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AN ASYMPTOTIC VANISHING THEOREM FOR GENERIC FAT POINTS IN POSITIVE CHARACTERISTIC

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Abstract: Let X be an integral projective variety defined over an algebraically closed field $\mathbb{K}$ such that $p := \mathrm{char}(\mathbb{K}) > 0$. Fix an integer $0 < m < p$ and $M, L \in \mathrm{Pic}(X)$ with L ample. Here we prove (following a paper by Alexander and Hirschowitz) the existence of an integer $d(m, X, L, M)$ such that for all integers $d \geq d(M, X, L, M)$ either $h^0(X, \mathcal{I}_Z \otimes M \otimes L^{\otimes d}) = 0$ or $h^1(X, \mathcal{I}_Z \otimes M \otimes L^{\otimes d}) = 0$, where Z is a general union of i-points of X with $i \leq m$.

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Let X be an integral projective variety defined over an algebraically closed field $\mathbb{K}$ such that $p := \mathrm{char}(\mathbb{K}) > 0$. Set $n := \dim(X)$ and assume $n > 0$. For any $P \in X_{reg}$ and any integer $i > 0$ let iP denote the closed subscheme of X with $(\mathcal{I}_P)^i$ as its ideal sheaf. Hence $(iP)_{red} = \{P\}$ and $\mathrm{length}(iP) = \binom{n+i-1}{n}$. The scheme iP will be called an i-point of X. Here we will prove the following positive characteristic version of [1].

Theorem 1. *Let X be an integral projective variety defined over an algebraically closed field $\mathbb{K}$ such that $p := char(\mathbb{K}) > 0$. Fix an integer $0 < m < p$ and $M, L \in Pic(X)$ such that L is ample. There is an integer $d(m, X, L, M)$ such that for all integers $d \geq d(M, X, L, M)$ either $h^0(X, \mathcal{I}_Z \otimes M \otimes L^{\otimes d}) = 0$ or $h^1(X, \mathcal{I}_Z \otimes M \otimes L^{\otimes d}) = 0$, where Z is a general union of i-points of X with*

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$i \leq m$.

Without some assumptions on X or m and p Theorem 1 is not true even when $n = 1$ (see [1], Remark 7.5). Our proofs are just a small modification of the proofs in [1]. We use the assumption $p > m$ both in the inductive step and in the proof of Proposition 1. To get the case $m \geq p$ if $n = 1$ (as in [1] for smooth curve) we need an assumption on the curve X: $\mathrm{Pic}^0(X)$ must be an extension of an Abelian variety by a trus $\mathbb{C}_m^k$, i.e. $\mathrm{Pic}^0(X)$ contains no copy of $\mathbb{C}_a$, i.e. the p-power map $\mathrm{Pic}^0(X) \to \mathrm{Pic}^0(X)$ should have finite kernel. This assumption allows us to avoid the quotation of Lemma 2 in the proof of Proposition 1 and hence to get its thesis for the curve X with no restriction on p and m.

Remark 1. Let X be an integral projective curve. Set $g := p_a(X)$. Since $\chi(\mathcal{O}_X) = -\chi(\omega_X) = 1 - g$ by duality (see [2]), Riemann-Roch (see [3], pp. 130-131) gives $\deg(\omega_X) = 2g - 2$ even if X is not Gorenstein.

Lemma 1. *Let X be an integral projective curve. Set $g := p_a(X)$. Fix an integer d and a rank 1 torsion free sheaf F on X such that $\deg(F) = 0$. Let L be a general member of $\mathrm{Pic}^d(X)$.*

(i) If $d \leq g - 1$, then $h^0(X, F \otimes L) = 0$.

(ii) If $d \geq g - 1$, then $h^1(X, F \otimes L) = 0$.

Proof. Since X is integral, $\mathrm{Pic}^d(X)$ is integral and of dimension g. We will first check part (i). Taking $L(D)$ instead of L with D effective and Cartier we reduce to the case $d = g - 1$. There is an integer e and an effective degree e Cartier divisor E such that $h^1(X, F \otimes L(E)) = 0$. Hence $h^0(X, F \otimes L(E)) = e$ (Rieman-Roch for torsion free sheaves (see [3], pp. 130-131)). Take a general $(P_1, \dots, P_e) \in X^e$. Hence $P_j \in X_{reg}$ for all j. Thus each sheaf $\mathcal{O}_X(-P_j)$ is invertible. For any integer f such that $0 \leq f \leq e$ set $A_f := F \otimes L(E)(-P_1 - \cdots - P_f)$. Hence $A_0 := F \otimes L(E)$ and $h^0(X, A_0) = e$. Now we use induction on f to prove $h^0(X, A_f) = e - f$ for all f. Fix an integer f such that $1 \leq f \leq e$. Assume $h^0(X, A_{f-1}) = e - f + 1$. Since $e - f + 1 > 0$ and X is an integral curve, the base locus of the sheaf A_{f-1} is zero-dimensional or empty. Hence it does not contain a general point $P \in X$. Since P_f is general after fixing $P_1, \dots, P_{f-1}$, we get $h^0(X, A_f) = e - f$. In this way we get $h^0(X, A_e) = 0$. Since $\deg(A_e) = g - 1$ and $\mathrm{Pic}^{g-1}(X)$ is irreducible, part (i) follows from semicontinuity. Now we will check part (ii). By Serre's duality (see [2]) it is sufficient to prove $h^0(X, \omega_X \otimes L^* \otimes F^*) = 0$. Use the previous proof using the torsion free sheaf $\omega_X \otimes L^* \otimes F^*$ instead of the sheaf $F \otimes L$. $\square$

Lemma 2. *Assume $n = 1$. Fix integers t, r such that $r > 0$ and $(r, p) = 1$. Then the map $u_t : Pic^t(X) \to Pic^{tr}(X)$ defined by $u(L) := L^{\otimes r}$ is dominant.*

Proof. If $g = 0$, then $X \cong \mathbb{P}^1$ and hence the proposition is obvious. From now on we assume $g > 0$. Since $\mathrm{Pic}^t(X)$ and $\mathrm{Pic}^{tr}(X)$ are integral and with the same dimension, it is sufficient to prove that a general fiber of u_t is zero-dimensional. We easily reduce to the case $t = 0$. $\mathrm{Pic}^0(X)$ is a commutative, reduced and irreducible algebraic group, u_0 is a group homomorphism and $\mathrm{Ker}(u_0)$ is the set of all r-torsion line bundles. It is sufficient to use that any commutative algebraic group G over $\mathbb{K}$ has only finitely many r-torsion points if r and $\mathrm{char}(\mathbb{K})$ are coprime. This assertion is well-known, but we will sketch a proof of it when $p := \mathrm{char}(\mathbb{K}) > 0$ following [4], pp. 40-41. We may assume that G is connected. There is a connected commutative linear algebraic subgroup R of G such that G/R is an Abelian variety (see [4], Proposition 11 at p. 40). R is the product of a torus and a unipotent group U, i.e. (for some $n > 0$) a subgroup of the group of the triangular $n \times n$ matrices having only 1's on the diagonal (see [4], p. 40). Hence it is sufficient to show that the r-torsion of U is finite. We claim that U has no r-torsion. For any $y \in U$ we have $(y - \mathrm{Id})^{p^n} = 0$ and hence $y^{p^n} = 1$. Since $(r, p) = 1$, $y^r = 1$ if and only if $y = 1$. □

Proposition 1. *Assume $n = 1$ and set $g := p_a(X)$. As in Theorem 1 assume $p > m$. Fix $M, L \in Pic(X)$ such that L is ample: Set $a := \deg(M)$ and $b := \deg(L)$. Fix $m > 0$ and let $d > 0$ be an integer such that $a + bd \geq g + m(m-1)(g-1)/2$. Let Z be a general union of arbitrary numbers of i-points of X for all $1 \leq i \leq m$. Then either $h^0(X, \mathcal{I}_Z \otimes M \otimes L^{\otimes d}) = 0$ or $h^1(X, \mathcal{I}_Z \otimes M \otimes L^{\otimes d}) = 0$.*

Proof. We recall that for every integer z the scheme $\mathrm{Pic}^z(X)$ is an integral g-dimensional quasi-projective variety. Since $\deg(M \otimes L^d) \geq 2g - 1$, $h^1(X, M \otimes L^{\otimes d}) = 0$ and hence $h^0(X, M \otimes L^{\otimes d}) = a + bd + 1 - g$. Let n_i be the number of i-points of Z. First assume $\deg(Z) \geq h^0(X, M \otimes L^{\otimes d})$. We need to check the vanishing of $H^0(X, \mathcal{I}_Z \otimes M \otimes L^{\otimes d})$. Since $\deg(Z) = \sum_{i=1}^m in_i$, the assumption $a + bd \geq g + m(m-1)(g-1)/2$ implies the existence of an integer i such that $1 \leq i \leq m$ and $n_i \geq g$. Fix $(P_1, \dots, P_h)$ general in X^g. Write $Z = W \cup \cup_{h=1}^g iP_h$, where W is a general union of n_j j-points, $1 \leq j \leq m$, $j \neq i$, and $n_i - g$ i-points. W and Z are Cartier divisors and $\mathcal{I}_Z \otimes M \otimes L^{\otimes d} = R(-i(P_1 + \cdots + P_g))$, where $R := M \otimes L^{\otimes d}(-W)$. To get a general Z we may first fix W and then take a general $(P_1, \dots, P_g) \in X^g$. It is easy to check that for general $(P_1, \dots, P_g) \in X^g$ the line bundle $\mathcal{O}_X(P_1 + \cdots + P_g)$ is a general member of $\mathrm{Pic}^g(X)$. Lemma 2 for $t = g$ and $r = i$ gives that $\mathcal{O}_X(i(P_1 + \cdots + P_g))$

is a general member of $\mathrm{Pic}^{ig}(X)$; here we use the assumption $m > p$ to get $(i,p) = 1$, a condition needed to apply Lemma 2. Hence $M \otimes L^{\otimes d}(-Z)$ is a general member of $\mathrm{Pic}^c(X)$ with $c := a + db - \deg(Z) \le g - 1$. Apply Lemma 1. Now assume $\epsilon := h^0(X, M \otimes L^{\otimes d}) - \deg(Z) > 0$. Take the new integers $n'_1 := n_1 + \epsilon$ and $n'_j := n_j$ for all $j \ge 2$. Let Z' (resp. Z) be the union of n'_i (resp. n_i) general i-points for all $i \in \{1, \dots, m\}$. Thus Z' is the union of Z and of ϵ general points of X. Since we proved $h^0(X, \mathcal{I}_{Z'} \otimes M \otimes L^{\otimes d}) = 0$, we get $h^0(X, \mathcal{I}_Z \otimes M \otimes L^{\otimes d}) \le \epsilon$. Since Riemann-Roch gives the opposite inequality, we get $h^0(X, \mathcal{I}_Z \otimes M \otimes L^{\otimes d}) = \epsilon$. Riemann-Roch gives $h^1(X, \mathcal{I}_Z \otimes M \otimes L^{\otimes d}) = 0$. □

Proof of Theorem 1. The case $n = 1$ is true by Proposition 1. The case $n > 1$ is proved by induction on n as in [1]. We only remark that even for the inductive step we need the assumption $p > m$, because in the proofs in [1], §8, a change of variable $t \mapsto t^r$, r an unknown integer $\le m$, is used and we need that r and p are coprime. □

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