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STRONG CONVERGENCE THEOREM OF A HYBRID PROJECTION ALGORITHM FOR A FAMILY OF QUASI-ϕ-ASYMPTOTICALLY NONEXPANSIVE MAPPINGS (I)

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Abstract: The main purpose of this paper is by using a new hybrid projection iterative algorithm to prove some strong convergence theorems for a family of quasi-ϕ-asymptotically nonexpansive mappings. The results presented in the paper improve and extend the corresponding results announced by some authors.

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1. Introduction

Throughout this paper, we assume that E is a real Banach space, E^* is the dual space of E, C is a nonempty closed convex subset of E, and $\langle \cdot, \cdot \rangle$ is the

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pairing between E and E^*. Recall that a mapping $T : C \to C$ is said to be asymptotically nonexpansive (see [1]) if there exists a sequence $\{k_n\} \subset [1, \infty)$ with $k_n \to 1$ such that

$$||T^n x - T^n y|| \leq k_n ||x - y|| \ \forall x, y \in C \text{ and } \forall n \geq 1. \tag{1.1}$$

In recent years, nonexpansive mappings and asymptotically nonexpansive mappings have been studies extensively by many authors. In 2003, Nakajo and Takahashi [5] proposed the following modification of the Mann iteration method for a nonexpansive mapping T in a Hilbert space H:

$$\begin{cases} x_0 \in C, \\ y_n = \alpha_n x_n + (1 - \alpha_n) T x_n, \\ C_n = \{z \in C : ||y_n - z|| \leq ||x_n - z||\}, \\ Q_n = \{z \in C :< x_n - z, x_0 - x_n >\geq 0\}, \\ x_{n+1} = P_{C_n \cap Q_n}(x_0), \end{cases} \tag{1.2}$$

where C is a closed convex subset of H, and P_K is the metric projection from H onto a closed convex subset K of H. They proved that if the sequence $\{\alpha_n\}$ is bounded above from one, then the sequence $\{x_n\}$ generated by (1.2) converges strongly to $P_{F(T)}(x_0)$.

In 2006, Kim and Xu [3] proposed the modification of the Mann iteration method for a asymptotically nonexpansive mapping T in a Hilbert space H. In 2005, Matsushita and Takahashi [4] proposed the hybrid iteration method with generalized projection for a relatively nonexpansive mapping T in a Banach space E. In 2009, Zhou and Gao [7] proposed the following modified hybrid iteration method with generalized projection for a family of closed and quasi-ϕ-asymptotically nonexpansive mappings which are asymptotically regular in a Banach space E:

$$\begin{cases} x_0 \in C, \\ y_{n,i} = J^{-1}(\alpha_n J x_n + (1 - \alpha_n) J T_i^n x_n), \\ C_{n,i} = \{z \in C : \phi(z, y_{n,i}) \leq \phi(z, x_n) + \xi_{n,i}\}, \\ C_n = \cap_{i \in I} C_{n,i}, \\ Q_0 = C, \\ Q_n = \{z \in Q_{n-1} :< x_n - z, J x_0 - J x_n >\geq 0\}, \\ x_{n+1} = \Pi_{C_n \cap Q_n}(x_0). \end{cases} \tag{1.3}$$

Under suitable conditions they proved that the sequence $\{x_n\}$ generated by (1.3) converges strongly to $\Pi_{\cap_{i \in I} F(T_i)}(x_0)$.

Motivated and inspired by the researches going on in this direction, the purpose of this paper is to introduce a hybrid projection iterative algorithm and prove strong some convergence theorems for a family of quasi-ϕ-asymptotically nonexpansive mappings in the setting of Banach spaces. The results presented in the paper improve and extend the corresponding results in [5], [4], [3], [7].

2. Preliminaries

Let E be a Banach space with a dual E^* and C be a nonempty closed convex subsets of E. We denote by $J : E \to 2^{E^*}$ the normalized duality mapping defined by

$$Jx = \{f \in E^* : \langle x, f\rangle = ||x||^2 = ||f||^2\}, \quad x \in E.$$

It is well known that if E is uniformly convex and uniformly smooth, then J and J^{-1} both are uniformly continuous on bounded subsets of E and E^*, respectively.

In the sequel, we always denote by $\phi : E \times E \to R^+$ the Lyapunov functional defined by

$$\phi(x, y) = ||x||^2 - 2\langle x,\ Jy\rangle + ||y||^2,\ \forall x, y \in E. \tag{2.1}$$

From the definition of ϕ, it is obvious that

$$(||x|| - ||y||)^2 \le \phi(x, y) \le (||x|| + ||y||)^2,\ \forall x, y \in E. \tag{2.2}$$

The *generalized projection* $\Pi_C : E \to C$ is defined by

$$\Pi_C(x) = z \Leftrightarrow \phi(z, x) = \inf_{y \in C} \phi(y, x), \quad \forall x \in E. \tag{2.3}$$

Lemma 2.1. (see [2]) *Let E be a smooth, strict convex and reflexive Banach space and C be a nonempty closed convex subset of E. Then, the following conclusions hold:*

(i) $\phi(x, \Pi_C y) + \phi(\Pi_C y, y) \le \phi(x, y),\ \forall x \in C, y \in E$.

(ii) Let $x \in E$ and $z \in C$, then

$$z = \Pi_C x \Longleftrightarrow \langle z - y,\ Jx - Jz\rangle \ge 0,\ \forall y \in C. \tag{2.4}$$

Let C be a closed convex subset of E, and T a mapping from C into itself. T is said to be *ϕ-asymptotically nonexpansive*, if there exists a sequence $\{k_n\} \subset [1, \infty)$ with $k_n \to 1$ such that $\phi(T^n x, T^n y) \le k_n \phi(x, y)$ for all $n \ge 1$ and $x, y \in C$. T is said to be *quasi-ϕ-nonexpansive*, if $F(T) \ne \emptyset$ and $\phi(p, Tx) \le \phi(p, x)$ for all $x \in C$ and $p \in F(T)$. T is said to be *quasi-ϕ-asymptotically nonexpansive*, if $F(T) \ne \emptyset$ and there exist a sequence $\{k_n\} \subset [1, \infty)$ with $k_n \to 1$ such that

$\phi(p, T^n x) \leq k_n \phi(p, x)$ for all $n \geq 1$, $x \in C$ and $p \in F(T)$. T is said to be *closed*, if for any $\{x_n\}$ with $x_n \to x$ and $Tx_n \to y$, then we have $Tx = y$.

T is said to be asymptotically regular on C if, for any bounded subset D of C, the following equality holds:

$$\lim_{n\to\infty} \sup\{||T^{n+1}x - T^n x|| : x \in D\} = 0.$$

The following lemmas will play an important role in the proof of the main results in this paper.

Lemma 2.2 (see [2]) *Let E be a uniformly convex and smooth Banach space and $\{x_n\}$, $\{y_n\}$ be sequences of E. If $\phi(x_n, y_n) \to 0$ (as $n \to \infty$) and either $\{x_n\}$ or $\{y_n\}$ is bounded, then $x_n - y_n \to 0$(as $n \to \infty$).*

Lemma 2.3 (see [7]) *Let E be a uniformly convex and smooth Banach space, C be a closed convex subset of E, and T be a closed and quasi-ϕ-asymptotically nonexpansive mapping from C into itself. Then $F(T)$ is a closed convex subset of C.*

Lemma 2.4 (see [6]) *Let E be a uniformly convex Banach space, $r > 0$ be a positive number, and $B_r(0)$ be a closed ball of E. For any given points $\{x_1, x_2, \cdots, x_n, \cdots\} \subset B_r(0)$ and for any given positive numbers $\{\lambda_1, \lambda_2, \cdots, \lambda_n, \cdots\}$ with $\Sigma_{n=1}^{\infty}\lambda_n = 1$. There exists a continuous, strictly increasing and convex function $g : [0, 2r) \to [0, \infty)$ with $g(0) = 0$ such that for any $i, j \in \{1, 2, \cdots, \}, i < j$,*

$$||\sum_{n=1}^{\infty} \lambda_n x_n||^2 \leq \sum_{n=1}^{\infty} \lambda_n ||x_n||^2 - \lambda_i \lambda_j g(||x_i - x_j||). \tag{2.5}$$

3. Main Results

Theorem 3.1. *Let C be a nonempty bounded closed convex subset of a uniformly convex and uniformly smooth Banach space E. Let $S : C \to C$ be a closed and quasi-ϕ nonexpansive mapping. For each $i = 1, 2, \cdots$, let $T_i : C \to C$ be a closed and quasi-ϕ-asymptotically nonexpansive mapping with a sequence $\{k_{n,i}\} \subset [1, \infty)$ such that $k_n := \sup_{i\geq 1} k_{n,i} \to 1$ $(n \to \infty)$ and $F = F(S) \bigcap \bigcap_{i=1}^{\infty} F(T_i) \neq \emptyset$. Suppose further that for each $i = 1, 2, \cdots$, T_i is* *asymptotically regular on C. Let $\{x_n\}$ be the sequence in C defined by:*

$$\begin{cases} x_0 \in C, \quad C_0 = C, \\ y_n = J^{-1}(\alpha_n JS(x_n) + \beta_n Jx_n + \sum\limits_{i=1}^{\infty}\gamma_{ni} JT_i^n x_n), \\ C_{n+1} = \{z \in C_n : \phi(z, y_n) \le \phi(z, x_n) + \xi_n\}, \\ x_{n+1} = \Pi_{C_{n+1}}(x_0), \end{cases} \tag{3.1}$$

where $J : E \to E^*$ is the normalized duality mapping, $M = \sup_{z\in F,\, n\ge 1} \phi(z, x_n)$. If $\xi_n = \sum\limits_{i=1}^{\infty}\gamma_{ni}(k_n - 1)M \to 0$, and $\{\alpha_n\}, \{\beta_n\}, \{\gamma_{ni}\}$ are the sequences in $[0, 1]$ satisfying the following conditions:

(a) $\alpha_n + \beta_n + \sum\limits_{i=0}^{\infty}\gamma_{ni} = 1, \quad \forall n \ge 0$;

(b) $\liminf\limits_{n\to\infty} \alpha_n\beta_n > 0; \liminf\limits_{n\to\infty} \beta_n\gamma_{ni} > 0, \quad i = 1, 2, \cdots$.

Then $\{x_n\}$ converges strongly to $\Pi_F x_0$, where Π_F is the generalized projection of E onto F.

Proof. We divide the proof of Theorem 3.1 into six steps:

(I) Because $\phi(z, y_n) \le \phi(z, x_n) + \xi_n$ is equivalent to $2\langle z,\ Jx_n - Jy_n\rangle \le ||x_n||^2 - ||y_n||^2 + \xi_n$. This implies that C_n is a closed and convex subset of C for all $n \ge 0$.

(II) Next, we prove that $F := F(S) \bigcap \bigcap\limits_{i=1}^{\infty} F(T_i) \subset C_n,\ \forall n \ge 0$.

Indeed, it is obvious that $F \subset C_0$. Suppose that $F \subset C_n$ for some $n \in \mathbb{N}$. Noting that $|| \cdot ||^2$ is convex and using (2.1), for any $z \in F \subset C_n$ and for any $\forall j \in \{0, 1, 2, \cdots\}$, we have that

$$\begin{aligned} \phi(z, y_n) &= \phi(z, J^{-1}(\alpha_n JS(x_n) + \beta_n Jx_n + \sum_{i=1}^{\infty}\gamma_{ni} JT_i^n x_n)) \\ &= ||z||^2 - 2\langle z,\ \alpha_n JS(x_n) + \beta_n Jx_n + \sum_{i=1}^{\infty}\gamma_{ni} JT_i^n x_n\rangle \\ &\quad + \ ||\alpha_n JS(x_n) + \beta_n Jx_n + \sum_{i=1}^{\infty}\gamma_{ni} JT_i^n x_n||^2 \\ &\le ||z||^2 - 2\alpha_n\langle z, JS(x_n)\rangle - 2\beta_n\langle z, Jx_n\rangle - 2\sum_{i=1}^{\infty}\gamma_{ni}\langle z,\ JT_i^n x_n\rangle \end{aligned}$$

$$+ \alpha_n||S(x_n)||^2 + \beta_n||x_n||^2 + \sum_{i=1}^{\infty}\gamma_{ni}||T_i^n x_n||^2 - \beta_n\gamma_{nj}g(||Jx_n - JT_j^n x_n||)$$

$$= \alpha_n\phi(z, S(x_n)) + \beta_n\phi(z, x_n) + \sum_{i=1}^{\infty}\gamma_{ni}\phi(z, T_i^n x_n) - \beta_n\gamma_{nj}g(||Jx_n - JT_j^n x_n||)$$

$$\le \alpha_n\phi(z, x_n) + \beta_n\phi(z, x_n) + \sum_{i=1}^{\infty}\gamma_{ni}((k_{n,i} - 1) + 1)\phi(z, x_n)$$

$$- \beta_n\gamma_{nj}g(||Jx_n - JT_j^n x_n||)$$

$$\le \phi(z, x_n) + \xi_n - \beta_n\gamma_{nj}g(||Jx_n - JT_j^n x_n||) \le \phi(z, x_n) + \xi_n. \quad (3.2)$$

This implies that $z \in C_n$. Thereby, $F \subset C_n$, $\forall n \ge 0$.

(III) Now, we prove that $\{x_n\}$ is a Cauchy sequence.

Indeed, since $x_{n+1} = \Pi_{C_{n+1}}x_0$ and $x_n = \Pi_{C_n}x_0$, $x_{n+1} \in C_{n+1} \subset C_n$, from the definition of Π_{C_n} we have

$$\phi(x_n, x_0) \le \phi(x_{n+1}, x_0), \ \forall n \ge 0. \quad (3.3)$$

Therefore $\{\phi(x_n, x_0)\}$ is nondecreasing. By the assumption that C is bounded, hence from (2.2) we know that $\{\phi(x_n, x_0)\}$ is bounded. This together with (3.3) ensures that the limit $\{\phi(x_n, x_0)\}$ exists. Write

$$\lim_{n\to\infty}\phi((x_n, x_0) = d. \quad (3.4)$$

From Lemma 2.1, we have, for any positive integer $m \ge n$, that

$$\begin{aligned}\phi(x_m, x_n) &= \phi(x_m, \Pi_{C_n}x_0) \le \phi(x_m, x_0) - \phi(\Pi_{C_n}x_0, x_0) \\ &= \phi(x_m, x_0) - \phi(x_n, x_0).\end{aligned} \quad (3.5)$$

This implies that

$$\lim_{n,m\to\infty}\phi(x_m, x_n) = 0.$$

By Lemma 2.2, we know that $x_m - x_n \to 0 (n, m \to \infty)$, hence, $\{x_n\}$ is a Cauchy sequence. Without loss of generality, we can assume that

$$x_n \to p \in C(n \to \infty). \quad (3.6)$$

(IV) Now, we prove $||x_n - T_i^n x_n|| \to 0$ for each $i = 1, 2, \cdots$ and $||x_n - S(x_n)|| \to 0$.

In fact, taking $m = n + 1$ in (3.5) it yields that

$$\phi(x_{n+1}, x_n) \le \phi(x_{n+1}, x_0) - \phi(x_n, x_0) \to 0 \quad (n \to \infty), \quad (3.7)$$

and hence $x_{n+1} - x_n \to 0$ as $n \to \infty$ by Lemma 2.2. Since $x_{n+1} = \Pi_{C_{n+1}}x_0 \in C_{n+1}$, and by the assumption that $\xi_n \to 0 \quad (as \ n \to \infty)$, hence from the definition of C_{n+1}, we have

$$\phi(x_{n+1}, y_n) \leq \phi(x_{n+1}, x_n) + \xi_n \to 0 \quad (n \to \infty), \tag{3.8}$$

and so $x_{n+1} - y_n \to 0 (n \to \infty)$ by Lemma 2.2. Thus we have

$$||y_n - x_n|| \leq ||y_n - x_{n+1}|| + ||x_{n+1} - x_n|| \to 0 \quad (n \to \infty). \tag{3.9}$$

Since J is uniformly continuous on any bounded sets of E, we conclude that

$$\lim_{n\to\infty} ||Jx_n - Jy_n|| = 0. \tag{3.10}$$

On the other hand, taking $j = 1, 2, \cdots$ in (3.2), for any $n \geq 1$ and for any $z \in F$, we have

$$\phi(z, y_n) \leq \phi(z, x_n) + \xi_n - \beta_n \gamma_{nj} g(||Jx_n - JT_j^n x_n||),$$

i.e.,

$$\beta_n \gamma_{nj} g(||Jx_n - JT_j^n x_n||) \leq \phi(z, x_n) - \phi(z, y_n) + \xi_n. \tag{3.11}$$

Since

$$\begin{aligned} \phi(z, x_n) - \phi(z, y_n) + \xi_n &= ||x_n||^2 - ||y_n||^2 - 2 < z, Jx_n - Jy_n > +\xi_n \\ &\leq ||x_n||^2 - ||y_n||^2 + 2||z||||Jx_n - Jy_n|| + \xi_n \\ &\leq ||x_n - y_n||(||x_n + y_n||) + 2||z||||Jx_n - Jy_n|| + \xi_n \end{aligned} \tag{3.12}$$

from (3.9) and (3.10), it follows that $\phi(z, x_n) - \phi(z, y_n) + \xi_n \to 0$ $(n \to \infty)$. Hence, from (3.11) and the condition (b) in Theorem 3.1, it yields that

$$g(||Jx_n - JT_j^n x_n||) \to 0 (n \to \infty), \ \forall j = 1, 2, \cdots . \tag{3.13}$$

Since g is continuous and strictly increasing with $g(0) = 0$, it follows from (3.13) that

$$||Jx_n - JT_j^n x_n|| \to 0 \ (as\ n \to \infty) \text{ and for each } j = 1, 2, \cdots .$$

Again by the assumption that E is uniformly convex and so E^* is uniformly smooth, hence J^{-1} is uniformly continuous on any bounded subset of E^*. Therefore we have

$$||x_n - T_j^n x_n|| \to 0 (n \to \infty), \text{ for each } j = 1, 2, \cdots . \tag{3.14}$$

Similarly, we have

$$\begin{aligned} \phi(z, y_n) &= \phi(z, J^{-1}(\alpha_n JS(x_n) + \beta_n Jx_n + \sum_{i=1}^{\infty} \gamma_{ni} JT_i^n x_n)) \\ &= ||z||^2 - 2\langle z,\ \alpha_n JS(x_n) + \beta_n Jx_n + \sum_{i=1}^{\infty} \gamma_{ni} JT_i^n x_n \rangle \\ &\quad + \ ||\alpha_n JS(x_n) + \beta_n Jx_n + \sum_{i=1}^{\infty} \gamma_{ni} JT_i^n x_n||^2 \end{aligned}$$

$$\leq ||z||^2 - 2\alpha_n\langle z, JS(x_n)\rangle - 2\beta_n\langle z, Jx_n\rangle - 2\sum_{i=1}^{\infty}\gamma_{ni}\langle z,\ JT_i^n x_n\rangle$$

$$+ \alpha_n||S(x_n)||^2 + \beta_n||x_n||^2 + \sum_{i=1}^{\infty}\gamma_{ni}||T_i^n x_n||^2 - \alpha_n\beta_n g(||Jx_n - JS(x_n)||)$$

$$= \alpha_n\phi(z, S(x_n)) + \beta_n\phi(z, x_n) + \sum_{i=1}^{\infty}\gamma_{ni}\phi(z, T_i^n x_n) - \alpha_n\beta_n g(||Jx_n - JS(x_n)||)$$

$$\leq \alpha_n\phi(z, x_n) + \beta_n\phi(z, x_n) + \sum_{i=1}^{\infty}\gamma_{ni}((k_{n,i}-1)+1)\phi(z, x_n) - \alpha_n\beta_n g(||Jx_n - JS(x_n)||)$$

$$\leq \phi(z, x_n) + \xi_n - \alpha_n\beta_n g(||Jx_n - JS(x_n)||),$$

i.e.,

$$\alpha_n\beta_n g(||Jx_n - JS(x_n)||) \leq \phi(z, x_n) - \phi(z, y_n) + \xi_n.$$

Hence, from the condition (b) in Theorem 3.1, it yields that

$$g(||Jx_n - JS(x_n)||) \to 0 \quad (n \to \infty).$$

Therefore, from the property of g, we have

$$||Jx_n - JS(x_n)|| \to 0 \quad (n \to \infty).$$

Since J^{-1} is uniformly norm-to-norm continuous on bounded sets, we have

$$||x_n - S(x_n)|| \to 0 \quad (n \to \infty).$$

(V) Now, we prove $p \in F = F(S) \bigcap \bigcap_{i=1}^{\infty} F(T_i)$.

In fact, from (III), (IV) and S is closed, we have $Sp = p$, i.e., $p \in F(S)$.

From $x_n \to p(n \to \infty)$ and (3.14), we have

$$T_j^n x_n \to p(n \to \infty) \text{ for each } j = 1, 2, \cdots . \tag{3.15}$$

Noting that

$$||T_i^{n+1} x_n - p|| \leq ||T_i^{n+1} x_n - T_i^n x_n|| + ||T_i^n x_n - p||, \tag{3.16}$$

using (3.15) and the asymptotic regularity of T_i, from (3.16) we have

$$T_i^{n+1} x_n \to p(n \to \infty),\ \text{i.e., } T_i T_i^n x_n \to p \quad (n \to \infty). \tag{3.17}$$

By virtue of the closeness of T_i, it follows from (3.15) and (3.17) that p is a fixed point of T_i, $\forall i \geq 1$, i.e., $p \in \bigcap_{i=1}^{\infty} F(T_i)$.

(VI) Now, we prove $x_n \to p = \Pi_F x_0$ $(n \to \infty)$.

Let $w = \Pi_F x_0$. From $w \in F \subset C_{n+1}$ and $x_{n+1} = \Pi_{C_{n+1}} x_0$, we have

$\phi(x_{n+1}, x_0) \leq \phi(w, x_0)$, $\forall n \geq 0$. This implies that

$$\phi(p, x_0) = \lim_{n\to\infty} \phi(x_n, x_0) \leq \phi(w, x_0) \tag{3.18}$$

By the definition of $\Pi_F x_0$ and (3.18), we have $p = w$. Therefore, $x_n \to p = \Pi_F x_0$ $(n \to \infty)$.

This completes the proof of Theorem 3.1. □

Remark. The asymptotic regularity assumption on T_i in Theorem 3.1 can be weaken to the assumption that $T_i^{n+1}x_n - T_i^n x_n \to 0$ as $n \to \infty$. The assumption that $T_i^{n+1}x_n - T_i^n x_n \to 0$ as $n \to \infty$ can be replaced by the uniform Lipschitz continuous of T_i.

Therefore, we have the following convergence result.

Corollary 3.1. *Let C be a nonempty bounded closed convex subset of a uniformly convex and uniformly smooth Banach space E, $S : C \to C$ be a closed and quasi-ϕ nonexpansive mapping and $\{T_i\}_{i=1}^\infty : C \to C$ be a family of closed and uniformly Lipschitz continuous and quasi-ϕ-asymptotically nonexpansive mappings with sequence $\{k_{n,i}\}_1^\infty \subset [1, \infty)$ such that $k_n := \sup_{i\geq 1} k_{n,i} \to 1$ $(n \to \infty)$ and $F = F(S) \cap \bigcap_{i=1}^{\infty} F(T_i) \neq \emptyset$. Let $\{x_n\}$, $\{\alpha_n\}, \{\beta_n\}, \{\gamma_{ni}\}$ and $\{\xi_n\}$ be the same sequences as given in Theorem 3.1. Then $\{x_n\}$ converges strongly to $\Pi_F x_0$.*

Because each quasi-ϕ nonexpansive mapping is a quasi-ϕ-asymptotically nonexpansive mapping with sequence $\{k_{n,i} = 1\}$, therefore we have the following

Corollary 3.2. *Let C be a nonempty bounded closed convex subset of a uniformly convex and uniformly smooth Banach space E. Let S be a closed and quasi-ϕ nonexpansive mapping and $\{T_i\}_{i=1}^\infty : C \to C$ be a family of closed and quasi-ϕ nonexpansive mappings such that $F = F(S) \cap \bigcap_{i=1}^{\infty} F(T_i) \neq \emptyset$. Let $\{x_n\}$ be the sequence defined by:*

$$\begin{cases} x_0 \in C, \quad C_0 = C, \\ y_n = J^{-1}(\alpha_n JS(x_n) + \beta_n Jx_n + \sum_{i=1}^{\infty} \gamma_{ni} JT_i x_n), \\ C_{n+1} = \{z \in C_n : \phi(z, y_n) \leq \phi(z, x_n)\}, \\ x_{n+1} = \Pi_{C_{n+1}}(x_0), \end{cases} \tag{3.19}$$

where $\{\alpha_n\}, \{\beta_n\}, \{\gamma_{ni}\} \subset [0, 1]$ is the sequence satisfying the conditions (a), (b) in Theorem 3.1. Then $\{x_n\}$ converges strongly to $\Pi_F x_0$.

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