International Journal of Pure and Applied Mathematics
Volume 61 No. 2 2010, 231-240

SIMPLIFIED SQUARE EQUATION IN THE QUATERNION ALGEBRA

Dušan Pagon
Faculty of Natural Sciences and Mathematics
University of Maribor
Koroška Cesta 160, Maribor, SI-2000, SLOVENIA
e-mail: dusan.pagon@uni-mb.si

Abstract: Quaternion arithmetics is nowadays broadly used in electrodynamics, general relativity theory Trifonov [4] and for graphical programming, where it successfully replaces tensors and other similar models. The description of the solutions of linear equation was obtained already by classics, who introduced the quaternion algebra (W.R. Hamilton, J. Sylvester), but very little is still known about the structure of the solutions of higher order algebraic equations. We give a complete description of the solutions of quaternion square equation in the case when the unknown is not mixed with the coefficients, but always stands behind them (a simplified case).

AMS Subject Classification: 11R52, 20G20
Key Words: quaternion, matrix equation, complex matrix, conjugate matrix

1. Basic Facts about Quaternions

Quaternions were invented by Sir William R. Hamilton on 16-th October 1843 as a 4-dimensional real algebra with the basis $\{1, i, j, k\}$ satisfying the following conditions: $i^2 = j^2 = k^2 = ijk = -1$ (see, for instance, Conway, Smith [1]). The quaternion H with coordinates (h_1, h_2, h_3, h_4) in this standard basis of the quaternion algebra $\mathbb{H}$ can be represented by a 2×2 complex matrix

$$\begin{pmatrix} h_1 + ih_2 & h_3 + ih_4 \\ -h_3 + ih_4 & h_1 - ih_2 \end{pmatrix} = \begin{pmatrix} z & w \\ -\overline{w} & \overline{z} \end{pmatrix}.$$

This mapping is a monomorphism (an embedding) of the quaternion algebra

Received: May 10, 2010

into the algebra of complex matrices $\mathrm{M}_{2\times 2}(\mathbb{C})$ with usual addition and multiplication, see Ward [5].

Due to the non-commutative multiplication in $\mathbb{H}$ the general form of a square equation is

$$\sum_l A_{p_l} X A_{q_l} X A_{r_l} + \sum_m B_{s_m} X B_{t_m} + C = 0.$$

Geometrically the process of finding the solutions of such an equation is equivalent to the look for intersections of four quadric surfaces in $\mathbb{R}$ (see Littlewood [3]). So we can expect from 0 to 16 isolated solutions or an infinite 1- to 3-parametric family of solutions.

As in the following investigations we will mainly represent quaternions with complex matrices, we first recall some classical results from the matrix algebra.

2. Solving Matrix Equations

Theorem A. (The Generalized Bezout's Theorem) *Let* $p(\lambda) = \sum_{k=0}^{n} A_k \lambda^k$ *be an arbitrary polynomial with matrix coefficients* $A_k \in M_{m\times m}(F)$ *(we can consider* p *as a polynomial matrix) and* M *a matrix, satisfying the condition* $p(M) = 0$*. Then there exists a unique polynomial* q*, such that* $p(\lambda) = q(\lambda)(\lambda I - M)$*.*

Corollary. *Similarly we have* $\overline{p}(\lambda) = (\lambda I - N)\, r(\lambda)$ *with a unique polynomial* r*, when the matrix* N *is a solution of the equation* $\overline{p}(\lambda) = \sum_{k=0}^{n} \lambda^k A_k = 0$*.*

These classical results of linear algebra can be found, for instance, in Gantmacher [2]. Here the main difference between scalar and matrix equations becomes clear; having a solution N of a square equation $\lambda^2 + B\lambda + C = 0$ an rewriting the left side of the last as a product of two binomials $(\lambda I - M)(\lambda I - N) = 0$, we will generally not obtain the second solution of this equation: while $\lambda^2 I - (M+N)(\lambda I) + MN = 0$ yields $N^2 - MN - N^2 + MN = 0$, $M^2 - M^2 - NM + MN = [M, N] \neq 0$ and will equal 0 only in a particular case, when both matrices commute.

Another well known result from matrix algebra that we will use is

Theorem B. *Every matrix solution* X *of a polynomial equation* $p(X) = \sum_{k=0}^{n} a_k X^k = 0$ *(with scalar coefficients* a_k*) is similar to a block diagonal matrix* $[J_{d_1}(\lambda_{i_1}), J_{d_2}(\lambda_{i_2}), \ldots, J_{d_m}(\lambda_{i_m})]$*, where* λ_{i_k}*,* $k = 1, \ldots, m$ *are roots of the scalar* *equation* $p(\lambda) = 0$ *and every submatrix* $J_{d_k}(\lambda_{i_k})$ *is a Jordan cell of the form:*

$$\begin{pmatrix} \lambda_{i_k} & 1 & 0 & \dots & 0 & 0 \\ 0 & \lambda_{i_k} & 1 & \dots & 0 & 0 \\ 0 & 0 & \lambda_{i_k} & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \\ 0 & 0 & 0 & \dots & \lambda_{i_k} & 1 \\ 0 & 0 & 0 & \dots & 0 & \lambda_{i_k} \end{pmatrix}$$

which size d_k *does not exceed the power of the binomial* $\lambda - \lambda_{i_k}$ *in polynomial* p *(the algebraic degree of* λ_{i_k}*).*

Combining propositions A and B we get the following theorem, describing the solutions of polynomial equations with matrix coefficients:

Theorem 1. *Every solution* M *of a matrix polynomial equation* $p(X) = \sum\limits_{k=0}^{n} A_k X^k = 0$ *satisfies the condition* $r(M) = 0$, *where* $r = \det(\sum\limits_{k=0}^{n} A_k \lambda^k)$ *is a polynomial of a scalar variable* λ.

Proof. According to the Bezout's Theorem for every solution M of the given equation we have $p(\lambda) = \sum\limits_{k=0}^{n} A_k \lambda^k = q(\lambda)\,(\lambda I - M)$, yielding

$$r(\lambda) = \det p(\lambda) = \det q(\lambda) \det(\lambda I - M). \tag{1}$$

At the same time (by Cayley-Hamilton's Theorem) a matrix M is a zero of its characteristic polynomial $p_M(\lambda) = \det(\lambda I - M)$. Substituting $\lambda = M$ in the left hand side of the equation (1), we thus obtain 0.

3. Square Equation over Quaternions

Let $AX^2 + BX + C = 0$ be a square equation with quaternion coefficients

$$A = \begin{pmatrix} a_1 + ia_2 & a_3 + ia_4 \\ -a_3 + ia_4 & a_1 - ia_2 \end{pmatrix}, \quad B = \begin{pmatrix} b_1 + ib_2 & b_3 + ib_4 \\ -b_3 + ib_4 & b_1 - ib_2 \end{pmatrix},$$

$$C = \begin{pmatrix} c_1 + ic_2 & c_3 + ic_4 \\ -c_3 + ic_4 & c_1 - ic_2 \end{pmatrix}$$

and solution

$$X = \begin{pmatrix} x_1 + ix_2 & x_3 + ix_4 \\ -x_3 + ix_4 & x_1 - ix_2 \end{pmatrix}.$$

Then the matrix $\lambda^2 A + \lambda B + C$ equals to

$$\begin{pmatrix} a_1x^2 + b_1x + c_1 + i(a_2\lambda^2 + b_2\lambda + c_2) & a_3\lambda^2 + b_3\lambda + c_3 + i(a_4\lambda^2 + b_4\lambda + c_4) \\ -a_3\lambda^2 - b_3\lambda - c_3 + i(a_4x^\lambda + b_4\lambda + c_4) & a_1x^\lambda + b_1\lambda + c_1 - i(a_2x^\lambda + b_2\lambda + c_2) \end{pmatrix}$$

and its determinant is

$$(a_1^2 + a_2^2 + a_3^2 + a_4^2)\lambda^4 + 2(a_1b_1 + a_2b_2 + a_3b_3 + a_4b_4)\lambda^3 + (2a_1c_1 + 2a_2c_2 + 2a_3c_3 + 2a_4c_4 + b_1^2 + b_2^2 + b_3^2 + b_4^2)\lambda^2 + 2(b_1c_1 + b_2c_2 + b_3c_3 + b_4c_4)\lambda + c_1^2 + c_2^2 + c_3^2 + c_4^2.$$

To write this expression in a more compact form we define the following vectors in $\mathbb{R}^4$:

$$a = (a_1, a_2, a_3, a_4),\ b = (b_1, b_2, b_3, b_4) \text{ and } c = (c_1, c_2, c_3, c_4).$$

Then the previous equation turns into

$$|a|^2\lambda^4 + 2\langle a|b\rangle\lambda^3 + (2\langle a|c\rangle + |b|^2)\lambda^2 + 2\langle b|c\rangle\lambda + |c|^2 = 0.$$

If we first limit our search to real solutions $\lambda \in \mathbb{R}$, this is even equivalent to $|\lambda^2 a + \lambda b + c|^2 = 0$.

4. The Case with Real Solutions

It follows that in this case the vector c (and the matrix C) is linearly dependent of the other two. With the assumption of the connection $c = -t^2a - tb$, the original equation will reduce to

$$(\lambda - t)^2(|a|^2\lambda^2 + 2(|a|^2t + \langle a|b\rangle)\lambda + |a|^2t^2 + 2\langle a|b\rangle t + |b|^2) = 0,$$

or shorter $(\lambda - t)^2|a(\lambda + t) + b|^2 = 0$, when another pair of real solutions exists. We will distinguish two subcases:

I. If we have a double solution $\lambda_{1,2} = t \in \mathbb{R}$ the remaining two solutions will be real exactly when B is a multiple of matrix A. Suppose that $B = -sA$ (yielding $C = (s - t)tA$). Then we obtain the condition: $|a(\lambda + t) + b|^2 = |a|^2(\lambda + t - s)^2 = 0$ and we remain with another double solutions $\lambda_{3,4} = s - t$. So in this case after division with matrix A, which is common to all coefficients of our equation, we arrive to a scalar equation with two real scalar matrices (corresponding to real numbers) as the only solutions.

Example 1. Square equation $p(X) = X^2 + 3X + 2 = 0$ leads to the scalar polynomial $(x + 1)^2(x + 2)^2$ and has exactly 2 solutions in $\mathbb{H}$:

$$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \text{ and } \begin{pmatrix} -2 & 0 \\ 0 & -2 \end{pmatrix}.$$

According to Theorems 1 and B we should also consider an infinite family of matrices, conjugated to non scalar diagonal matrix (with both eigenvalues on the diagonal):

$$\begin{pmatrix} k & l \\ m & n \end{pmatrix}\begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}\begin{pmatrix} k & l \\ m & n \end{pmatrix}^{-1} = \frac{1}{kn-lm}\begin{pmatrix} \lambda kn-\mu lm & (\mu-\lambda)kl \\ (\lambda-\mu)mn & \mu kn-\lambda lm \end{pmatrix},$$

where $kn - lm \neq 0$.

Lemma 1. *In the above family there are no quaternion matrices.*

Proof. To represent a quaternion, such matrix should fulfill the following conditions:

$$\frac{\lambda kn-\mu lm}{kn-lm} = \overline{\frac{\mu kn-\lambda lm}{kn-lm}} \text{ and } \frac{(\lambda-\mu)kl}{kn-lm} = \frac{(\lambda-\mu)\overline{mn}}{\overline{kn}-\overline{lm}}.$$

The first equation is equivalent to $(\lambda-\mu)(|kn|^2-|lm|^2)=0$, while multiplying the second connection with $|kn-lm|^2$ and dividing the resulting equation by $\lambda-\mu$ we get: $(|k|^2+|m|^2)l\overline{n} = (|l|^2+|n|^2)k\overline{m}$. Next, we multiply both sides with $m\overline{l}$ and apply the first connection between absolute values to obtain

$$(|k|^2+|m|^2)|n|^2k\overline{l} = (|l|^2+|n|^2)|k|^2m\overline{n}.$$

Substituting here $|n|$ by $\frac{|l|\,|m|}{|k|}$ and simplifying we finally have: $(|k|^2+|m|^2)|l|^2\overline{ml}$ $= (|l|^2+|n|^2)|k|^2\overline{kn}$ or $|k|^2(|l|^2+|n|^2)\overline{kn-lm} = 0 \Rightarrow kn-lm=0$, contrary to our original assumption that $\begin{vmatrix} k & l \\ m & n \end{vmatrix} \neq 0$.

Lemma 2. *Matrix solutions (of the discussed square equation) similar to Jordan' cell, that are theoretically possible in the case of coinciding eigenvalues ($\lambda=\mu$), never represent a quaternion.*

Proof. This matrices look like $\begin{pmatrix} k & l \\ m & n \end{pmatrix}\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}\begin{pmatrix} k & l \\ m & n \end{pmatrix}^{-1} = \frac{1}{kn-lm}$ $\times \begin{pmatrix} k(n\lambda-m)-lm\lambda & k^2 \\ -m^2 & k(n\lambda+m)-lm\lambda \end{pmatrix}$, where the determinant $D = kn-$ $lm \neq 0$, and to represent a quaternion should fulfill the conditions:

$$\frac{k^2}{D} = \frac{\overline{m}^2}{\overline{D}} \text{ and } \frac{km}{D} + \frac{\overline{km}}{\overline{D}} = 0.$$

The first equation yields $|k|=|m|$ or, more precisely: $k = \pm\overline{m}\frac{D}{|D|}$. Substituting this value for k in the second equation we immediately get a contradiction: $2\frac{|m|^2}{|D|} = 0$.

II. Let us now suppose that our equation has a real double solution $\lambda_{1,2} = t$

while the other two roots are conjugated complex numbers $\lambda_{3,4} = u \pm iv, v \neq 0$. Comparing both forms of the square polynomial

$$|a|^2x^2 + 2(|a|^2t + \langle a|b\rangle)x + |ta+b|^2$$
$$= |a|^2(x-u-iv)(x-u+iv) = |a|^2(x^2 - 2ux + u^2 + v^2),$$

we get the conditions: $u = -\frac{|a|^2t+\langle a|b\rangle}{|a|^2}$ and $v = \pm\frac{\sqrt{|a|^2|b|^2-\langle a|b\rangle^2}}{|a|^2}$.

Therefore, in this case our equation has one real solution (a scalar matrix) and a quaternion solution of the form

$$\begin{pmatrix} k & l \\ m & n \end{pmatrix} \begin{pmatrix} u+iv & 0 \\ 0 & u-iv \end{pmatrix} \begin{pmatrix} k & l \\ m & n \end{pmatrix}^{-1},$$

with u, v satisfying the above conditions.

Example 2. Starting with the equation

$$\begin{pmatrix} 1-2i & -3+2i \\ 3+2i & 1+2i \end{pmatrix} X^2 + \begin{pmatrix} 4+i & -i \\ -i & 4-i \end{pmatrix} X + \begin{pmatrix} -12+6i & 12-6i \\ -12-6i & -12-6i \end{pmatrix} = 0,$$

we have a corresponding scalar polynomial:

$$\det(\lambda^2 A + \lambda B + C)$$
$$= \begin{vmatrix} \lambda^2+4\lambda-12-i(2\lambda^2-\lambda-6) & -3\lambda^2+12\lambda+i(2\lambda^2-\lambda-6) \\ 3\lambda^2-12\lambda+i(2\lambda^2-\lambda-6) & \lambda^2+4\lambda-12+i(2\lambda^2-\lambda-6) \end{vmatrix}$$
$$= 18\lambda^4 - 126\lambda^2 - 72\lambda + 360 = 18(\lambda-2)^2(\lambda+2+i)(\lambda+2-i),$$

and the only quaternion solutions of the original equation are the matrices $\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$ and $\frac{1}{3}\begin{pmatrix} -6-i & -2+2i \\ 2+2i & -6+i \end{pmatrix}$.

Summarizing the results of this section, we obtain the following

Theorem 2. *Real roots of a square equation $AX^2 + BX + C = 0$ with quaternion coefficients correspond to (double) real roots of the 4-th-order equation det$(\lambda^2 A + \lambda B + C) = 0$ (with real coefficients).*

5. Square Equations without Real Solutions

Finally, we will assume that the obtained scalar equation has no real roots. Then for both pairs of conjugated complex roots $u \pm iv$ the following dot product must vanish:

$$\langle (u+iv)^2a + (u+iv)b + c | (u-iv)^2a + (u-iv)b + c\rangle$$
$$= |a|^2(u^4+v^4) + 2\langle a|b\rangle u^3 - 6|a|^2u^2v^2 + (2\langle a|c\rangle + |b|^2)(u^2-v^2) + 2\langle b|c\rangle u$$

$$- 6\langle a|b\rangle uv^2 + |c|^2 + 2i[2|a|^2u^3v+3\langle a|b\rangle u^2v-2|a|^2uv^3 + (2\langle a|c\rangle+|b|^2)uv-\langle a|b\rangle v^3+\langle b|c\rangle v]. \quad (2)$$

The imaginary part of the last expression can be divided by $v \neq 0$.

I. Assuming that $\langle a|2ua + b\rangle \neq 0$ from here we obtain:

$$v^2 = \frac{2|a|^2u^3 + 3\langle a|b\rangle u^2 + (2\langle a|c\rangle + |b|^2)u + \langle b|c\rangle}{2|a|^2u + \langle a|b\rangle}.$$

As we are looking for $v \in \mathbb{R}^+$, the last expression must be positive. Substituting the above value of v^2 in the real part of the expression (2) we obtain a 6-th-order equation for u in which real solutions we are interested:

$$|\langle 2ua + b|a\rangle(u^2a + ub + c) - \langle u^2a + ub + c|2ua + b\rangle a|^2 = \langle u^2a + ub + c|2ua + b\rangle\langle 2ua + b|a\rangle|2ua + b|^2.$$

Remark. The last equation can be also rewritten in a more compact form: $|\langle f'|f''\rangle f-\langle f|f'\rangle f''|-\sqrt{2\langle f|f'\rangle\langle f'|f''\rangle}|f'| = 0$ if we denote by f a vector function $u \mapsto u^2a + ub + c$.

Example 3. $\begin{pmatrix} 3i & i \\ i & -3i \end{pmatrix} X^2 + \begin{pmatrix} 2+i & 1-3i \\ -1-3i & 2-i \end{pmatrix} X + \begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix} = 0.$ Then

$$\det(\lambda^2 A + \lambda B + C) = \begin{vmatrix} 2\lambda + 1 + i(3\lambda^2 + \lambda) & \lambda - 2 + i(\lambda^2 - 3\lambda) \\ -\lambda + 2 + i(\lambda^2 - 3\lambda) & 2\lambda + 1 - i(3\lambda^2 + \lambda) \end{vmatrix} = 10\lambda^4 + 15\lambda^2 + 5 = 5(\lambda^2 + 1)(2\lambda^2 + 1) = 0,$$

yielding $\lambda_{1,2} = \pm i$ and $\lambda_{3,4} = \pm i\frac{\sqrt{2}}{2}$.

Finally, the only solutions of the original quaternion equation are:

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad \text{and} \quad \frac{1}{6}\begin{pmatrix} -i & 4-i \\ -4-i & i \end{pmatrix}.$$

There indeed exist another 4 matrix solutions (similar to nonquaternion diagonal matrices), but they do not represent quaternions:

$$\frac{1}{4}\begin{pmatrix} 1+\sqrt{2}+i(3+2\sqrt{2}) & 1-2\sqrt{2}+i(1+\sqrt{2}) \\ -3+i(1+\sqrt{2}) & -1-\sqrt{2}+i \end{pmatrix},$$

$$\frac{1}{4}\begin{pmatrix} 1-\sqrt{2}+i(3-2\sqrt{2}) & 1+2\sqrt{2}+i(1-\sqrt{2}) \\ -3+i(1-\sqrt{2}) & -1+\sqrt{2}+i \end{pmatrix},$$

$$\frac{1}{4}\begin{pmatrix} -\sqrt{2}-i(1+\sqrt{2}) & 3+i(1+\sqrt{2}) \\ -1+2\sqrt{2}+i(1+\sqrt{2}) & 1+\sqrt{2}-i(3+2\sqrt{2}) \end{pmatrix}$$

and

$$\frac{1}{4}\begin{pmatrix} \sqrt{2}-i(1-\sqrt{2}) & 3+i(1-\sqrt{2}) \\ -1-2\sqrt{2}+i(1-\sqrt{2}) & 1-\sqrt{2}-i(3-2\sqrt{2}) \end{pmatrix}.$$

II. If the vectors a and b are not collinear, there exists a unique $u \in \mathbb{R}$ such that $2ua + b \perp a$. It is $u = -\frac{\langle a|b\rangle}{2|a|^2}$. Substituting this value in the remaining imaginary part of expression (2) we obtain the necessary condition on the coefficients of the initial equation for the existence of solutions of this form:

$$\langle a|b\rangle^3 - (2\langle a|c\rangle + |b|^2)\langle a|b\rangle|a|^2 + 2\langle b|c\rangle|a|^4 = 0.$$

Finally, equaling to 0 the real part of the expression (2), in which we have substituted u by $-\frac{\langle a|b\rangle}{2|a|^2}$, we find possible values of v from the biquadratic equation:

$$2(|a|v)^4 + (3(\frac{\langle a|b\rangle}{|a|})^2 - 4\langle a|c\rangle - 2|b|^2)(|a|v)^2$$
$$-\frac{3}{8}(\frac{\langle a|b\rangle}{|a|})^4 + (\langle a|c\rangle + \frac{|b|^2}{2})(\frac{\langle a|b\rangle}{|a|})^2 - 2\langle a|b\rangle\langle b|c\rangle + 2|a|^2|c|^2 = 0.$$

III. Now, let us suppose that $b = 2ta$ for some $t \in \mathbb{R}$ and $u \neq -t$. Than from the equivalence of the imaginary part of expression (2) to 0 it follows:

$$v = \pm\sqrt{u^2 + 2tu + \frac{\langle a|c\rangle}{|a|^2}},$$

while u is a real solution of the following equation, obtained by substituting the above value of v into the real part of (2):

$$4|a|^4u^4 + 16t|a|^4u^3 + (20t^2|a|^2 + 4\langle a|c\rangle)|a|^2u^2$$
$$+ (4t^2|a|^2 + 8\langle a|c\rangle)t|a|^2u + (4t^2|a|^2 + \langle a|c\rangle)\langle a|c\rangle - |a|^2|c|^2 = 0.$$

Thus we can obtain up to 4 pairs of conjugated complex numbers $u \pm iv$ and, correspondingly, up to 8 solutions of the original quaternion equation.

IV. The last case, to be considered separately, is when the relation $B = -2uA$ holds among the coefficients of our quaternion square equation, where u is a real part of the solution of the related scalar equation. Then the imaginary part in (2) vanishes and v^2 is determined by the real part of this expression

$$|a|^2v^4 + 2(|a|^2u^2 - \langle a|c\rangle)v^2 + (|a|^2u^2 - 2\langle a|c\rangle)u^2 + |c|^2 = 0,$$

which can be rewritten in the form

$$(|a|v^2 + |a|u^2 - \frac{\langle a|c\rangle}{|a|})^2 = \frac{\langle a|c\rangle^2 - |a|^2|c|^2}{|a|^2} \leq 0.$$

Real solutions of this equation exist only when vectors a, c are colinear ($c =$

$sa,\ s \in \mathbb{R}$), so the original quaternion square equation is again equivalent to one with the real coefficients: $X^2 - 2uX + s = 0$ (compare with case I in Section 4).

Finally, we obtain two (double) solutions $v = \pm\sqrt{s-u^2}$ (if $s > u^2$), yielding an infinite family of solutions of the original quaternion equation, similar to a diagonal matrix with conjugated complex numbers on the diagonal.

Example 4. Consider the equation $X^2 - 4X + 13 = 0,\ X \in \mathbb{H}$. Then

$$\det\,(\lambda^2 A + \lambda B + C) = (\lambda^2 - 4\lambda^2 + 13)^2 = (\lambda - 2 - 3i)^2(\lambda - 2 + 3i)^2 = 0$$

and every matrix, conjugated with $D = \begin{pmatrix} 2+3i & 0 \\ 0 & 2-3i \end{pmatrix}$ will be a solution of our equation:

$$X = P^{-1}DP = \begin{pmatrix} 2 + \frac{3i(ps+qr)}{ps-qr} & \frac{6iqs}{ps-qr} \\ -\frac{6ipr}{ps-qr} & 2 - \frac{3i(ps+qr)}{ps-qr} \end{pmatrix}, \text{ where } P = \begin{pmatrix} p & q \\ r & s \end{pmatrix}.$$

To obtain a quaternion matrix, $\frac{ps+qr}{ps-qr}$ must be a real number and the equality $\overline{\frac{pr}{ps-qr}} = -\frac{qs}{ps-qr}$ must hold. This conditions are fulfilled exactly when $p\overline{r} + q\overline{s} = 0$, so the obtained family of solutions is

$$\left\{ \begin{pmatrix} 2 + 3i\frac{|p|^2-|q|^2}{|p|^2+|q|^2} & 6i\frac{\overline{p}q}{|p|^2+|q|^2} \\ 6i\frac{p\overline{q}}{|p|^2+|q|^2} & 2 - 3i\frac{|p|^2-|q|^2}{|p|^2+|q|^2} \end{pmatrix},\quad p, q \in \mathbb{C} \right\}.$$

6. Conclusion

In the previous section we have proved the following

Theorem 3. *A simplified square equation $AX^2 + BX + C = 0$ with quaternion coefficients will have from 2 to 8 or an infinite family of (nonreal) quaternion solutions exactly when all roots of the related scalar equation (with real coefficients) are complex numbers with nonzero imaginary component: $u + iv,\ u \neq 0$.*

As we have already mentioned in the introductory Section 1, it follows from the geometric discussion of square equation over the algebra $\mathbb{H}$, that in general case (when mixed products of coefficients and unknown X are present) the set of its solutions is a variety, depending on several parameters, or it consists of up to 16 different quaternions. There is no convenient algorithm, based on the coefficient of the given equation, to classify all appearing cases.

References

[1] J.H. Conway, D.A. Smith, *On Quaternions and Octonions: Their Geometry, Arithmetic and Symmetry*, A.K. Peters, Natick, MA (2003).

[2] F.R. Gantmacher, *The Theory of Matrices*, Volumes I and II, Chelsea, New York (1959).

[3] D.E. Littlewood, The quadratic equation in quaternions, *Proc. London Math. Soc.* (1929), 40-46.

[4] V. Trifonov, A linear solution of the four-dimensionality problem, *Europhysics Letters*, **32**, No. 8 (1995), 621-626.

[5] J.P. Ward, *Quaternions and Cayley Numbers: Algebra and Applications*, Kluwer Academic Publishers, Dordrecht (1997).