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REPRESENTATIONS THEOREMS FOR SCALAR FUNCTIONS IN A 4-DIMENSIONAL EUCLIDEAN SPACE

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Abstract: In a 4-dimensional Euclidean space, representations theorems are obtained here for scalar valued isotropic functions depending on an arbitrary number of scalars, skew-symmetric second order tensors and symmetric second order tensors; at least one of these last ones is assumed to have an eigenvalue with multiplicity 1. The case with at least a non null vector, among the independent variables, has already been treated in literature; so it is here not treated. The result is a finite, but long, set of scalar valued isotropic functions such that every other scalar function of the same variables can be expressed as a function of the elements of this set. The methodology used to obtain this set is directed in trying to use similar representation theorems, already known in literature for the 3-dimensional case.

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1. Introduction

Representation theorems are the mathematical tool by which to impose the physical principle requiring that the laws of physics do not depend on the observer. Now these laws are expressed in terms of tensors $F^{\alpha_1 \cdots \alpha_n}$ which, in turns, are functions of other tensors $X_{\beta_1 \cdots \beta_m}$; moreover, we know the trans-

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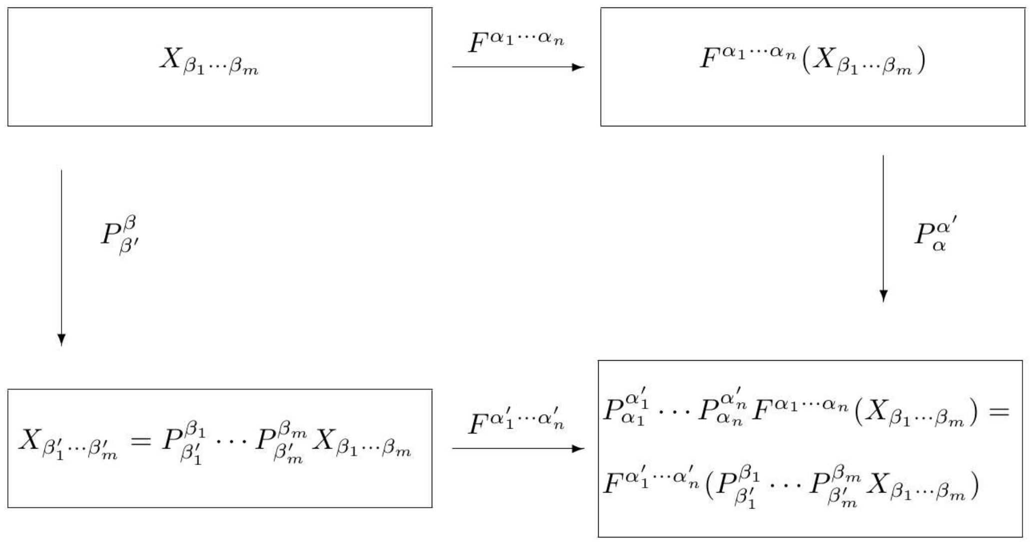

Figure 1: The condition defining isotropic functions

formation laws of the components of a tensor, when the basis of the vectorial space is changed. Therefore, the above mentioned requirement amounts in imposing that the diagram in Figure 1 is commutative, where $P^{\alpha'}_{\alpha}$ is the matrix of the change of basis. In other words, proceeding from the upper left corner of the figure and moving to the right hand side, we start from the independent variables $X_{\beta_1\cdots\beta_m}$ in the reference frame Σ, then we apply to them the function $F^{\alpha_1\cdots\alpha_n}$, in Σ; after that, proceeding to the lower side, we transform the result in the reference Σ'. Following the other side of the diagram, we transform the independent variables in Σ'; on the transformed variables, we apply the function $F^{\alpha'_1\cdots\alpha'_n}$, in Σ'. We require that the result is the same.

What has already been done, about this, in literature? Regarding the framework of a 3-dimensional Euclidean vectorial space, in [5], [7] the case has been considered with $n = 0$ (an arbitrary number of scalars), $n = 1$ (an arbitrary number of vectors), $n = 2$ (an arbitrary number of second order tensors, some of which are symmetric and the remaining ones are skew-symmetric); similar values have been considered for m, that is to distinguish the different types of tensorial functions. The result is a set S^0 of particular scalar functions such that every other scalar function of the same variables can be expressed as a function of the elements of S^0; similarly, for the other values of m we obtain a set S^m of particular tensorial functions of order m such that every other tensorial function of the same order, and depending on the same variables, can be expressed as a linear combination of the elements of S^m through scalar coefficients. The sets S^m are called "representations"; they are called "irreducible" if no proper subset satisfies the same property. In [1] Boheler proved that the representations exposed in [5] are not irreducible, showing also some their redundant elements. In [3] Pennisi and Trovato proved that once eliminated the elements indicated by Boheler, the remaining elements furnish a complete and irreducible representation; but in [6] Telega had expressed a different result, so that the located they mistake in Telega's argument.

In [2] complete representations were found also for the case $m = 3$, that is third order tensorial functions.

Regarding the case of a 4-dimensional vectorial space, Pennisi and Trovato furnished in [4] complete representations, but only for the case of a pseudo-Euclidean vectorial space and with the hypothesis that, among the independent variables, there is a time-like 4-vector. Obviously, their result holds also in the case of a 4-dimensional Euclidean vectorial space when, among the independent variables, there is a 4-vector different from zero. There remain to exploit the other cases, where there are no vectors among the independent variables. Of these cases, for the sake of brevity, we will consider only that where, among the independent variables, there is at least a symmetric second order tensor $\mathcal{A}$ endowed with an eigenvalue a with multiplicity 1. We purpose to exploit in the future the remaining cases.

We will list the results, that is the set S^0, in Section 2 in order not to lose the thread of what one is saying, while we will describe now how it has been obtained. To this end, let $\vec{v}$ be a real eigenvector corresponding to the eigenvalue a of $\mathcal{A}$, with multiplicity 1; in the reference frames whose x_1 axis is directed as $\vec{v}$, the tensor $\mathcal{A}$ takes the form

$$\mathcal{A} = \begin{pmatrix} a & \vec{0}^T \\ \vec{0} & A \end{pmatrix} \tag{1}$$

with A symmetric second order tensor belonging to the 3-dimensional Euclidean space orthogonal to $\vec{v}$. Let

$$\lambda^4 - S_1\lambda^3 + S_2\lambda^2 - S_3\lambda^4 + S_4 = 0 \tag{2}$$

be the characteristic equation of $\mathcal{A}$, that is the equation whose roots are the eigenvalues. Obviously, the scalars S_i can be expressed in terms of the scalars $tr\mathcal{A}^i$ for $i = 1,..,4$ belonging to the set S^0 of Section 2 (the abbreviation tr denotes the trace of the subsequent tensor). If now we define

$$s_1 = S_1 - a \quad , \quad s_2 = S_2 - as_1 \quad , \quad s_3 = S_3 - as_2 \quad , \tag{3}$$

we have that $S_4 = as_3$ and $a^3 - s_1 a^2 + s_2 a - s_3 \neq 0$ because a is root of (2) with multiplicity 1; moreover, $\lambda^3 - s_1\lambda^2 + s_2\lambda - s_3 = 0$ is the characteristic equation of A, so that, thanks to the Hamilton-Kayley's Theorem, we have

$$A^3 = s_1 A^2 - s_2 A + s_3 I.$$

This fact suggests us to define

$$I_1 = \frac{\mathcal{A}^3 - s_1\mathcal{A}^2 + s_2\mathcal{A} - s_3 I}{a^3 - s_1 a^2 + s_2 a - s_3} \quad \text{obtaining that} \quad I_1 = diag(1,0,0,0). \tag{4}$$

This tensor, in turns, allows us to split every tensor into a part parallel to $\vec{v}$ and into a part belonging to the 3-dimensional Euclidean space orthogonal to $\vec{v}$. More precisely, every symmetric second order tensor $\mathcal{M}$ may be decomposed in

$$\mathcal{M} = \begin{pmatrix} m_{11} & \vec{m}^T \\ \vec{m} & M \end{pmatrix}. \tag{5}$$

Moreover, we can use the change of variables, from $\mathcal{M}$ to

$$\begin{aligned} & m_{11} = trI_1\mathcal{M}, \\ & \vec{M} = \mathcal{M}I_1 - m_{11}I_1 = \begin{pmatrix} 0 & \vec{0}^T \\ \vec{m} & 0 \end{pmatrix}, \\ & M^* = \mathcal{M} - \vec{M} - \vec{M}^T - m_{11}I_1 \\ & \qquad = \mathcal{M} - \mathcal{M}I_1 - I_1\mathcal{M} + m_{11}I_1 = \begin{pmatrix} 0 & \vec{0}^T \\ \vec{0} & M \end{pmatrix} \text{ if } \mathcal{M} \text{ is symmetric}, \\ & M^* = \mathcal{M} - \vec{M} + \vec{M}^T \\ & \qquad = \mathcal{M} - \mathcal{M}I_1 - I_1\mathcal{M} = \begin{pmatrix} 0 & \vec{0}^T \\ \vec{0} & M \end{pmatrix}, \text{ if } \mathcal{M} \text{ is skew-symmetric}. \end{aligned} \tag{6}$$

Moreover, for every change of frames which leave unchanged the x_1 axis, we have that m_{11} behaves as a scalar, while $\vec{M}$ and M^* maintain their form except for substituting $\vec{m}$ and M with the expressions obtained for 3-dimensional vectors and second order tensors, with a 3-dimensional transformation of the reference frame. Can we apply the known representation theorems in a 3-dimensional space? We can say yes, for two reasons:

— First of all, we may rewrite, in Figure 2, the diagram in Figure 1 but for the particular case $n = 0$, that is for scalar functions;

it shows that the scalar function F is determined if we know it in the reference frames whose x_1 axis is directed as $\vec{v}$. On the other hand, in these frames it is determined as an arbitrary function of the scalars in the 3-dimensional representation.

— We can convert the scalars of the 3-dimensional representation in the

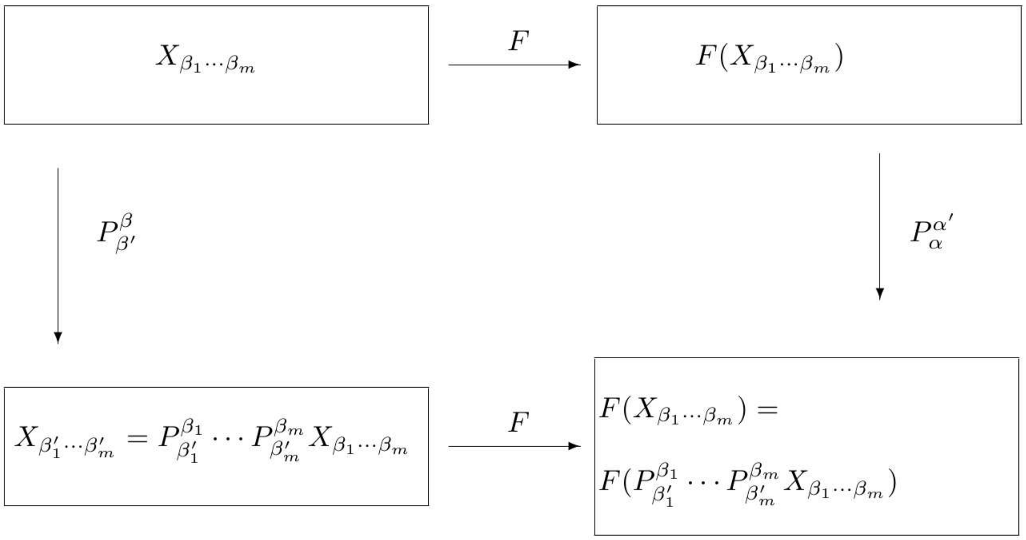

Figure 2: The condition defining isotropic scalar functions

present 4-dimensional form; it suffices to observe that

$$M^* N^* = \begin{pmatrix} 0 & \vec{0}^T \\ \vec{0} & MN \end{pmatrix} ; \; N^* \vec{M} = \begin{pmatrix} 0 & \vec{0}^T \\ N\vec{m} & 0 \end{pmatrix} ; \qquad (7)$$

$$\vec{P}^T N^* \vec{M} = \begin{pmatrix} \vec{p}^T N \vec{m} & \vec{0}^T \\ \vec{0} & 0 \end{pmatrix} ;$$

from which $\quad trM^* N^* = trMN \qquad$ and $\qquad tr\vec{P}^T N^* \vec{M} = \vec{p}^T N \vec{m}$.

Let us now apply the above described guidelines. Let us start from the representation S^0 for 3-dimensional scalar functions, which is reported in [3]. We observe that in [3] the second order tensors are indicated in correspondence with the indexes i, j, k going from 1 to N. But here we have to explicitate the terms containing the tensor A which comes from $\mathcal{A}$. Consequently, we can consider the indexes i, j, k going from 0 to N; after that we can write explicitly the terms with the index 0 and substitute A_0 with A. In this way we obtain a set of scalars S; we do not report them for the sake of brevity, but effectively they can be identified with those reported in Table 1 of Section 3, raws from 1 to 18, and with $\vec{a}_\alpha$, $\vec{a}_\beta$ instead of the original $\vec{v}_\alpha$, $\vec{v}_\beta$. We have now to take into account of the fact that the vectors $\vec{v}_\alpha$, $\vec{v}_\beta$ may originate from the 4-dimensional symmetric second order tensors, or from the skew-symmetric ones. Therefore, let us write 3 copies of the set S: In the first one we substitute $\vec{v}_\alpha$, $\vec{v}_\beta$ with $\vec{a}_\alpha$,

$\vec{a}_\beta$ (which originate from the 4-dimensional symmetric second order tensors) obtaining just the raws from 1 to 18 of Table 1 in Section 3. In the second copy we leave the terms not depending on vectors (because already considered above) and, in the remaining ones, we substitute $\vec{v}_\alpha$, $\vec{v}_\beta$ with $\vec{\omega}_\gamma$, $\vec{\omega}_\delta$ (which originate from the 4-dimensional skew-symmetric second order tensors), so obtaining the raws from 19 to 29 of Table 1 in Section 3. From the third copy we take only the terms where both $\vec{v}_\alpha$, $\vec{v}_\beta$ appear (because the remaining ones have already been considered above) and substitute $\vec{v}_\alpha$ with $\vec{a}_\alpha$ and $\vec{v}_\beta$ with $\vec{\omega}_\gamma$ so obtaining the raws from 30 to 35 of Table 1 in Section 3.

In Section 4 we will use the above mentioned method to express in 4-dimensional notation the scalars which we have listed in Section 3. As it can be seen from the result, they are affected unfortunately by the tensor I_1 of equation (4); to overcome this drawback, we have substituted I_1 from equation (4) in the scalars of Section 4 and, for every term containing a linear combination of the variables $s_i/(a^3 - s_1 a^2 + s_2 a - s_3)$ we have taken the coefficients of this combination and the eventual zero degree term, and inserted them in the final representation. In this way we have obtained the scalars listed in the following section, where we have also eliminated some terms already appearing in the list. In this way we are sure that every scalar valued function, of our independent variables, can be expressed as a function of the scalars of the set S^0 of in Section 2; in other words, we can say that S^0 furnishes a complete representation. Obviously, we do not know if S^0 is an irreducible representation, in the sense that no of its proper subsets furnishes a complete representation; we refrain from exploiting this aspect of irreducibility for the sake of brevity, because S^0 has many elements, as it can be seen in Section 2. On the other hand, in applications the numbers H, N and M are not generic, so that it will be easier to detect eventual reduntant elements in S^0, and leave them out.

2. The Final Representation for 4-Dimensional Scalar Valued Isotropic Functions

We will describe in this section the final result of the present paper. It amounts in the following

Representation Theorem. *Every scalar valued isotropic function depending on the scalars λ_h, on the symmetric second order tensor $\mathcal{A}$ endowed at least with an eigenvalue of multiplicity 1, on the further symmetric second order tensors $\mathcal{A}_i$ and on the skew-symmetric second order tensors Ω_γ for $h = 1, ..., H$,*

$i = 1, ..., N$, $\gamma = 1, ..., M$, *can be expressed as a function of the elements of the following set* S^0.

We describe this set in terms of particular scalars depending on some indexes whose variability range is $h = 1, ..., H$; $i, j, k = 1, ..., N$ with $i < j < k$; $p, q, r = 1, ..., M$ with $p < q < r$; $\alpha, \beta = 1, ..., N$ with $\alpha < \beta$, and $\gamma, \delta = 1, ..., M$ with $\gamma < \delta$. The set S^0 can be obtained, in the above mentioned way, from the following scalars

The set S^0

λ_h, $tr\mathcal{A}$, $tr\mathcal{A}^2$, $tr\mathcal{A}^3$, $tr\mathcal{A}^4$, $tr\mathcal{A}_i$, $tr\mathcal{A}_\alpha^2$, $tr\mathcal{A}_\alpha\mathcal{A}_\beta$, $tr\mathcal{A}\mathcal{A}_i$, $tr\mathcal{A}^2\mathcal{A}_i$,
$tr\mathcal{A}_i^2$, $tr\mathcal{A}_\alpha\mathcal{A}^2\mathcal{A}_\alpha$, $tr\mathcal{A}\mathcal{A}_i^2$, $tr\mathcal{A}_\alpha\mathcal{A}_i\mathcal{A}_\alpha$, $tr\mathcal{A}_\alpha\mathcal{A}\mathcal{A}_\beta$,
$tr\mathcal{A}_\alpha\mathcal{A}\mathcal{A}_i\mathcal{A}_\beta$, $tr\mathcal{A}_i^3$, $tr\mathcal{A}_\alpha\mathcal{A}_i^2\mathcal{A}_\alpha$, $tr\mathcal{A}_\alpha\mathcal{A}^2\mathcal{A}_\beta$, $tr\mathcal{A}_\alpha\mathcal{A}_i\mathcal{A}_\beta$,
$tr\mathcal{A}_i^2\mathcal{A}_j$, $tr\mathcal{A}_i\mathcal{A}_j^2$, $tr\mathcal{A}_\alpha\mathcal{A}_i^2\mathcal{A}_\beta$, $tr\mathcal{A}_\alpha(\mathcal{A}_i\mathcal{A}_j - \mathcal{A}_j\mathcal{A}_i)\mathcal{A}_\beta$, $tr\mathcal{A}_\alpha\mathcal{A}_i\mathcal{A}_j\mathcal{A}_\alpha$,
$tr\mathcal{A}_\alpha(\mathcal{A}\mathcal{A}_i - \mathcal{A}_i\mathcal{A})\mathcal{A}_\beta$, $tr\mathcal{A}_\alpha\Omega_p\mathcal{A}_\beta$, $tr\mathcal{A}_\alpha\mathcal{A}\Omega_\gamma$,
$tr\Omega_p^2$, $tr\Omega_\gamma\Omega_\delta$, $tr\Omega_\gamma\Omega_p\Omega_\delta$, $tr\mathcal{A}_\alpha\Omega_p\Omega_\gamma$,
$tr\mathcal{A}_\alpha\Omega_p^2\mathcal{A}_\alpha$, $tr\mathcal{A}_\alpha\Omega_p\Omega_q\mathcal{A}_\alpha$, $tr\mathcal{A}_\alpha\Omega_p^2\Omega_\gamma$, $tr\mathcal{A}_\alpha\Omega_p^2\Omega_q\mathcal{A}_\alpha$, $tr\mathcal{A}_\alpha\Omega_q^2\Omega_p\mathcal{A}_\alpha$,
$tr\mathcal{A}_\alpha\Omega_p^2\mathcal{A}_\beta$, $tr\mathcal{A}_\alpha(\Omega_p\Omega_q - \Omega_q\Omega_p)\mathcal{A}_\beta$, $tr\Omega_\gamma\mathcal{A}\Omega_\gamma$, $tr\Omega_\gamma\mathcal{A}\Omega_\delta$, $tr\mathcal{A}\Omega_p^2$, $tr\mathcal{A}\Omega_p\Omega_q$,
$tr\Omega_\gamma\mathcal{A}^2\Omega_\gamma$, $tr\Omega_\gamma\mathcal{A}^2\Omega_\delta$, $tr\mathcal{A}^2\Omega_p^2$, $tr\Omega_\gamma\mathcal{A}\Omega_p\Omega_\gamma$, $tr\Omega_\gamma\mathcal{A}^2\Omega_p\Omega_\gamma$,
$tr\Omega_\gamma(\mathcal{A}\Omega_p - \Omega_P\mathcal{A})\Omega_\delta$, $tr\mathcal{A}^2\mathcal{A}_i\Omega_p$, $tr\mathcal{A}_\alpha\mathcal{A}_i\Omega_\gamma$, $tr\mathcal{A}_\alpha\mathcal{A}\Omega_p\mathcal{A}_\alpha$,
$tr\mathcal{A}_\alpha(\mathcal{A}_i\mathcal{A}_j - \mathcal{A}_j\mathcal{A}_i)\Omega_\gamma$, $tr\mathcal{A}_\alpha(\mathcal{A}\mathcal{A}_i - \mathcal{A}_i\mathcal{A})\Omega_\gamma$,
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$tr\mathcal{A}\mathcal{A}_\alpha\Omega_p\mathcal{A}\Omega_p^2\mathcal{A}_\alpha$, $tr\mathcal{A}\Omega_p(\Omega_p\mathcal{A}-\mathcal{A}\Omega_p)\Omega_p\mathcal{A}_\alpha$,
$tr\mathcal{A}_i\Omega_p^2\mathcal{A}_j\Omega_p$, $tr\mathcal{A}\Omega_p^2\mathcal{A}_i\Omega_p$, $tr\mathcal{A}_i^2\Omega_p^2\mathcal{A}_i\Omega_p$, $tr\mathcal{A}^2\Omega_p^2\mathcal{A}\Omega_p$,
$tr\mathcal{A}(\mathcal{A}_i\Omega_p^2\mathcal{A}_j\Omega_p+\Omega_p\mathcal{A}_i\Omega_p^2\mathcal{A}_j+\mathcal{A}_j\Omega_p\mathcal{A}_i\Omega_p^2+\Omega_p\mathcal{A}_j\Omega_p\mathcal{A}_i\Omega_p+\Omega_p^2\mathcal{A}_j\Omega_p\mathcal{A}_i)$,
$tr\mathcal{A}(\mathcal{A}\Omega_p^2\mathcal{A}_i\Omega_p+\Omega_p\mathcal{A}\Omega_p^2\mathcal{A}_i+\mathcal{A}_i\Omega_p\mathcal{A}\Omega_p^2+\Omega_p\mathcal{A}_i\Omega_p\mathcal{A}\Omega_p+\Omega_p^2\mathcal{A}_i\Omega_p\mathcal{A})$,
$tr\mathcal{A}(\mathcal{A}_i^2\Omega_p^2\mathcal{A}_i\Omega_p+\Omega_p\mathcal{A}_i^2\Omega_p^2\mathcal{A}_i+\mathcal{A}_i\Omega_p\mathcal{A}_i^2\Omega_p^2+\Omega_p\mathcal{A}_i\Omega_p\mathcal{A}_i^2\Omega_p+\Omega_p^2\mathcal{A}_i\Omega_p\mathcal{A}_i^2+$
$+\mathcal{A}_i\Omega_p^2\mathcal{A}_i\Omega_p\mathcal{A}_i)$,
$tr\mathcal{A}(\mathcal{A}^2\Omega_p^2\mathcal{A}\Omega_p+\Omega_p\mathcal{A}^2\Omega_p^2\mathcal{A}+\mathcal{A}\Omega_p\mathcal{A}^2\Omega_p^2+\Omega_p\mathcal{A}\Omega_p\mathcal{A}^2\Omega_p+\Omega_p^2\mathcal{A}\Omega_p\mathcal{A}^2+\mathcal{A}\Omega_p^2\mathcal{A}\Omega_p\mathcal{A})$,
$tr\mathcal{A}^2\mathcal{A}_\alpha\mathcal{A}^2\mathcal{A}_\alpha$, $tr\mathcal{A}^2\mathcal{A}_\alpha\mathcal{A}_i\mathcal{A}_\alpha$, $tr\mathcal{A}^2\mathcal{A}_\alpha\mathcal{A}\mathcal{A}_i\mathcal{A}_\beta$, $tr\mathcal{A}^2\mathcal{A}_\alpha\mathcal{A}_i^2\mathcal{A}_\alpha$,
$tr\mathcal{A}^2\mathcal{A}_\alpha\mathcal{A}^2\mathcal{A}_\beta$, $tr\mathcal{A}^2\mathcal{A}_\alpha\mathcal{A}_i\mathcal{A}_\beta$, $tr\mathcal{A}^2\mathcal{A}_\alpha\mathcal{A}_i^2\mathcal{A}_\beta$, $tr\mathcal{A}^2\mathcal{A}_\alpha(\mathcal{A}_i\mathcal{A}_j-\mathcal{A}_j\mathcal{A}_i)\mathcal{A}_\beta$,
$tr\mathcal{A}^2\mathcal{A}_\alpha\mathcal{A}_i\mathcal{A}_j\mathcal{A}_\alpha$, $tr\mathcal{A}^2\mathcal{A}_\alpha(\mathcal{A}\mathcal{A}_i-\mathcal{A}_i\mathcal{A})\mathcal{A}_\beta$, $tr\mathcal{A}^2\mathcal{A}_\alpha\Omega_\gamma$,
$tr\mathcal{A}^2\mathcal{A}_\alpha\Omega_p\mathcal{A}_\beta$, $tr\mathcal{A}^2\mathcal{A}_\alpha\mathcal{A}\Omega_\gamma$, $tr\mathcal{A}^2\Omega_\gamma\Omega_p\Omega_\delta$, $tr\mathcal{A}^2\mathcal{A}_\alpha\Omega_p\Omega_\gamma$,
$tr\mathcal{A}^2\mathcal{A}_\alpha\Omega_p^2\mathcal{A}_\alpha$, $tr\mathcal{A}^2\mathcal{A}_\alpha\Omega_p\Omega_q\mathcal{A}_\alpha$, $tr\mathcal{A}^2\mathcal{A}_\alpha\Omega_p^2\Omega_\gamma$, $tr\mathcal{A}^2\mathcal{A}_\alpha\Omega_p^2\Omega_q\mathcal{A}_\alpha$,
$tr\mathcal{A}^2\mathcal{A}_\alpha\Omega_q^2\Omega_p\mathcal{A}_\alpha$, $tr\mathcal{A}^2\mathcal{A}_\alpha\Omega_p^2\mathcal{A}_\beta$, $tr\mathcal{A}^2\mathcal{A}_\alpha(\Omega_p\Omega_q-\Omega_q\Omega_p)\mathcal{A}_\beta$,
$tr\mathcal{A}^2\Omega_\gamma\mathcal{A}\Omega_\gamma$, $tr\mathcal{A}^2\Omega_\gamma\mathcal{A}\Omega_\delta$, $tr\mathcal{A}^2\Omega_\gamma\mathcal{A}^2\Omega_\gamma$, $tr\mathcal{A}^2\Omega_\gamma\mathcal{A}^2\Omega_\delta$, $tr\mathcal{A}^2\Omega_\gamma\mathcal{A}\Omega_p\Omega_\gamma$,
$tr\mathcal{A}^2\Omega_\gamma\mathcal{A}^2\Omega_p\Omega_\gamma$, $tr\mathcal{A}^2\Omega_\gamma(\mathcal{A}\Omega_p-\Omega_P\mathcal{A})\Omega_\delta$, $tr\mathcal{A}^2\mathcal{A}_\alpha\mathcal{A}_i\Omega_\gamma$,
$tr\mathcal{A}^2\mathcal{A}_\alpha\mathcal{A}\Omega_p\mathcal{A}_\alpha$, $tr\mathcal{A}^2\mathcal{A}_\alpha(\mathcal{A}_i\mathcal{A}_j-\mathcal{A}_j\mathcal{A}_i)\Omega_\gamma$, $tr\mathcal{A}^2\mathcal{A}_\alpha(\mathcal{A}\mathcal{A}_i-\mathcal{A}_i\mathcal{A})\Omega_\gamma$,
$tr\mathcal{A}^2\Omega_\gamma(\mathcal{A}_i\mathcal{A}_j-\mathcal{A}_j\mathcal{A}_i)\Omega_\delta$, $tr\mathcal{A}^2\Omega_\gamma(\mathcal{A}\mathcal{A}_i-\mathcal{A}_i\mathcal{A})\Omega_\delta$, $tr\mathcal{A}^2\Omega_\gamma\mathcal{A}\mathcal{A}_i\Omega_\gamma$,
$tr\mathcal{A}^2\Omega_\gamma\mathcal{A}_i\mathcal{A}_j\Omega_\gamma$, $tr\mathcal{A}^2\Omega_\gamma\mathcal{A}_i\Omega_\delta$, $tr\mathcal{A}^2\Omega_\gamma\mathcal{A}_i\Omega_\gamma$, $tr\mathcal{A}^2\Omega_\gamma\mathcal{A}_i^2\Omega_\delta$,
$tr\mathcal{A}^2\Omega_\gamma\mathcal{A}_i^2\Omega_\gamma$, $tr\mathcal{A}^2\Omega_\gamma\Omega_p^2\Omega_\delta$, $tr\mathcal{A}^2\Omega_\gamma\Omega_p^2\Omega_\gamma$, $tr\mathcal{A}^2\mathcal{A}_\alpha\mathcal{A}_i^2\Omega_\gamma$,
$tr\mathcal{A}^2\mathcal{A}_\alpha\mathcal{A}^2\Omega_\gamma$, $tr\mathcal{A}^2\mathcal{A}_\alpha(\Omega_p\Omega_q-\Omega_q\Omega_p)\Omega_\gamma$, $tr\mathcal{A}^2\mathcal{A}_\alpha(\mathcal{A}\Omega_p-\Omega_p\mathcal{A})\Omega_\gamma$,
$tr\mathcal{A}^2\mathcal{A}_\alpha(\mathcal{A}_i\Omega_p-\Omega_p\mathcal{A}_i)\Omega_\gamma$, $tr\mathcal{A}^2\Omega_\gamma(\Omega_p\Omega_q-\Omega_q\Omega_p)\Omega_\delta$, $tr\mathcal{A}^2\Omega_\gamma(\mathcal{A}_i\Omega_p-\Omega_p\mathcal{A}_i)\Omega_\delta$,
$tr\mathcal{A}^2\Omega_\gamma\Omega_p\Omega_q\Omega_\gamma$, $tr\mathcal{A}^2\Omega_\gamma\mathcal{A}_i\Omega_p\Omega_\gamma$, $tr\mathcal{A}^2\Omega_\gamma\Omega_p^2\Omega_q\Omega_\gamma$, $tr\mathcal{A}^2\Omega_\gamma\Omega_p\Omega_q^2\Omega_\gamma$,
$tr\mathcal{A}^2\Omega_\gamma\mathcal{A}_i^2\Omega_p\Omega_\gamma$, $tr\mathcal{A}^2\mathcal{A}_\alpha(\mathcal{A}\Omega_p-\Omega_p\mathcal{A})\mathcal{A}_\beta$, $tr\mathcal{A}^2\mathcal{A}_\alpha(\mathcal{A}_i\Omega_p-\Omega_p\mathcal{A}_i)\mathcal{A}_\beta$,
$tr\mathcal{A}^2\mathcal{A}_\alpha\mathcal{A}_i\Omega_p\mathcal{A}_\alpha$, $tr\mathcal{A}^2\mathcal{A}_\alpha\mathcal{A}_i^2\Omega_p\mathcal{A}_\alpha$, $tr\mathcal{A}^2\mathcal{A}_\alpha\mathcal{A}^2\Omega_p\mathcal{A}_\alpha$, $tr\mathcal{A}^2\Omega_\gamma\Omega_p\mathcal{A}_i\Omega_p^2\Omega_\gamma$,
$tr\mathcal{A}^2\Omega_\gamma\Omega_p\mathcal{A}\Omega_p^2\Omega_\gamma$, $tr\mathcal{A}^2\mathcal{A}_\alpha\Omega_p\mathcal{A}_i\Omega_p^2\mathcal{A}_\alpha$, $tr\mathcal{A}^2\Omega_p(\Omega_p\mathcal{A}_i-\mathcal{A}_i\Omega_p)\Omega_p\mathcal{A}_\alpha$,
$tr\mathcal{A}^2\mathcal{A}_\alpha\Omega_p\mathcal{A}\Omega_p^2\mathcal{A}_\alpha$, $tr\mathcal{A}^2\Omega_p(\Omega_p\mathcal{A}-\mathcal{A}\Omega_p)\Omega_p\mathcal{A}_\alpha$,

$tr\mathcal{A}^2(\mathcal{A}_i\Omega_p^2\mathcal{A}_j\Omega_p + \Omega_p\mathcal{A}_i\Omega_p^2\mathcal{A}_j + \mathcal{A}_j\Omega_p\mathcal{A}_i\Omega_p^2 + \Omega_p\mathcal{A}_j\Omega_p\mathcal{A}_i\Omega_p + \Omega_p^2\mathcal{A}_j\Omega_p\mathcal{A}_i)$,
$tr\mathcal{A}^2(\mathcal{A}\Omega_p^2\mathcal{A}_i\Omega_p + \Omega_p\mathcal{A}\Omega_p^2\mathcal{A}_i + \mathcal{A}_i\Omega_p\mathcal{A}\Omega_p^2 + \Omega_p\mathcal{A}_i\Omega_p\mathcal{A}\Omega_p + \Omega_p^2\mathcal{A}_i\Omega_p\mathcal{A})$,
$tr\mathcal{A}^2(\mathcal{A}_i^2\Omega_p^2\mathcal{A}_i\Omega_p + \Omega_p\mathcal{A}_i^2\Omega_p^2\mathcal{A}_i + \mathcal{A}_i\Omega_p\mathcal{A}_i^2\Omega_p^2 + \Omega_p\mathcal{A}_i\Omega_p\mathcal{A}_i^2\Omega_p +$
$+ \Omega_p^2\mathcal{A}_i\Omega_p\mathcal{A}_i^2 + \mathcal{A}_i\Omega_p^2\mathcal{A}_i\Omega_p\mathcal{A}_i)$,
$tr\mathcal{A}^2(\mathcal{A}^2\Omega_p^2\mathcal{A}\Omega_p + \Omega_p\mathcal{A}^2\Omega_p^2\mathcal{A} + \mathcal{A}\Omega_p\mathcal{A}^2\Omega_p^2 + \Omega_p\mathcal{A}\Omega_p\mathcal{A}^2\Omega_p + \Omega_p^2\mathcal{A}\Omega_p\mathcal{A}^2 + \mathcal{A}\Omega_p^2\mathcal{A}\Omega_p\mathcal{A})$,
$tr\mathcal{A}^3\mathcal{A}_i$, $tr\mathcal{A}^3\mathcal{A}_\alpha^2$, $tr\mathcal{A}^3\mathcal{A}_\alpha\mathcal{A}_\beta$, $tr\mathcal{A}^3\mathcal{A}_\alpha\mathcal{A}\mathcal{A}_\alpha$, $tr\mathcal{A}^3\mathcal{A}_\alpha\mathcal{A}^2\mathcal{A}_\alpha$, $tr\mathcal{A}^3\mathcal{A}_\alpha\mathcal{A}_i\mathcal{A}_\alpha$,
$tr\mathcal{A}^3\mathcal{A}_\alpha\mathcal{A}\mathcal{A}_\beta$, $tr\mathcal{A}^3\mathcal{A}_\alpha\mathcal{A}\mathcal{A}_i\mathcal{A}_\beta$, $tr\mathcal{A}^3\mathcal{A}_\alpha\mathcal{A}_i^2\mathcal{A}_\alpha$, $tr\mathcal{A}^3\mathcal{A}_\alpha\mathcal{A}^2\mathcal{A}_\beta$, $tr\mathcal{A}^3\mathcal{A}_\alpha\mathcal{A}_i\mathcal{A}_\beta$,
$tr\mathcal{A}^3\mathcal{A}_\alpha\mathcal{A}_i^2\mathcal{A}_\beta$, $tr\mathcal{A}^3\mathcal{A}_\alpha(\mathcal{A}_i\mathcal{A}_j - \mathcal{A}_j\mathcal{A}_i)\mathcal{A}_\beta$, $tr\mathcal{A}^3\mathcal{A}_\alpha\mathcal{A}_i\mathcal{A}_j\mathcal{A}_\alpha$,
$tr\mathcal{A}^3\mathcal{A}_\alpha(\mathcal{A}\mathcal{A}_i - \mathcal{A}_i\mathcal{A})\mathcal{A}_\beta$, $tr\mathcal{A}^3\mathcal{A}_\alpha\Omega_\gamma$, $tr\mathcal{A}^3\mathcal{A}_\alpha\Omega_p\mathcal{A}_\beta$, $tr\mathcal{A}^3\mathcal{A}_\alpha\mathcal{A}\Omega_\gamma$,
$tr\mathcal{A}^3\Omega_p^2$, $tr\Omega_p^2$, $tr\mathcal{A}^3\Omega_\gamma\Omega_\delta$, $tr\mathcal{A}^3\Omega_\gamma\Omega_p\Omega_\delta$, $tr\mathcal{A}^3\mathcal{A}_\alpha\Omega_p\Omega_\gamma$, $tr\mathcal{A}^3\mathcal{A}_\alpha\Omega_p^2\mathcal{A}_\alpha$,
$tr\mathcal{A}^3\mathcal{A}_\alpha\Omega_p\Omega_q\mathcal{A}_\alpha$, $tr\mathcal{A}^3\mathcal{A}_\alpha\Omega_p^2\Omega_\gamma$, $tr\mathcal{A}^3\mathcal{A}_\alpha\Omega_p^2\Omega_q\mathcal{A}_\alpha$, $tr\mathcal{A}^3\mathcal{A}_\alpha\Omega_q^2\Omega_p\mathcal{A}_\alpha$,
$tr\mathcal{A}^3\mathcal{A}_\alpha\Omega_p^2\mathcal{A}_\beta$, $tr\mathcal{A}^3\mathcal{A}_\alpha(\Omega_p\Omega_q - \Omega_q\Omega_p)\mathcal{A}_\beta$, $tr\mathcal{A}^3\Omega_\gamma\mathcal{A}\Omega_\gamma$, $tr\mathcal{A}^3\Omega_\gamma\mathcal{A}\Omega_\delta$,
$tr\mathcal{A}^3\Omega_\gamma\mathcal{A}^2\Omega_\gamma$, $tr\mathcal{A}^3\Omega_\gamma\mathcal{A}^2\Omega_\delta$, $tr\mathcal{A}^3\Omega_\gamma\mathcal{A}\Omega_p\Omega_\gamma$, $tr\mathcal{A}^3\Omega_\gamma\mathcal{A}^2\Omega_p\Omega_\gamma$,
$tr\mathcal{A}^3\Omega_\gamma(\mathcal{A}\Omega_p - \Omega_P\mathcal{A})\Omega_\delta$, $tr\mathcal{A}^3\mathcal{A}_\alpha\mathcal{A}_i\Omega_\gamma$, $tr\mathcal{A}^3\mathcal{A}_\alpha\mathcal{A}\Omega_p\mathcal{A}_\alpha$,
$tr\mathcal{A}^3\mathcal{A}_\alpha(\mathcal{A}_i\mathcal{A}_j - \mathcal{A}_j\mathcal{A}_i)\Omega_\gamma$, $tr\mathcal{A}^3\mathcal{A}_\alpha(\mathcal{A}\mathcal{A}_i - \mathcal{A}_i\mathcal{A})\Omega_\gamma$,
$tr\mathcal{A}^3\Omega_\gamma(\mathcal{A}_i\mathcal{A}_j - \mathcal{A}_j\mathcal{A}_i)\Omega_\delta$, $tr\mathcal{A}^3\Omega_\gamma(\mathcal{A}\mathcal{A}_i - \mathcal{A}_i\mathcal{A})\Omega_\delta$, $tr\mathcal{A}^3\Omega_\gamma\mathcal{A}\mathcal{A}_i\Omega_\gamma$,
$tr\mathcal{A}^3\Omega_\gamma\mathcal{A}_i\mathcal{A}_j\Omega_\gamma$, $tr\mathcal{A}^3\Omega_\gamma\mathcal{A}_i\Omega_\delta$, $tr\mathcal{A}^3\Omega_\gamma\mathcal{A}_i\Omega_\gamma$, $tr\mathcal{A}^3\Omega_\gamma\mathcal{A}_i^2\Omega_\delta$,
$tr\mathcal{A}^3\Omega_\gamma\mathcal{A}_i^2\Omega_\gamma$, $tr\mathcal{A}^3\Omega_\gamma\Omega_p^2\Omega_\delta$, $tr\mathcal{A}^3\Omega_\gamma\Omega_p^2\Omega_\gamma$, $tr\mathcal{A}^3\mathcal{A}_\alpha\mathcal{A}_i^2\Omega_\gamma$,
$tr\mathcal{A}^3\mathcal{A}_\alpha\mathcal{A}^2\Omega_\gamma$, $tr\mathcal{A}^3\mathcal{A}_\alpha(\Omega_p\Omega_q - \Omega_q\Omega_p)\Omega_\gamma$, $tr\mathcal{A}^3\mathcal{A}_\alpha(\mathcal{A}\Omega_p - \Omega_p\mathcal{A})\Omega_\gamma$,
$tr\mathcal{A}^3\mathcal{A}_\alpha(\mathcal{A}_i\Omega_p - \Omega_p\mathcal{A}_i)\Omega_\gamma$, $tr\mathcal{A}^3\Omega_\gamma(\Omega_p\Omega_q - \Omega_q\Omega_p)\Omega_\delta$, $tr\mathcal{A}^3\Omega_\gamma(\mathcal{A}_i\Omega_p - \Omega_p\mathcal{A}_i)\Omega_\delta$,
$tr\mathcal{A}^3\Omega_\gamma\Omega_p\Omega_q\Omega_\gamma$, $tr\mathcal{A}^3\Omega_\gamma\mathcal{A}_i\Omega_p\Omega_\gamma$, $tr\mathcal{A}^3\Omega_\gamma\Omega_p^2\Omega_q\Omega_\gamma$, $tr\mathcal{A}^3\Omega_\gamma\Omega_p\Omega_q^2\Omega_\gamma$,
$tr\mathcal{A}^3\Omega_\gamma\mathcal{A}_i^2\Omega_p\Omega_\gamma$, $tr\mathcal{A}^3\mathcal{A}_\alpha(\mathcal{A}\Omega_p - \Omega_p\mathcal{A})\mathcal{A}_\beta$, $tr\mathcal{A}^3\mathcal{A}_\alpha(\mathcal{A}_i\Omega_p - \Omega_p\mathcal{A}_i)\mathcal{A}_\beta$,
$tr\mathcal{A}^3\mathcal{A}_\alpha\mathcal{A}_i\Omega_p\mathcal{A}_\alpha$, $tr\mathcal{A}^3\mathcal{A}_\alpha\mathcal{A}_i^2\Omega_p\mathcal{A}_\alpha$, $tr\mathcal{A}^3\mathcal{A}_\alpha\mathcal{A}^2\Omega_p\mathcal{A}_\alpha$, $tr\mathcal{A}^3\Omega_\gamma\Omega_p\mathcal{A}_i\Omega_p^2\Omega_\gamma$,
$tr\mathcal{A}^3\Omega_\gamma\Omega_p\mathcal{A}\Omega_p^2\Omega_\gamma$, $tr\mathcal{A}^3\mathcal{A}_\alpha\Omega_p\mathcal{A}_i\Omega_p^2\mathcal{A}_\alpha$, $tr\mathcal{A}^3\Omega_p(\Omega_p\mathcal{A}_i - \mathcal{A}_i\Omega_p)\Omega_p\mathcal{A}_\alpha$,
$tr\mathcal{A}^3\mathcal{A}_\alpha\Omega_p\mathcal{A}\Omega_p^2\mathcal{A}_\alpha$, $tr\mathcal{A}^3\Omega_p(\Omega_p\mathcal{A} - \mathcal{A}\Omega_p)\Omega_p\mathcal{A}_\alpha$,
$tr\mathcal{A}^3(\mathcal{A}_i\Omega_p^2\mathcal{A}_j\Omega_p + \Omega_p\mathcal{A}_i\Omega_p^2\mathcal{A}_j + \mathcal{A}_j\Omega_p\mathcal{A}_i\Omega_p^2 + \Omega_p\mathcal{A}_j\Omega_p\mathcal{A}_i\Omega_p + \Omega_p^2\mathcal{A}_j\Omega_p\mathcal{A}_i)$,
$tr\mathcal{A}^3(\mathcal{A}\Omega_p^2\mathcal{A}_i\Omega_p + \Omega_p\mathcal{A}\Omega_p^2\mathcal{A}_i + \mathcal{A}_i\Omega_p\mathcal{A}\Omega_p^2 + \Omega_p\mathcal{A}_i\Omega_p\mathcal{A}\Omega_p + \Omega_p^2\mathcal{A}_i\Omega_p\mathcal{A})$,
$tr\mathcal{A}^3(\mathcal{A}_i^2\Omega_p^2\mathcal{A}_i\Omega_p + \Omega_p\mathcal{A}_i^2\Omega_p^2\mathcal{A}_i + \mathcal{A}_i\Omega_p\mathcal{A}_i^2\Omega_p^2 + \Omega_p\mathcal{A}_i\Omega_p\mathcal{A}_i^2\Omega_p +$
$+ \Omega_p^2\mathcal{A}_i\Omega_p\mathcal{A}_i^2 + \mathcal{A}_i\Omega_p^2\mathcal{A}_i\Omega_p\mathcal{A}_i)$,
$tr\mathcal{A}^3(\mathcal{A}^2\Omega_p^2\mathcal{A}\Omega_p + \Omega_p\mathcal{A}^2\Omega_p^2\mathcal{A} + \mathcal{A}\Omega_p\mathcal{A}^2\Omega_p^2 + \Omega_p\mathcal{A}\Omega_p\mathcal{A}^2\Omega_p + \Omega_p^2\mathcal{A}\Omega_p\mathcal{A}^2 + \mathcal{A}\Omega_p^2\mathcal{A}\Omega_p\mathcal{A})$.

We note that, in these 4 last scalars, each term in round brackets can be obtained from the previous one with a cyclic permutation of his factors.

3. The 3-Dimensional Scalars which Will be Converted in the 4-Dimensional Notation

In the introduction we have already described how to obtain the set of 3-dimensional scalars to convert later in the 4-dimensional notation. We will list now the result in the following table. The scalars here listed depend on some indexes which link them to the N+1 symmetric second order tensor ($\mathcal{A}$ and $\mathcal{A}_i$) and to the M skew-symmetric tensors (Ω_γ) from which they come. In particular, the indexes used below are $i, j, k = 1, ..., N$ with $i < j < k$; $p, q, r = 1, ..., M$ with $p < q < r$; $\alpha, \beta = 1, ..., N$ with $\alpha < \beta$, and $\gamma, \delta = 1, ..., M$ with $\gamma < \delta$.

1. $\vec{a}_\alpha \cdot \vec{a}_\alpha, \vec{a}_\alpha \cdot \vec{a}_\beta$,
2. $trA, trA^2, trA^3, trAA_i, trA^2A_i, trAA_i^2, trA^2A_i^2, trAA_iA_j$,
3. $trW_p^2, trW_pW_q, trW_pW_qW_r$,
4. $\vec{a}_\alpha \cdot A\vec{a}_\alpha, \vec{a}_\alpha \cdot A^2\vec{a}_\alpha, \vec{a}_\alpha \cdot AA_i\vec{a}_\alpha$,
5. $\vec{a}_\alpha \cdot A\vec{a}_\beta, \vec{a}_\alpha \cdot A^2\vec{a}_\beta, \vec{a}_\alpha \cdot (AA_i - A_iA)\vec{a}_\beta$,
6. $\vec{a}_\alpha \cdot W_p^2\vec{a}_\alpha, \vec{a}_\alpha \cdot W_pW_q\vec{a}_\alpha, \vec{a}_\alpha \cdot W_p^2W_q\vec{a}_\alpha, \vec{a}_\alpha \cdot W_pW_q^2\vec{a}_\alpha$,
7. $\vec{a}_\alpha \cdot W_p\vec{a}_\beta, \vec{a}_\alpha \cdot W_p^2\vec{a}_\beta, \vec{a}_\alpha \cdot (W_pW_q - W_qW_p)\vec{a}_\beta$,
8. $trAW_p^2, trA^2W_p^2, trA^2W_p^2AW_p, trAW_pW_q$, $trAW_pW_q^2$, $trAW_p^2W_q$,
9. $trAA_iW_p, trAW_p^2A_iW_p, trAA_i^2W_p, trA^2A_iW_p$,
10. $\vec{a}_\alpha \cdot AW_p\vec{a}_\alpha, \vec{a}_\alpha \cdot W_pAW_p^2\vec{a}_\alpha, \vec{a}_\alpha \cdot A^2W_p\vec{a}_\alpha$,
11. $\vec{a}_\alpha \cdot (AW_p - W_pA)\vec{a}_\beta$,
12. $trA_i, trA_i^2, trA_i^3, trA_iA_j, trA_i^2A_j, trA_iA_j^2, trA_i^2A_j^2, trA_iA_jA_k$,
13. $\vec{a}_\alpha \cdot A_i\vec{a}_\alpha, \vec{a}_\alpha \cdot A_i^2\vec{a}_\alpha, \vec{a}_\alpha \cdot A_iA_j\vec{a}_\alpha$,
14. $\vec{a}_\alpha \cdot A_i\vec{a}_\beta, \vec{a}_\alpha \cdot A_i^2\vec{a}_\beta, \vec{a}_\alpha \cdot (A_iA_j - A_jA_i)\vec{a}_\beta$,
15. $trA_iW_p^2, trA_i^2W_p^2, trA_i^2W_p^2A_iW_p, trA_iW_pW_q$, $trA_iW_pW_q^2$, $trA_iW_p^2W_q$,
16. $trA_iA_jW_p, trA_iW_p^2A_jW_p, trA_iA_j^2W_p, trA_i^2A_jW_p$,
17. $\vec{a}_\alpha \cdot A_iW_p\vec{a}_\alpha, \vec{a}_\alpha \cdot W_pA_iW_p^2\vec{a}_\alpha, \vec{a}_\alpha \cdot A_i^2W_p\vec{a}_\alpha$,
18. $\vec{a}_\alpha \cdot (A_iW_p - W_pA_i)\vec{a}_\beta$,
19. $\vec{\omega}_\gamma \cdot \vec{\omega}_\gamma, \vec{\omega}_\gamma \cdot \vec{\omega}_\delta$,
20. $\vec{\omega}_\gamma \cdot A\vec{\omega}_\gamma, \vec{\omega}_\gamma \cdot A^2\vec{\omega}_\gamma, \vec{\omega}_\gamma \cdot AA_i\vec{\omega}_\gamma$,
21. $\vec{\omega}_\gamma \cdot A\vec{\omega}_\delta, \vec{\omega}_\gamma \cdot A^2\vec{\omega}_\delta, \vec{\omega}_\gamma \cdot (AA_i - A_iA)\vec{\omega}_\delta$,
22. $\vec{\omega}_\gamma \cdot W_p^2\vec{\omega}_\gamma, \vec{\omega}_\gamma \cdot W_pW_q\vec{\omega}_\gamma, \vec{\omega}_\gamma \cdot W_p^2W_q\vec{\omega}_\gamma, \vec{\omega}_\gamma \cdot W_pW_q^2\vec{\omega}_\gamma$,
23. $\vec{\omega}_\gamma \cdot W_p\vec{\omega}_\delta, \vec{\omega}_\gamma \cdot W_p^2\vec{\omega}_\delta, \vec{\omega}_\gamma \cdot (W_pW_q - W_qW_p)\vec{\omega}_\delta$,
24. $\vec{\omega}_\gamma \cdot AW_p\vec{\omega}_\gamma, \vec{\omega}_\gamma \cdot W_pAW_p^2\vec{\omega}_\gamma, \vec{\omega}_\gamma \cdot A^2W_p\vec{\omega}_\gamma$,
25. $\vec{\omega}_\gamma \cdot (AW_p - W_pA)\vec{\omega}_\delta$,
26. $\vec{\omega}_\gamma \cdot A_i\vec{\omega}_\gamma, \vec{\omega}_\gamma \cdot A_i^2\vec{\omega}_\gamma, \vec{\omega}_\gamma \cdot A_iA_j\vec{\omega}_\gamma$,
27. $\vec{\omega}_\gamma \cdot A_i\vec{\omega}_\delta, \vec{\omega}_\gamma \cdot A_i^2\vec{\omega}_\delta, \vec{\omega}_\gamma \cdot (A_iA_j - A_jA_i)\vec{\omega}_\delta$,
28. $\vec{\omega}_\gamma \cdot A_iW_p\vec{\omega}_\gamma, \vec{\omega}_\gamma \cdot W_pA_iW_p^2\vec{\omega}_\gamma, \vec{\omega}_\gamma \cdot A_i^2W_p\vec{\omega}_\gamma$,
29. $\vec{\omega}_\gamma \cdot (A_iW_p - W_pA_i)\vec{\omega}_\delta$,
30. $\vec{a}_\alpha \cdot \vec{\omega}_\gamma$,
31. $\vec{a}_\alpha \cdot A\vec{\omega}_\gamma, \vec{a}_\alpha \cdot A^2\vec{\omega}_\gamma, \vec{a}_\alpha \cdot (AA_i - A_iA)\vec{\omega}_\gamma$,
32. $\vec{a}_\alpha \cdot W_p\vec{\omega}_\gamma, \vec{a}_\alpha \cdot W_p^2\vec{\omega}_\gamma, \vec{a}_\alpha \cdot (W_pW_q - W_qW_p)\vec{\omega}_\gamma$,
33. $\vec{a}_\alpha \cdot (AW_p - W_pA)\vec{\omega}_\gamma$,

34. $\vec{a}_\alpha \cdot A_i\vec{\omega}_\gamma, \vec{a}_\alpha \cdot A_i^2\vec{\omega}_\gamma, \vec{a}_\alpha \cdot (A_iA_j - A_jA_i)\vec{\omega}_\gamma\,,$
35. $\vec{a}_\alpha \cdot (A_iW_p - W_pA_i)\vec{\omega}_\gamma\,,$

Table 1

4. A First Conversion of the 3-Dimensional Scalars, in the 4-Dimensional Notation

We note firstly that, among the scalars to enclose in the representation S^0 reported in Section 2, we have surely to take $tr\mathcal{A}$, $tr\mathcal{A}^2$, $tr\mathcal{A}^3$, $tr\mathcal{A}^4$ (which were useful in the definition (4) of I_1 and $trI_1\mathcal{A}_i$ (which needs to determine the component 11 of $\mathcal{A}_i$); we write them in the first raw of the following table. Moreover, as it can be seen from equation $(7)_{4,5}$, we have to modify the scalars of Section 3 in the following way:

— First of all, we append an apex * to each second order tensor.

— After that we substitute each vector with its expression in capital letters, taking care to substitute the vector allocated in the left hand side, with its transposition.

– Then, in the terms containing vectors, instead of the symbol denoting the scalar product, we put the symbol tr (trace) at the beginning of all the expression.

— Lastly, we use equations (6).

In this way we find the scalars of the following table (the first number at the beginning of each raw of the table denotes the number of raw in the table of Section 3, from which we are now obtaining the 4-dimensional corresponding one; the index denotes the position of that scalar in its raw). Obviously, we leave out eventual added terms which are functions of previously obtained scalars. We also take into account, sometimes and to speed up calculations, of the following identities

$$trI_1\mathcal{M}I_1 = trI_1\mathcal{M}, I_1^2 = I_1, tr\vec{M} = 0, I_1\vec{M} = 0, \vec{M}I_1 = \vec{M},$$
$$\mathcal{A}I_1 = aI_1 = I_1\mathcal{A},\ tr\mathcal{A}\mathcal{B}I_1 = trI_1\mathcal{A}\mathcal{B},\ trI_1\mathcal{A}_iI_1\mathcal{A}_i = (a^i_{11})^2,\ tr\mathcal{A}\mathcal{B} = tr\mathcal{B}\mathcal{A}.$$

Sometimes it may be convenient to start from the scalars on the right hand side of the table, towards the corresponding left side; in this case we use passages like the following ones

$$tr\mathcal{M}\mathcal{N}\mathcal{P} = (\mathcal{M}\mathcal{N}\mathcal{P})_{\mu\mu} = \mathcal{M}_{\mu\nu}\mathcal{N}_{\nu\vartheta}\mathcal{P}_{\vartheta\mu} = \mathcal{M}_{1\nu}\mathcal{N}_{\nu\vartheta}\mathcal{P}_{\vartheta 1} + \mathcal{M}_{a\nu}\mathcal{N}_{\nu\vartheta}\mathcal{P}_{\vartheta a}$$

$$
\begin{aligned}
&= \mathcal{M}_{11}\mathcal{N}_{1\vartheta}\mathcal{P}_{\vartheta 1} + \mathcal{M}_{1a}\mathcal{N}_{a\vartheta}\mathcal{P}_{\vartheta 1} + \mathcal{M}_{a1}\mathcal{N}_{1\vartheta}\mathcal{P}_{\vartheta a} + \mathcal{M}_{ab}\mathcal{N}_{b\vartheta}\mathcal{P}_{\vartheta a} \\
&= \mathcal{M}_{11}\mathcal{N}_{11}\mathcal{P}_{11} + \mathcal{M}_{11}\mathcal{N}_{1a}\mathcal{P}_{a1} + \mathcal{M}_{1a}\mathcal{N}_{a1}\mathcal{P}_{11} + \mathcal{M}_{1a}\mathcal{N}_{ab}\mathcal{P}_{b1} \\
&+ \mathcal{M}_{a1}\mathcal{N}_{11}\mathcal{P}_{1a} + \mathcal{M}_{a1}\mathcal{N}_{1b}\mathcal{P}_{ba} + \mathcal{M}_{ab}\mathcal{N}_{b1}\mathcal{P}_{1a} + \mathcal{M}_{ab}\mathcal{N}_{bc}\mathcal{P}_{ca} \\
&= \mathcal{M}_{11}\mathcal{N}_{11}\mathcal{P}_{11} + \mathcal{M}_{11}\vec{n}\cdot\vec{p} + \mathcal{N}_{11}\vec{m}\cdot\vec{p} + \mathcal{P}_{11}\vec{m}\cdot\vec{n} \\
&+ \vec{m}\cdot N\vec{p} + \vec{m}\cdot P\vec{n} + \vec{n}\cdot M\vec{p} + trMNP\,;
\end{aligned}
$$

the last of these passages has to be intended correct except for the signs, which depend on the symmetry or skew-symmetry of $\mathcal{M}$, $\mathcal{N}$, $\mathcal{P}$.

In other words, we have adopted the convention on repeated indexes, from 1 to 4 for Greek indexes and from 2 to 4 for Latin indexes; after that, we have explicitated the terms with 1 as index. Lastly, we have used equation (5).

Some other times it may be convenient to reach the result with still different methods; in these cases we have enclosed a note in the table, which we now explain:

Notes. 1) The first 3 scalars of $(2)_{1-5}$ give $tr\mathcal{A}$, $tr\mathcal{A}^2$, $tr\mathcal{A}^3$ which are already written in the non numbered raw.

2) For the previous 2 scalars it is better to perform the calculations through the components, instead of following the general procedure.

3) For the last scalar, follow the same method of note 2.

4) Through explicit calculations, we see that this 4-dimensional scalar is just equal to

$$(\mathcal{A}_{11}^{\alpha}\mathcal{A}_{11}^{i} + \vec{v}_{\alpha}\cdot\vec{v}_i)^2 + 2\mathcal{A}_{11}^{\alpha}\vec{v}_i\cdot A_i\vec{v}_{\alpha} + (\mathcal{A}_{11}^{\alpha})^2\vec{v}_i\cdot\vec{v}_i + \vec{v}_{\alpha}\cdot\mathcal{A}_i^2\vec{v}_{\alpha}.$$

5) Through calculations we see that this last scalar is equal to $A_{11}^{i}A_{11}^{j} + 2\vec{a}_i\cdot\vec{a}_j + trA_iA_j$.

In this way we obtain just the table which now follows

	Table for 4-dimensional scalars
	$tr\mathcal{A}$, $tr\mathcal{A}^2$, $tr\mathcal{A}^3$, $tr\mathcal{A}^4$, $trI_1\mathcal{A}_i$
$(1)_{1,2}$, $(2)_{1-5}$	$trI_1\mathcal{A}_\alpha^2$, $trI_1\mathcal{A}_\alpha\mathcal{A}_\beta$, Note 1, $tr\mathcal{A}\mathcal{A}_i$, $tr\mathcal{A}^2\mathcal{A}_i$
$(12)_{1,2}$, $(4)_{1,2}$	$tr\mathcal{A}_i$, $tr\mathcal{A}_i^2$, $trI_1\mathcal{A}_\alpha\mathcal{A}\mathcal{A}_\alpha$, $trI_1\mathcal{A}_\alpha\mathcal{A}^2\mathcal{A}_\alpha$,
$(2)_{6,7}$, $(13)_1$	$tr\mathcal{A}\mathcal{A}_i^2$, $tr\mathcal{A}^2\mathcal{A}_i^2$, Note 2, $trI_1\mathcal{A}_\alpha\mathcal{A}_i\mathcal{A}_\alpha$,
$(5)_1$, $(4)_3$, $(12)_3$	$trI_1\mathcal{A}_\alpha\mathcal{A}\mathcal{A}_\beta$, $trI_1\mathcal{A}_\alpha\mathcal{A}\mathcal{A}_i\mathcal{A}_\beta$, $tr\mathcal{A}_i^3$, Note 3
$(13)_2$, $(5)_2$	$trI_1\mathcal{A}_\alpha\mathcal{A}_i^2\mathcal{A}_\alpha$ Note 4, $trI_1\mathcal{A}_\alpha\mathcal{A}^2\mathcal{A}_\beta$,
$(12)_4$, $(2)_8$, $(14)_1$	$tr\mathcal{A}_i\mathcal{A}_j$ Note 5, $tr\mathcal{A}\mathcal{A}_i\mathcal{A}_j$, $trI_1\mathcal{A}_\alpha\mathcal{A}_i\mathcal{A}_\beta$,
$(12)_{5,6}$, $(14)_2$	$tr\mathcal{A}_i^2\mathcal{A}_j$, $tr\mathcal{A}_i\mathcal{A}_j^2$, $trI_1\mathcal{A}_\alpha\mathcal{A}_i^2\mathcal{A}_\beta$,
$(14)_3$, $(13)_3$	$trI_1\mathcal{A}_\alpha(\mathcal{A}_i\mathcal{A}_j-\mathcal{A}_j\mathcal{A}_i)\mathcal{A}_\beta$, $trI_1\mathcal{A}_\alpha\mathcal{A}_i\mathcal{A}_j\mathcal{A}_\alpha$,
$(5)_3$, $(12)_8$	$trI_1\mathcal{A}_\alpha(\mathcal{A}\mathcal{A}_i-\mathcal{A}_i\mathcal{A})\mathcal{A}_\beta$, $tr\mathcal{A}_i\mathcal{A}_j\mathcal{A}_k$,
$(30)_1$, $(7)_1$, $(31)_1$	$trI_1\mathcal{A}_\alpha\Omega_\gamma$, $trI_1\mathcal{A}_\alpha\Omega_p\mathcal{A}_\beta$, $trI_1\mathcal{A}_\alpha\mathcal{A}\Omega_\gamma$,
$(9)_1$, $(19)_1$, $(3)_1$, $(19)_2$	$tr\mathcal{A}\mathcal{A}_i\Omega_p$, $trI_1\Omega_p^2$, $tr\Omega_p^2$, $trI_1\Omega_\gamma\Omega_\delta$,
$(3)_2$, $(23)_1$, $(3)_3$,	$tr\Omega_p\Omega_q$, $trI_1\Omega_\gamma\Omega_p\Omega_\delta$, $tr\Omega_p\Omega_q\Omega_r$,
$(32)_1$, $(6)_1$, $(6)_2$	$trI_1\mathcal{A}_\alpha\Omega_p\Omega_\gamma$, $trI_1\mathcal{A}_\alpha\Omega_p^2\mathcal{A}_\alpha$, $trI_1\mathcal{A}_\alpha\Omega_p\Omega_q\mathcal{A}_\alpha$,
$(32)_2$, $(6)_3$	$trI_1\mathcal{A}_\alpha\Omega_p^2\Omega_\gamma$, $trI_1\mathcal{A}_\alpha\Omega_p^2\Omega_q\mathcal{A}_\alpha$,
$(6)_4$, $(7)_2$	$trI_1\mathcal{A}_\alpha\Omega_q^2\Omega_p\mathcal{A}_\alpha$, $trI_1\mathcal{A}_\alpha\Omega_p^2\mathcal{A}_\beta$,
$(7)_3$, $(20)_1$	$trI_1\mathcal{A}_\alpha(\Omega_p\Omega_q-\Omega_q\Omega_p)\mathcal{A}_\beta$, $trI_1\Omega_\gamma\mathcal{A}\Omega_\gamma$,
$(21)_1$, $(8)_1$, $(8)_4$, $(20)_2$	$trI_1\Omega_\gamma\mathcal{A}\Omega_\delta$, $tr\mathcal{A}\Omega_p^2$, $tr\mathcal{A}\Omega_p\Omega_q$, $trI_1\Omega_\gamma\mathcal{A}^2\Omega_\gamma$,
$(21)_2$, $(8)_2$, $(24)_1$	$trI_1\Omega_\gamma\mathcal{A}^2\Omega_\delta$, $tr\mathcal{A}^2\Omega_p^2$, $trI_1\Omega_\gamma\mathcal{A}\Omega_p\Omega_\gamma$,
$(24)_3$, $(25)_1$	$trI_1\Omega_\gamma\mathcal{A}^2\Omega_p\Omega_\gamma$, $trI_1\Omega_\gamma(\mathcal{A}\Omega_p-\Omega_P\mathcal{A})\Omega_\delta$,
$(9)_4$, $(34)_1$	$tr\mathcal{A}^2\mathcal{A}_i\Omega_p$, $trI_1\mathcal{A}_\alpha\mathcal{A}_i\Omega_\gamma$,
$(10)_1$, $(34)_3$	$trI_1\mathcal{A}_\alpha\mathcal{A}\Omega_p\mathcal{A}_\alpha$, $trI_1\mathcal{A}_\alpha(\mathcal{A}_i\mathcal{A}_j-\mathcal{A}_j\mathcal{A}_i)\Omega_\gamma$,
$(31)_3$	$trI_1\mathcal{A}_\alpha(\mathcal{A}\mathcal{A}_i-\mathcal{A}_i\mathcal{A})\Omega_\gamma$,
$(27)_3$	$trI_1\Omega_\gamma(\mathcal{A}_i\mathcal{A}_j-\mathcal{A}_j\mathcal{A}_i)\Omega_\delta$,
$(21)_3$, $(20)_3$	$trI_1\Omega_\gamma(\mathcal{A}\mathcal{A}_i-\mathcal{A}_i\mathcal{A})\Omega_\delta$, $trI_1\Omega_\gamma\mathcal{A}\mathcal{A}_i\Omega_\gamma$,
$(26)_3$, $(27)_1$, $(26)_1$	$trI_1\Omega_\gamma\mathcal{A}_i\mathcal{A}_j\Omega_\gamma$, $trI_1\Omega_\gamma\mathcal{A}_i\Omega_\delta$, $trI_1\Omega_\gamma\mathcal{A}_i\Omega_\gamma$,
$(15)_4$, $(27)_2$, $(26)_2$	$tr\mathcal{A}_i\Omega_p\Omega_q$, $trI_1\Omega_\gamma\mathcal{A}_i^2\Omega_\delta$, $trI_1\Omega_\gamma\mathcal{A}_i^2\Omega_\gamma$,
$(23)_2$, $(22)_1$, $(34)_2$	$trI_1\Omega_\gamma\Omega_p^2\Omega_\delta$, $trI_1\Omega_\gamma\Omega_p^2\Omega_\gamma$, $trI_1\mathcal{A}_\alpha\mathcal{A}_i^2\Omega_\gamma$,
$(31)_2$, $(32)_3$	$trI_1\mathcal{A}_\alpha\mathcal{A}^2\Omega_\gamma$, $trI_1\mathcal{A}_\alpha(\Omega_p\Omega_q-\Omega_q\Omega_p)\Omega_\gamma$,
$(33)_1$	$trI_1\mathcal{A}_\alpha(\mathcal{A}\Omega_p-\Omega_p\mathcal{A})\Omega_\gamma$,
$(35)_1$	$trI_1\mathcal{A}_\alpha(\mathcal{A}_i\Omega_p-\Omega_p\mathcal{A}_i)\Omega_\gamma$,
$(23)_3$	$trI_1\Omega_\gamma(\Omega_p\Omega_q-\Omega_q\Omega_p)\Omega_\delta$,
$(29)_1$	$trI_1\Omega_\gamma(\mathcal{A}_i\Omega_p-\Omega_p\mathcal{A}_i)\Omega_\delta$,
$(22)_2$, $(28)_1$	$trI_1\Omega_\gamma\Omega_p\Omega_q\Omega_\gamma$, $trI_1\Omega_\gamma\mathcal{A}_i\Omega_p\Omega_\gamma$,
$(22)_3$, $(22)_4$	$trI_1\Omega_\gamma\Omega_p^2\Omega_q\Omega_\gamma$, $trI_1\Omega_\gamma\Omega_p\Omega_q^2\Omega_\gamma$,
$(28)_3$, $(11)_1$	$trI_1\Omega_\gamma\mathcal{A}_i^2\Omega_p\Omega_\gamma$, $trI_1\mathcal{A}_\alpha(\mathcal{A}\Omega_p-\Omega_p\mathcal{A})\mathcal{A}_\beta$,
$(18)_1$, $(15)_1$	$trI_1\mathcal{A}_\alpha(\mathcal{A}_i\Omega_p-\Omega_p\mathcal{A}_i)\mathcal{A}_\beta$, $tr\mathcal{A}_i\Omega_p^2$,
$(16)_1$, $(17)_1$	$tr\mathcal{A}_i\mathcal{A}_j\Omega_p$, $trI_1\mathcal{A}_\alpha\mathcal{A}_i\Omega_p\mathcal{A}_\alpha$,

Before proceeding with the other scalars, it is now useful to introduce three theorems in the 3-dimensional context; after that, we will see their implications for the 4-dimensional case.

Theorem 1. *The scalars*

$$\vec{u} \cdot AB\vec{v} \qquad \text{and} \qquad \vec{u} \cdot BA\vec{v}\,, \tag{8}$$

with A and B second order symmetric tensors, are functions of

$$\begin{aligned} &\vec{u}\cdot\vec{u}, \vec{u}\cdot\vec{v}, \vec{v}\cdot\vec{v}, \vec{u}\cdot A\vec{u}, \vec{u}\cdot A\vec{v}, \vec{v}\cdot A\vec{v}, \vec{u}\cdot B\vec{u}, \vec{u}\cdot B\vec{v}, \vec{v}\cdot B\vec{v}\,, \\ &\vec{u}\cdot A^2\vec{u}, \vec{u}\cdot A^2\vec{v}, \vec{v}\cdot A^2\vec{v}\,, \vec{u}\cdot B^2\vec{u}, \vec{u}\cdot B^2\vec{v}, \vec{v}\cdot B^2\vec{v}, \\ &\vec{u}\cdot AB\vec{u}, \vec{v}\cdot AB\vec{v}, \vec{u}\cdot(AB-BA)\vec{v}, tr A, tr B\,. \end{aligned} \tag{9}$$

In fact, if $\vec{u}=\vec{0}$, the theorem holds because the scalars (8) are zero.
If $\vec{u}\neq\vec{0}$ and $\vec{v}$ is parallel to $\vec{u}$, that is

$$\vec{v} = \frac{\vec{u}\cdot\vec{v}}{\vec{u}\cdot\vec{u}}\vec{u} \quad \textit{then we have} \quad \vec{u}\cdot AB\vec{v} = \frac{\vec{u}\cdot\vec{v}}{\vec{u}\cdot\vec{u}}\vec{u}\cdot AB\vec{u} = \vec{u}\cdot BA\vec{v}\,,$$

and the theorem is proved also in this case.

Lastly, if $\vec{u}$ and $\vec{v}$ are linearly independent, we can perform the following calculations in the reference frame with x_1 axis directed as $\vec{u}$ and x_2 axis directed as the component of $\vec{v}$ orthogonal to $\vec{u}$; in this frame we have $\vec{u}\equiv(u,0,0)$ and $\vec{v}\equiv(v_1,v_2,0)$ with $u>0$ and $v_2>0$. After that, we can do what follows:

From $(9)_{1-3}$ we obtain $\vec{u}$ and $\vec{v}$.

After that, from $(9)_{4-9}$ we obtain a_{11}, a_{12}, a_{22}, b_{11}, b_{12}, b_{22}.

Then from $(9)_{16,17}$ we obtain a_{33}, b_{33}.

Then from $(9)_{10-15}$ we obtain $(a_{13})^2$, $a_{13}\,a_{23}$, $(a_{23})^2$, $(b_{13})^2$, $b_{13}\,b_{23}$, $(b_{23})^2$.

Let us now distinguish some cases:

1. $(a_{13})^2\neq 0$. Then we can choose the verse of the x_3 axis so that $a_{13}>0$ and obtain it from $(a_{13})^2$; after that, we obtain a_{23} from $a_{13}\,a_{23}$, then we obtain b_{13} from $\vec{u}\cdot AB\vec{u}$ and, lastly, b_{23} from $\vec{u}\cdot(AB-BA)\vec{v}$. Therefore, in this case we have obtained $\vec{u}$, $\vec{v}$, A and B as functions of the scalars (9), so that also the scalars (8) are functions of the scalars (9). Obviously, we have proved this property in a particular reference frame; but the result holds in every other reference frame, because scalars do not depend on the frame.

2. $(a_{13})^2=0$, but $(b_{13})^2\neq 0$. In this case we repeat the passages of the previous case, but exchanging A and B.

3. $(a_{13})^2=0$, $(b_{13})^2=0$. In this case it remains to obtain a_{23} and b_{23}; but we note that neither $\vec{u}\cdot AB\vec{v}$, nor $\vec{u}\cdot BA\vec{v}$ depend on these variables; therefore, we have the same result of the previous cases.

Let us now prove the

Theorem 2. *The scalars*

$$\vec{u} \cdot AW\vec{v} \qquad \text{and} \qquad \vec{u} \cdot WA\vec{v}\,, \tag{10}$$

with A second order symmetric tensor and W second order skew-symmetric tensor, are functions of

$$\begin{aligned} &\vec{u}\cdot\vec{u}, \vec{u}\cdot\vec{v}, \vec{v}\cdot\vec{v}, \vec{u}\cdot A\vec{u}, \vec{u}\cdot A\vec{v}, \vec{v}\cdot A\vec{v}, \vec{u}\cdot W\vec{v}\,, \\ &\vec{u}\cdot A^2\vec{u}, \vec{u}\cdot A^2\vec{v}, \vec{v}\cdot A^2\vec{v}\,, \vec{u}\cdot W^2\vec{u}, \vec{u}\cdot W^2\vec{v}, \vec{v}\cdot W^2\vec{v}, \\ &\vec{u}\cdot AW\vec{u}, \vec{v}\cdot AW\vec{v}, \vec{u}\cdot(AW - WA)\vec{v}, tr A\,. \end{aligned} \tag{11}$$

In fact, if $\vec{u} = \vec{0}$, the theorem holds because the scalars (10) are zero.
If $\vec{u} \neq \vec{0}$ and $\vec{v}$ is directed like $\vec{u}$, that is

$$\vec{v} = \frac{\vec{u}\cdot\vec{v}}{\vec{u}\cdot\vec{u}}\vec{u} \quad \text{then we have} \qquad \vec{u}\cdot AW\vec{v} = \frac{\vec{u}\cdot\vec{v}}{\vec{u}\cdot\vec{u}}\vec{u}\cdot AW\vec{u} \quad \text{and}$$
$$\vec{u}\cdot WA\vec{v} = -\vec{u}\cdot AW\vec{v}\,,$$

so that the theorem is proved for this case.

Lastly, if $\vec{u}$ and $\vec{v}$ are linearly independent, we may perform the following passages in the reference frame characterized by $\vec{u} \equiv (u, 0, 0)$, $\vec{v} \equiv (v_1, v_2, 0)$, $u > 0$, $v_2 > 0$.

From $(11)_{1-3}$ we obtain $\vec{u}$ and $\vec{v}$.

After that, from $(11)_{4-6}$ and $(11)_{17}$ we obtain a_{11}, a_{12}, a_{22}, a_{33}.

Then from $(11)_7$ we obtain w_{12}.

Then from $(11)_{8-13}$ we obtain $(a_{13})^2$, $a_{13}\,a_{23}$, $(a_{23})^2$, $(w_{13})^2$, $w_{13}\,w_{23}$, $(w_{23})^2$.

Let us now distinguish some cases:

1. $(a_{13})^2 \neq 0$. Then we choose the versus of the x_3 axis so that $a_{13} > 0$ and obtain it from $(a_{13})^2$; after that, we obtain a_{23} from $a_{13}\,a_{23}$. After that, we obtain w_{13} from $\vec{u}\cdot AW\vec{u}$ and lastly w_{23} from $\vec{u}\cdot(AW - WA)\vec{v}$. Then, in this case we have obtained $\vec{u}$, $\vec{v}$, A and W as functions of the scalars (11) so that also the scalars (10) are functions of the scalars (11). Obviously, we have proved this property in a particular reference frame; but the result holds in every other reference frame, because scalars do not depend on the frame.

2. $(a_{13})^2 = 0$, but $(w_{13})^2 \neq 0$. Then we choose the versus of the x_3 axis so that $w_{13} > 0$ and obtain it from $(w_{13})^2$; after that, we obtain w_{23} from $w_{13}\,w_{23}$. After that, we obtain a_{23} from $\vec{u}\cdot(AW - WA)\vec{v}$. Then, in this case we have obtained $\vec{u}$, $\vec{v}$, A and W as functions of the scalars (11) so that also the scalars (10) are functions of the scalars (11). Obviously, we have proved this property in a particular reference frame; but the result holds in every other reference

frame, because scalars do not depend on the frame.

3. $(a_{13})^2 = 0$, $(w_{13})^2 = 0$. In this case it remains to obtain a_{23} and w_{23}; but we note that neither $\vec{u} \cdot AW\vec{v}$, nor $\vec{u} \cdot WA\vec{v}$ depend on these variables; therefore, we have the same result of the previous cases.

Let us now complete the set of theorems, in the 3-dimensional space, with

Theorem 3. *The scalars*

$$\vec{u} \cdot VW\vec{v} \qquad \text{and} \qquad \vec{u} \cdot WV\vec{v}\,, \tag{12}$$

with V and W skew-symmetric second order tensors, are functions of

$$\begin{aligned} &\vec{u} \cdot \vec{u}, \vec{u} \cdot \vec{v}, \vec{v} \cdot \vec{v}, \vec{u} \cdot V\vec{v}, \vec{u} \cdot W\vec{v}\,, \\ &\vec{u} \cdot V^2\vec{u}, \vec{u} \cdot V^2\vec{v}, \vec{v} \cdot V^2\vec{v}\,, \vec{u} \cdot W^2\vec{u}, \vec{u} \cdot W^2\vec{v}, \vec{v} \cdot W^2\vec{v}, \\ &\vec{u} \cdot VW\vec{u}, \vec{v} \cdot VW\vec{v}, \vec{u} \cdot (VW - WV)\vec{v}\,. \end{aligned} \tag{13}$$

In fact, if $\vec{u} = \vec{0}$, the theorem holds because the scalars (12) are zero.
If $\vec{u} \neq \vec{0}$ and $\vec{v}$ is directed like $\vec{u}$, that is

$$\vec{v} = \frac{\vec{u} \cdot \vec{v}}{\vec{u} \cdot \vec{u}}\vec{u} \quad \text{then we have} \quad \vec{u} \cdot VW\vec{v} = \vec{u} \cdot WV\vec{v} = \frac{\vec{u} \cdot \vec{v}}{\vec{u} \cdot \vec{u}}\vec{u} \cdot VW\vec{u}\,.$$

Lastly, if $\vec{u}$ and $\vec{v}$ are linearly independent, we may perform the following passages in the reference frame characterized by $\vec{u} \equiv (u, 0, 0)$, $\vec{v} \equiv (v_1, v_2, 0)$, $u > 0$, $v_2 > 0$.

From $(13)_{1-3}$ we obtain $\vec{u}$ and $\vec{v}$.

After that, from $(13)_{4-5}$ we obtain v_{12} and w_{12}.

Then from $(11)_{6-11}$ we obtain $(v_{13})^2$, $v_{13}\,v_{23}$, $(v_{23})^2$, $(w_{13})^2$, $w_{13}\,w_{23}$, $(w_{23})^2$.

Let us now distinguish some cases:

1. $(w_{13})^2 \neq 0$. Then we choose the versus of the x_3 axis so that $w_{13} > 0$ and obtain it from $(w_{13})^2$; after that, we obtain w_{23} from $w_{13}\,w_{23}$. After that, we obtain v_{13} from $\vec{u} \cdot VW\vec{u}$ and lastly v_{23} from $\vec{u} \cdot (VW - WV)\vec{v}$. Then, in this case we have obtained $\vec{u}$, $\vec{v}$, V and W as functions of the scalars (13) so that also the scalars (12) are functions of the scalars (13).

2. $(w_{13})^2 = 0$, but $(v_{13})^2 \neq 0$. In this case we repeat the passages of the previous case, but exchanging V and W.

3. $(w_{13})^2 = 0$, $(v_{13})^2 = 0$. In this case it remains to obtain w_{23} and v_{23}; but we note that neither $\vec{u} \cdot VW\vec{v}$, nor $\vec{u} \cdot WV\vec{v}$ depend on these variables; therefore, we have the same result of the previous cases.

We note that the scalars (8), (10), (12) don't belong to the set S^0 for 3-dimensional vectorial spaces; consequently, it is obvious that they are functions of the elements of S^0. But, for the application which we intend to do, it is necessary to know on what elements of S^0 they depend. For this end we have proved the theorems from 1 to 3.

Let us now apply to the theorems (from 1 to 3) the method we were using for the previous table; as a result, we obtain the following theorem which holds in the 4-dimensional vectorial space:

Theorem 4. *The scalars* $trI_1\mathcal{A}_\alpha\mathcal{A}\mathcal{A}_i\mathcal{A}_\beta$, $trI_1\mathcal{A}_\alpha\mathcal{A}_i\mathcal{A}\mathcal{A}_\beta$, $trI_1\mathcal{A}_\alpha\mathcal{A}_i\mathcal{A}_j\mathcal{A}_\beta$, $trI_1\mathcal{A}_\alpha\mathcal{A}_j\mathcal{A}_i\mathcal{A}_\beta$, $s\,trI_1\Omega_\gamma\mathcal{A}\mathcal{A}_i\Omega_\delta$, $trI_1\Omega_\gamma\mathcal{A}_i\mathcal{A}\Omega_\delta$, $trI_1\Omega_\gamma\mathcal{A}_i\mathcal{A}_j\Omega_\delta$, $trI_1\Omega_\gamma\mathcal{A}_j\mathcal{A}_i\Omega_\delta$, $trI_1\mathcal{A}_\alpha\mathcal{A}\mathcal{A}_i\Omega_\gamma$, $trI_1\mathcal{A}_\alpha\mathcal{A}_i\mathcal{A}\Omega_\gamma$, $trI_1\mathcal{A}_\alpha\mathcal{A}_i\mathcal{A}_j\Omega_\gamma$, $trI_1\mathcal{A}_\alpha\mathcal{A}_j\mathcal{A}_i\Omega_\gamma$, $trI_1\mathcal{A}_\alpha\mathcal{A}\Omega_p\mathcal{A}_\beta$, $trI_1\mathcal{A}_\alpha\Omega_p\mathcal{A}\mathcal{A}_\beta$, $trI_1\mathcal{A}_\alpha\mathcal{A}_i\Omega_p\mathcal{A}_\beta$, $trI_1\mathcal{A}_\alpha\Omega_p\mathcal{A}_i\mathcal{A}_\beta$, $trI_1\Omega_\gamma\mathcal{A}\Omega_p\Omega_\delta$, $trI_1\Omega_\gamma\Omega_p\mathcal{A}\Omega_\delta$, $trI_1\Omega_\gamma\mathcal{A}_i\Omega_p\Omega_\delta$, $trI_1\Omega_\gamma\Omega_p\mathcal{A}_i\Omega_\delta$, $trI_1\mathcal{A}_\alpha\mathcal{A}\Omega_p\Omega_\gamma$, $trI_1\mathcal{A}_\alpha\Omega_p\mathcal{A}\Omega_\gamma$, $trI_1\mathcal{A}_\alpha\mathcal{A}_i\Omega_p\Omega_\gamma$, $trI_1\mathcal{A}_\alpha\Omega_p\mathcal{A}_i\Omega_\gamma$, $trI_1\mathcal{A}_\alpha\Omega_p\Omega_q\mathcal{A}_\beta$, $trI_1\mathcal{A}_\alpha\Omega_q\Omega_p\mathcal{A}_\beta$, $trI_1\Omega_\gamma\Omega_p\Omega_q\Omega_\delta$, $trI_1\Omega_\gamma\Omega_q\Omega_p\Omega_\delta$, $trI_1\mathcal{A}_\alpha\Omega_p\Omega_q\Omega_\delta$, $trI_1\mathcal{A}_\alpha\Omega_q\Omega_p\Omega_\delta$,
are functions of the 4-dimensional scalars already written in the previous table.

(We note that every scalar of Theorem 4 with an even site can be obtained from that in the previous site, simply exchanging the two internal factors.)

By applying this theorem, we are now able to continue the transformation from 3-dimensional scalars into 4-dimensional ones, which we interrupted above.

	Table for 4-dimensional scalars (Continuation).
$(12)_7$, $(8)_5$, $(8)_6$	$tr\mathcal{A}_i^2\mathcal{A}_j^2$, $tr\mathcal{A}\Omega_p\Omega_q^2$, $tr\mathcal{A}\Omega_p^2\Omega_q$
$(9)_3$, $(15)_2$, $(15)_{5,6}$	$tr\mathcal{A}\mathcal{A}_i^2\Omega_p$, $tr\mathcal{A}_i^2\Omega_p^2$, $tr\mathcal{A}_i\Omega_p\Omega_q^2$, $tr\mathcal{A}_i\Omega_p^2\Omega_q$
$(16)_{3,4}$, $(17)_3$	$tr\mathcal{A}_i\mathcal{A}_j^2\Omega_p$, $tr\mathcal{A}_i^2\mathcal{A}_j\Omega_p$, $trI_1\mathcal{A}_\alpha\mathcal{A}_i^2\Omega_p\mathcal{A}_\alpha$
$(10)_3$, $(28)_2$	$trI_1\mathcal{A}_\alpha\mathcal{A}^2\Omega_p\mathcal{A}_\alpha$, $trI_1\Omega_\gamma\Omega_p\mathcal{A}_i\Omega_p^2\Omega_\gamma$
$(24)_2$	$trI_1\Omega_\gamma\Omega_p\mathcal{A}\Omega_p^2\Omega_\gamma$
$(17)_2$, Note 6	$trI_1\mathcal{A}_\alpha\Omega_p\mathcal{A}_i\Omega_p^2\mathcal{A}_\alpha$, $trI_1\Omega_p(\Omega_p\mathcal{A}_i-\mathcal{A}_i\Omega_p)\Omega_p\mathcal{A}_\alpha$
$(10)_2$, Note 7	$trI_1\mathcal{A}_\alpha\Omega_p\mathcal{A}\Omega_p^2\mathcal{A}_\alpha$, $trI_1\Omega_p(\Omega_p\mathcal{A}-\mathcal{A}\Omega_p)\Omega_p\mathcal{A}_\alpha$
$(16)_2$, Note 8	$tr\mathcal{A}_i\Omega_p^2\mathcal{A}_j\Omega_p - trI_1(\mathcal{A}_i\Omega_p^2\mathcal{A}_j\Omega_p+\Omega_p\mathcal{A}_i\Omega_p^2\mathcal{A}_j+$ $+\mathcal{A}_j\Omega_p\mathcal{A}_i\Omega_p^2+\Omega_p\mathcal{A}_j\Omega_p\mathcal{A}_i\Omega_p+\Omega_p^2\mathcal{A}_j\Omega_p\mathcal{A}_i)$
$(9)_2$, Note 8	$tr\mathcal{A}\Omega_p^2\mathcal{A}_i\Omega_p - trI_1(\mathcal{A}\Omega_p^2\mathcal{A}_i\Omega_p+\Omega_p\mathcal{A}\Omega_p^2\mathcal{A}_i+$ $+\mathcal{A}_i\Omega_p\mathcal{A}\Omega_p^2+\Omega_p\mathcal{A}_i\Omega_p\mathcal{A}\Omega_p+\Omega_p^2\mathcal{A}_i\Omega_p\mathcal{A})$
$(15)_3$, Note 8	$tr\mathcal{A}_i^2\Omega_p^2\mathcal{A}_i\Omega_p - trI_1(\mathcal{A}_i^2\Omega_p^2\mathcal{A}_i\Omega_p+\Omega_p\mathcal{A}_i^2\Omega_p^2\mathcal{A}_i+$ $+\mathcal{A}_i\Omega_p\mathcal{A}_i^2\Omega_p^2+\Omega_p\mathcal{A}_i\Omega_p\mathcal{A}_i^2\Omega_p+$ $+\Omega_p^2\mathcal{A}_i\Omega_p\mathcal{A}_i^2+\mathcal{A}_i\Omega_p^2\mathcal{A}_i\Omega_p\mathcal{A}_i)$
$(8)_3$, Note 8	$tr\mathcal{A}^2\Omega_p^2\mathcal{A}\Omega_p - trI_1(\mathcal{A}^2\Omega_p^2\mathcal{A}\Omega_p+\Omega_p\mathcal{A}^2\Omega_p^2\mathcal{A}+$ $+\mathcal{A}\Omega_p\mathcal{A}^2\Omega_p^2+\Omega_p\mathcal{A}\Omega_p\mathcal{A}^2\Omega_p+$ $+\Omega_p^2\mathcal{A}\Omega_p\mathcal{A}^2+\mathcal{A}\Omega_p^2\mathcal{A}\Omega_p\mathcal{A})$

Notes. 6) For this raw we have preferred to insert a scalar more than necessary (it would suffice the sum of the first one and of the second one premultiplied by $trI_1\mathcal{A}_i$), in order to have not long results; moreover, having both scalars will help us in the sequel for the transformation of the scalar $(15)_3$.

7) Also for this raw we have preferred to insert a scalar more than necessary (it would suffice the sum of the first one and of the second one premultiplied by $trI_1\mathcal{A}$), in order to have not long results; moreover, having both scalars will help us in the sequel for the transformation of the scalar $(8)_3$.

8) We note that, in these scalars, the first term in round brackets is the same whose trace is taken before the round brackets; the other ones may be obtained by cyclic permutation of the factors of the first term.

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