1 \end{bmatrix} + (c + di) \begin{bmatrix} \ln r_1 \\ -i \ln r_1 \end{bmatrix} \\ &\quad = \begin{bmatrix} c \ln r_1 - di \ln r_1 \\ ci \ln r_1 + d \ln r_1 \end{bmatrix} + \begin{bmatrix} c \ln r_1 + di \ln r_1 \\ -ci \ln r_1 + d \ln r_1 \end{bmatrix} = \begin{bmatrix} 2c \ln r_1 \\ 2d \ln r_1 \end{bmatrix}, \end{aligned} \tag{12}$$

$$\begin{aligned} 2c \ln r_1 &= Re[\ln r_2 e^{i\theta_2}]_1 + Re[\ln r_2 e^{i\theta_2}]_2 \\ &= \ln r_2 \cos\theta_2 + \ln r_2 \cos\theta_2 = 2r_2 \cos\theta_2 \end{aligned} \tag{13}$$

for all eigenvectors. Reducing,

$$c \ln r_1 = \cos\theta_2 \ln r_2. \tag{14}$$

Traditionally, eigenvectors are input vectors that have the same phase angle (direction) but with a different magnitude after transformation. In our case, since the eigenvalues are complex, the output vector is necessarily rotated from the input vector and therefore cannot have the same direction. It does, however, allow us to set the magnitudes of the input and output vector to be equal and the output phase angle to be constant, independent of the input vector. This is our new interpretation of eigenvectors.

Now, $r_2 = r_1$ and

$$c = \cos\theta_2 = \cos\left(d \ln\sqrt{a^2+b^2} + c\tan^{-1}\frac{b}{a}\right) = A\cdot B - C\cdot D, \tag{15}$$

where, $A = \cos(d\ln\sqrt{a^2+b^2})$, $B = \cos\left(c\tan^{-1}\frac{b}{a}\right)$, $C = \sin(d\ln\sqrt{a^2+b^2})$, and $D = \sin\left(c\tan^{-1}\frac{b}{a}\right)$.

We know that $a \geq 1$ and we will show later that $b$ and $d$ will always have opposite signs. So, for $a \geq 1$, $b > 0$, $d < 0$, and $c > 0$, $A > 0$, $B > 0$, $C < 0$, and $D > 0$, making $A\cdot B - C\cdot D > 0$. For $a \geq 1$, $b < 0$, $d > 0$, and $c > 0$, $A > 0$, $B > 0$, $C > 0$, and $D < 0$, making $A\cdot B - C\cdot D > 0$.

Therefore,

$$c = \cos\theta_2 > 0 \quad \text{and} \quad 0 \leq c \leq 1. \tag{16}$$

We have demonstrated that $c$ must fall in the critical strip as described by Riemann.

We would now like to attempt to show that $c = \frac{1}{2}$.

Consider the case:

$$\begin{bmatrix} \ln r_2 \\ \theta_2 \end{bmatrix} = \begin{bmatrix} \ln r_1 \\ i\ln r_1 \end{bmatrix} = \begin{bmatrix} c & -d \\ d & c \end{bmatrix} \begin{bmatrix} \ln r_1 \\ i\ln r_1 \end{bmatrix}. \tag{17}$$

We will see an example of this later. Then

$$\begin{aligned} \ln r_1 &= c\ln r_1 - di\ln r_1, \\ i\ln r_1 &= d\ln r_1 - ci\ln r_1, \quad \text{which become} \\ 1 &= c - di, \\ i &= d + ci. \end{aligned}$$

This does not help us because these are not independent equations.

However, when we substitute $\theta_1$ for $i\ln r_1$, we get

$$\begin{bmatrix} \ln r_1 \\ i\ln r_1 \end{bmatrix} = \begin{bmatrix} c & -d \\ d & c \end{bmatrix} \begin{bmatrix} \ln r_1 \\ \theta_1 \end{bmatrix}, \tag{18}$$

$$\ln r_1 = c\ln r_1 - d\theta_1,$$
$$i\ln r_1 = d\ln r_1 + c\theta_1.$$

We then solve for $d$ and substitute

$$d = \frac{(c-1)\ln r_1}{\theta_1}, \tag{19}$$

$$i\ln r_1 = \left(\frac{(c-1)\ln r_1}{\theta_1}\right)\ln r_1 + c\theta_1,$$
$$\theta_1 i\ln r_1 = (c-1)[\ln r_1]^2 + c\theta_1^2,$$
$$c\theta_1^2 - \theta_1 i\ln r_1 + (c-1)[\ln r_1]^2 = 0. \tag{20}$$

For the case when $\theta_1$ is unique, the discriminant of this equation must equal 0.

$$(-i\ln r_1)^2 - 4c(c-1)[\ln r_1]^2 = 0, \tag{21}$$
$$-[\ln r_1]^2 - 4c(c-1)[\ln r_1]^2 = 0, \quad -1 - 4c(c-1) = 0,$$
$$4c^2 - 4c + 1 = 0, \quad (2c-1)(2c-1) = 0, \quad c = \frac{1}{2},$$

which is consistent with the Riemann Hypothesis.

## 6. Selecting and Evaluating Terms of this Eta Function

Then $\cos\theta_2 = \frac{1}{2}$ and

$$\theta_2 = \cos^{-1}\left(\frac{1}{2}\right) = \pm\frac{\pi}{3}, \pm\frac{5\pi}{3}, \pm\frac{7\pi}{3}, \pm\frac{11\pi}{3}, \pm\frac{13\pi}{3}, \cdots. \tag{22}$$

Substituting in (6) and (7), we get

$$\ln\sqrt{a^2+b^2} = \frac{1}{2}\ln\sqrt{a^2+b^2} - d\tan^{-1}\frac{b}{a},$$

$$d = \frac{-\frac{1}{2}\ln\sqrt{a^2+b^2}}{\tan^{-1}\frac{b}{a}} = \frac{-.25\ln(a^2+b^2)}{\tan^{-1}\frac{b}{a}}. \tag{23}$$

As we see here, $b$ and $d$ will always have opposite signs.

$$\frac{\pi}{3} = d\ln\sqrt{a^2+b^2} + \frac{1}{2}\tan^{-1}\frac{b}{a},$$

or

$$\frac{\pi}{3} = \frac{-.125[\ln(a^2+b^2)]^2}{\tan^{-1}\frac{b}{a}} + .5\tan^{-1}\frac{b}{a}. \tag{24}$$

We see that the eta function's independent variable is a set of $b$ values. That is, the value of the eta function is determined by the particular value of $b$ selected for each value of $a$. Now the eta function can be written as

$$\eta(\{b\}) = \sum_{a=1}^{\infty} \frac{(-1)^{a+bi-1}}{(a+bi)^{(c+di)}}. \tag{25}$$

Referring back to our new eta function, we can simplify it to gain some insight into whether or not it is possible for it to converge to 0.

We need to evaluate $(-1)^{a-1+bi} = (-1)^{a-1}(-1)^{bi}$. The first term of the product tells us that the individual terms of $\eta$ will switch off between being positive and negative. All odd values of $a$ will yield positive terms and all even values of $a$ will yield negative terms.

For the second term of this product, let $N = (-1)^{bi}$ then $\ln N = bi\ln(-1)$ and since $\ln(-1) = i\pi$ (due to Euler's identity: $e^{i\pi} = -1$)

$$\ln N = -b\pi \quad \text{and} \quad N = e^{-b\pi} \quad \text{so} \quad (-1)^{a-1+bi} = (-1)^{a-1}e^{-b\pi}.$$

Rewriting the eta function

$$\eta(\{b\}) = \sum_{a=1}^{\infty}(-1)^{a-1}e^{-b\pi}(a+bi)^{-(c+di)}. \tag{26}$$

For each value of $a$, we can choose any phase angle previously identified $(\pm\frac{\pi}{3}, \pm\frac{5\pi}{3}, \cdots)$. The selection of one of these angles will determine values of $b$ and $d$ from the equations derived. Figure 1 plots the function for determining $b$ and Figure 2 plots the function for determining $d$.

Since we have added an imaginary component to the eta function, we can no longer expect its imaginary sum to be zero. In this plausibility argument for the Riemann Hypothesis, our focus will be that the real portion of the eta function converges to zero.

## 7. Convergence to Zero

Now, to show

$$Re[\eta(\{b\})] = \sum_{a=1}^{\infty}(-1)^{a-1}e^{-b\pi}Re[(a+bi)^{-(c+di)}] = 0. \tag{27}$$

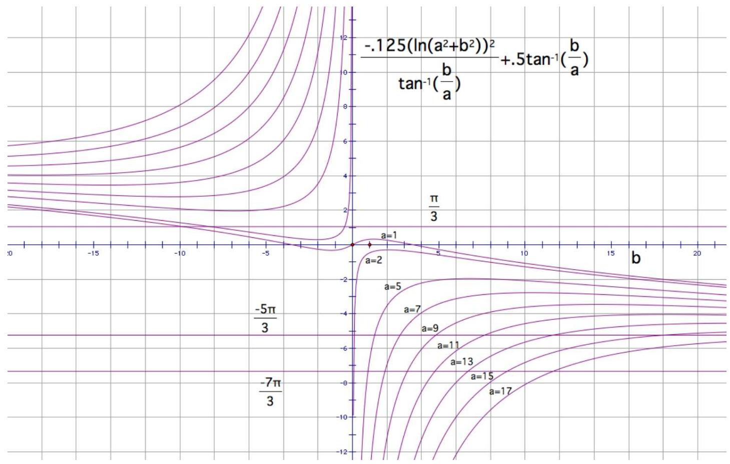


Figure 1

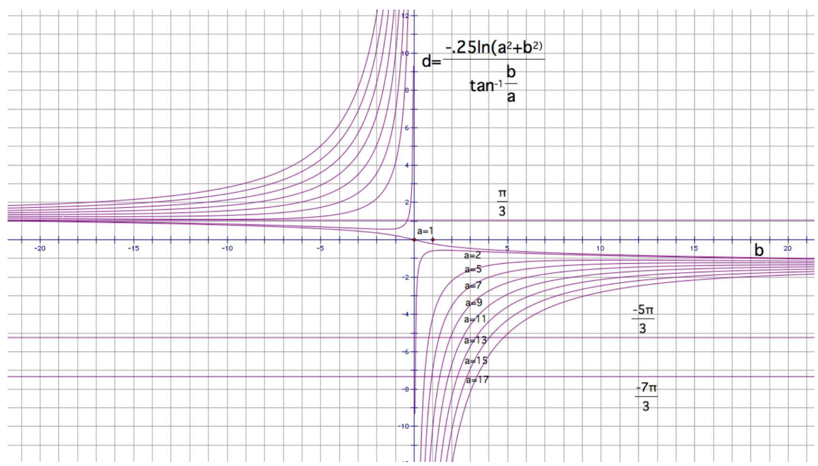


Figure 2

For each successive value of $a$, we choose a phase angle which determines $b$ and $d$. Once we find the values of $b$ and $d$, we find $Re[(a+bi)^{-(c+di)}]$, $e^{-b\pi}$ and their product. Remembering that the terms with odd values of $a$ are positive and those with even values are negative, we selected values of $a$ and phase angles to result in convergence of eta to zero.

In Table 1, we see that with the chosen values of $a$ and the phase angles, the sum of all of the terms is -.00000345. While this is not zero, it shows that we may venture as far as we wish and get closer and closer to zero.

To check our results, we can calculate $(a+bi)^{(c+di)}$ and get the same magnitude and a phase angle of $\frac{\pi}{3}$. For example, we can take $a = 5$ and $b = 1.28325520$, whose magnitude is 5.162048422. When we calculate $(a+bi)^{(c+di)}$ using $a = 5$, $b = 1.28325520$, $c = .5$, and $d = -3.26661348$, we get $(5 + 1.28325520i)^{(.5-3.26661348i)} = 2.581024297 + 4.470465023i$, whose magnitude is 5.162048425 and phase is 1.047197532$\approx \frac{\pi}{3}$. Figure 3 shows the plotted results for $a = 2, 5, 7$ to show that $(r_2, \theta_2)$ has a phase of $\pm\frac{\pi}{3}$ and a magnitude of $(r_1, \theta_1)$.

| $a$ | $\theta$ | $b$ | $d$ | $(-1)^{a-1}e^{-b\pi}Re(a+bi)^{-(c+di)}$ |
|---|---|---|---|---|
| 1 | 0 | 0 | 0 | +1 |
| 2 | $\frac{\pi}{3}$ | -.45265218 | 1.61320999 | -1.01083533 |
| 5 | $\frac{-5\pi}{3}$ | 1.28325520 | -3.26661348 | +.00171913 |
| 7 | $\frac{-7\pi}{3}$ | 1.88413270 | -3.76693197 | +.00018536 |
| 9 | $\frac{-35\pi}{3}$ | .59424240 | -16.67949596 | +.00857059 |
| 11 | $\frac{-19\pi}{3}$ | 1.6089900 | -8.29127616 | +.00028689 |
| 13 | $\frac{-19\pi}{3}$ | 2.18379140 | -7.74758726 | +.00003976 |
| 15 | $\frac{-23\pi}{3}$ | 2.31427960 | -8.88373057 | +.00002292 |
| 17 | $\frac{-25\pi}{3}$ | 2.64143680 | -9.22871990 | +.00000723 |
| | | | | Sum = -.00000345 |

Table 1: Different values of $a$ and phase angles

Notice that this table does not show some values of $a$, such as 3, 4, 6, 8, etc. This is because we have the flexibility of choosing our phase angle. We can always select a phase angle such that $b$ will be very large, making $e^{-b\pi}$ very small. This makes that term of the eta function insignificantly small. Figure 4 shows that with the chosen values of $a = 3$ and $\theta = \frac{-5\pi}{3}$, $b \approx 72.03$ and $e^{-b\pi} \approx 5.3 \times 10^{-99}$.

The first term of our variant Dirichlet eta function is a real number whose value is one. This is an anomaly since the phase angle is $0 \neq \frac{\pi}{3}$. This is caused by the Dirichlet eta function's cancellation of the Riemann zeta function's singularity at $s = 1$. This exception retains the consistency with the standard zeta and eta functions, rather than selecting a value of $b \approx \pm 9$, either of which would have distorted the results needlessly.

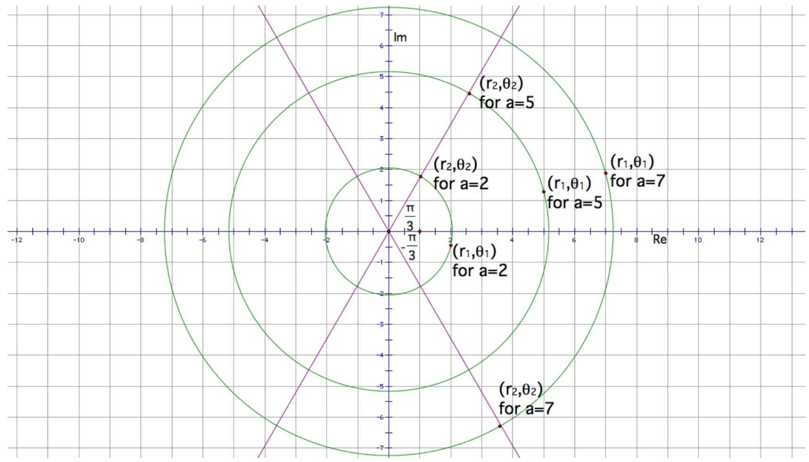


Figure 3

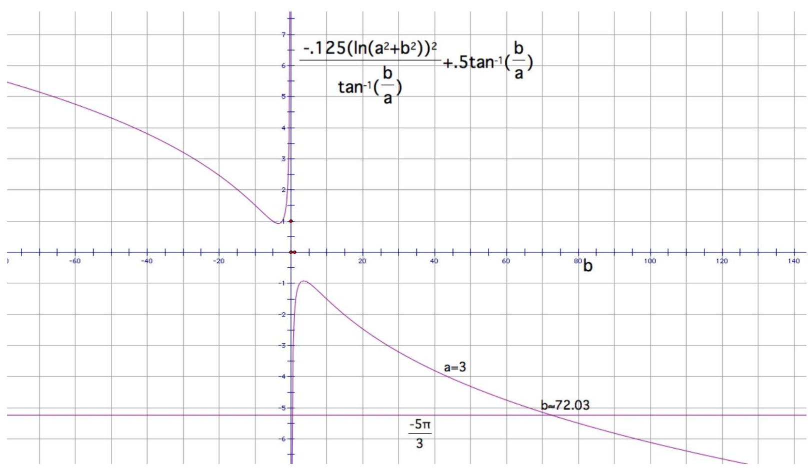


Figure 4

The values of $a = 1$, $b = 0$, $c = \frac{1}{2}$, and $d = 0$ are the same as in (17), where $\ln r_2 = \ln r_1$ and $\theta_2 = \theta_1 = i \ln r_1 = 0$.

We can use the formal definition of a limit (see [4]) that for every $\epsilon > 0$ there exists a corresponding $\delta > 0$, such that if $0 < |x - x_0| < \delta \implies |f(x) - L| < \epsilon$ then $\lim\limits_{x \to x_0} f(x) = L$. For our case $x = a$, $x_0 = a_\delta$ (where $a_\delta$ is the highest value of $a$ selected), $\delta = 1$, $f(x) = \eta(\{b\})$, $L = 0$, and $\epsilon$ is arbitrary. In Table 1, $a_\delta = 17$ and $\epsilon = .000005$.

## 8. Conclusion

While the Riemann Hypothesis remains unproven, this plausibility argument shows that when the variant Dirichlet eta function's exponent has a real part equal to one half, the function can converge to zero. As with other variant zeta and eta function studies, this study may hold additional insights into a future proof of the Riemann Hypothesis. Additionally, the methods of this study have been applied to the question of the Mass Gap Hypothesis in Appendix.

## Acknowledgments

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## Appendix: An Application to the Mass Gap Hypothesis

Devlin [5] writes, "The essential mechanism for producing mass is the converse of Einstein's famous equation $E = mc^2$, that is, $m = \frac{E}{c^2}$, which tells you that you can get mass from pure energy. Experiments, computer simulations, and some theoretical calculations have led physicists to believe that there must be a 'mass gap', i.e., there is a nonzero minimum energy level for excitations of the vacuum (no massless particle-waves)... To date, no one has been able to prove this property rigorously." Hawking [6] wrote that most of the universe may still exist in imaginary time and that imaginary time is at right angles to ordinary, real time. Combining these ideas with the techniques in this study may provide an approach and/or a plausibility argument of the Mass Gap Hypothesis.

Considering $E$ and $m$ as complex variables and $c^2$ is replaced by $(a+bi)^{(c+di)}$, where $a =$c (the speed of light) and $c = 2$, we can formulate

$$\begin{bmatrix} \ln E_r \\ E_\theta \end{bmatrix} = \begin{bmatrix} c & -d \\ d & c \end{bmatrix} \begin{bmatrix} \ln m_r \\ m_\theta \end{bmatrix}$$

as in (8).

Applying the inverse matrix to both sides yields

$$\begin{bmatrix} \ln m_r \\ m_\theta \end{bmatrix} = \begin{bmatrix} \frac{c}{c^2+d^2} & \frac{d}{c^2+d^2} \\ \frac{-d}{c^2+d^2} & \frac{c}{c^2+d^2} \end{bmatrix} \begin{bmatrix} \ln E_r \\ E_\theta \end{bmatrix}.$$

Using our previous eigenvector solutions, the solution for energy would be: $\ln E_r = c \ln m_r - id \ln m_r = (c - di) \ln m_r$ as in (10).

For mass, we multiply the equations by $i$ to reflect the transformation between real and imaginary time and the result is

$$\ln m_r = \frac{-d}{c^2+d^2} \ln E_r + \frac{ic}{c^2+d^2} \ln E_r$$
$$= (d - ci) \left[ \frac{-1}{c^2+d^2} \right] \ln E_r = (d - ci) \ln E_r^{-1/(c^2+d^2)}.$$

The equations for mass are in the same form as the equations for energy except that the real and imaginary parts of the exponents have been interchanged and the magnitude of the energy has been scaled. This suggests that the existence of quantum energy implies the existence of quantum mass. The similarity of plausibility arguments for both the Riemann Hypothesis and the Mass Gap Hypothesis suggest that this variant Dirichlet eta function may warrant additional study.