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AMPLIFICATION OF THERMAL NOISE BY AN ELECTROSTATIC FIELD

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Abstract: Recently a formula for the variance of the thermal voltage between the ends of a conductor has been proposed. We discuss this formula with regard to possibilities of manipulating the thermal voltage. In particular, utilizing the dependence of the variance of the voltage on the density of the electronic gas confined to the conductor, we point out a possibility of voltage amplification by the exposition of the conductor to an electrostatic field. This possibility is noteworthy insofar as it necessitates only negligible input of electric power.

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1. Thermal Noise in a Modified Drude Model

In [1] the electronic gas in a conductor is modelled as a system of charged mass points obeying Newtonian dynamics. The velocities of the mass points are distributed according to a centered normal distribution whose variance σ^2 can be interpreted thermally according to

$$\sigma^2 = \frac{k_B \cdot T}{m}, \tag{1.1}$$

where k_B, T and m denote Boltzmann constant, temperature of the gas and

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mass of an electron, respectively; cf. [2], Chapter 1.

Let L and a^2 denote length and cross area of the conductor, respectively.

It is natural to model the thermal voltage signal at the ends of the conductor as a trajectory of a weakly stationary stochastic process $(U(t))_{t\in\mathbb{R}}$.

In [1] a formula for the variance $\mathrm{Var}(U(t))$ is obtained; this formula involves an integral whose approximation entails that

$$\mathrm{Var}(U(t)) \approx \varrho\, e^2 s^2 \sigma^2 \cdot \frac{L}{a^2} \qquad (t \in \mathbb{R}) \tag{1.2}$$

holds where ϱ, e and s denote the density of the electronic gas, the charge of an electron and the resistivity of the material, respectively.

Example 1.1. Let us consider a copper wire of length $L = 300$ m and diameter $d = 0.8$ mm at temperature $T = 300$ K. Under the assumption that each copper atom contributes one electron to the gas, electronic density ϱ is given by

$$\varrho = \overline{\varrho} \cdot N_A/m_r = 8.4723 \cdot 10^{28}\ \mathrm{m}^{-3},$$

where $\overline{\varrho} = 8940.0\ \mathrm{kg/m}^3$, $N_A = 6.022 \cdot 10^{26}\ \mathrm{kg}^{-1}$ and $m_r = 63.546$ denote the mass density of copper, the modified Avogadro number and the relative atomic mass, respectively.

It follows from (1.2) for the dispersion of the thermal voltage

$$\big(\mathrm{Var}(U(t))\big)^{1/2} \approx 1.3 \cdot 10^{-3}\ \mathrm{V} \qquad (t \in \mathbb{R}). \tag{1.3}$$

The fact that the theoretical value in (1.3) has also been obtained in measurements, can be interpreted as an empirical confirmation of (1.2).

Remark 1.2. (1.2) suggests that the variance of the thermal voltage can be manipulated by adjusting temperature T of the conductor; cf. (1.1). The main possibility of amplifying the thermal voltage between the ends of a conductor is, however, to increase locally the density ϱ of the electronic gas by exposing the conductor to an electrostatic field, which is discussed in Section 3.

2. Estimation of the Voltage Dispersion

Let us suppose that the thermal voltage signal is modeled by a stochastic process $(U(t))_{t\in\mathbb{R}}$ with almost surely continuous paths. Since the signal can be interpreted as noise, we assume

$$\mathbb{E}(U(t)) = 0 \quad (t \in \mathbb{R}),$$

where $\mathbb{E}$ denotes the expectation.

We additionally assume that the process $(U^2(t))_{t\in\mathbb{R}}$ of squared voltages is weakly stationary, that is the value

$$v := \operatorname{Var}(U(t)) = \mathbb{E}(U^2(t)) \qquad (t \in \mathbb{R}) \tag{2.1}$$

does not depend on time parameter t and the auto-covariance function

$$\gamma(h) := \operatorname{Cov}(U^2(t), U^2(t+h)) \qquad (t, h \in \mathbb{R})$$

is well-defined, bounded and continuous.

With a digital oscilloscope it is possible to sample averages

$$V_j := \frac{1}{\tau} \int\limits_{(j-1)\tau}^{j\tau} U^2(t)\, dt \qquad (j = 1, 2, \ldots)$$

of squared voltage where $\tau > 0$ is an averaging time. Under the described assumptions we have

$$\mathbb{E}(V_j) = \frac{1}{\tau} \int\limits_{(j-1)\tau}^{j\tau} \mathbb{E}(U^2(t))\, dt = v \qquad (j = 1, 2, \ldots). \tag{2.2}$$

A natural estimator of parameter v defined in (2.1) is given by

$$\overline{V}_N := \frac{1}{N} \sum_{j=1}^{N} V_j \qquad (j = 1, 2, \ldots).$$

In view of (2.2) $\overline{V}_N$ is unbiased and we have, moreover, the following consistency result:

Lemma 2.1. *Suppose that the auto-covariance function γ of the weakly stationary process $(U^2(t))_{t\in\mathbb{R}}$ is bounded and continuous and that $\lim_{h\to\infty} \gamma(h) = 0$. Then*

$$\lim_{N\to\infty} \operatorname{Var}\big(\overline{V}_N\big) = 0.$$

In particular, by Chebyshev inequality, $\big(\overline{V}_N\big)$ is weakly consistent:

$$\overline{V}_N \to v \qquad \textit{in probability for } N \to \infty.$$

Proof. Put

$$W(t) := U^2(t) - v \qquad (t \in \mathbb{R}).$$

By Fubini Theorem it follows that

$$\operatorname{Var}(V_j) = \operatorname{Var}(V_1) = \frac{1}{\tau^2} \int\limits_0^{\tau} \int\limits_0^{\tau} \mathbb{E}(W(t)W(s))\, dtds$$

$$= \frac{2}{\tau^2} \int_0^{\tau} (\tau - h) \cdot \gamma(h) \, dh < \infty, \quad (2.3)$$

and that

$$\begin{aligned} \mathrm{Cov}(V_j, V_{j+k}) &= \mathrm{Cov}(V_1, V_{1+k}) \\ &= \frac{1}{\tau^2} \int_0^{\tau} \int_{k\tau}^{(k+1)\tau} \mathbb{E}\big(W(t)W(s)\big) \, ds dt = \frac{1}{\tau} \int_{(k-1)\tau}^{(k+1)\tau} \gamma(h) \, dh \end{aligned}$$

hold for $j, k = 1, 2, \ldots$. Under the assumptions of the lemma we obtain:

$$\lim_{k \to \infty} \mathrm{Cov}(V_1, V_{1+k}) = 0. \quad (2.4)$$

Variance $\mathrm{Var}\big(\overline{V}_N\big)$ has the representation

$$\mathrm{Var}\big(\overline{V}_N\big) = \frac{1}{N^2} \sum_{j=1}^{N} \mathrm{Var}(V_j) \; + \; \frac{2}{N^2} \sum_{k=1}^{N-1} (N - k) \cdot \mathrm{Cov}(V_1, V_{1+k});$$

it follows that the inequality

$$\mathrm{Var}\big(\overline{V}_N\big) \leq \frac{1}{N} \mathrm{Var}(V_1) + \frac{2}{N} \sum_{k=1}^{N-1} |\mathrm{Cov}(V_1, V_{1+k})| \quad (2.5)$$

holds for $N = 1, 2, \ldots$. Combining (2.3), (2.4) and (2.5) proves the assertion. □

Remark 2.2. A weakly consistent estimator of the voltage dispersion $\sqrt{v}$ is given by $\big(\sqrt{\overline{V}_N}\big)$.

3. The Experiment and its Evaluation

When dc voltage u is imposed on a parallel-plate capacitor, then an approximatively homogeneous electrostatic field is induced between the plates.

When a copper rod is placed inside the capacitor orthogonally to the plates, then a charge transfer occurs inside the rod, which means that the electronic gas confined to the rod is compressed in the proximity of the positively charged plate. According to (1.2) the variance of the thermal voltage between appropriate points of the rod should increase.

Figure 1 shows the parallel-plates capacitor with the rod involved in the experiment.

Let us assume that the the thermal voltage between appropriate points

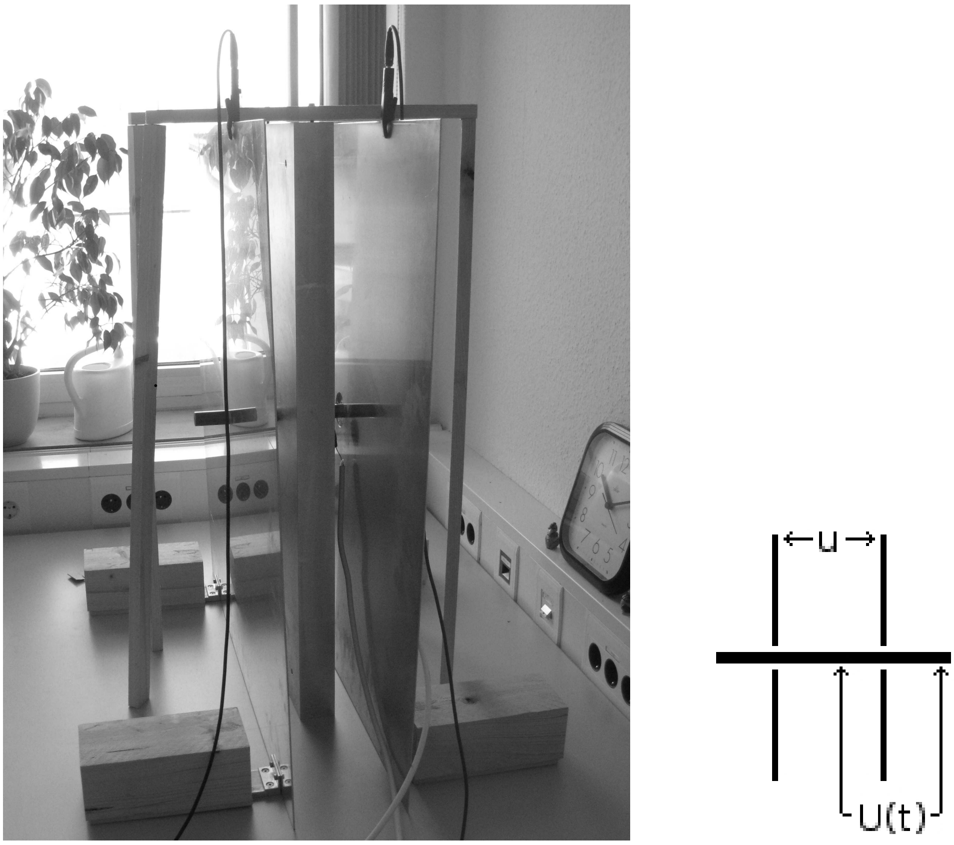

Figure 1: Arrangement of the experiment

of the rod placed orthogonally to the plates of the charged capacitor can be modelled by a stochastic process $(U(t))_{t\in\mathbb{R}}$ fulfilling the conditions formulated in Section 2. A digital oscilloscope measures the averages

$$V_j = \frac{1}{\tau} \int\limits_{(j-1)\tau}^{j\tau} U^2(t)\,dt \qquad (j = 1, 2, \ldots)$$

of the squared thermal voltage where $\tau > 0$ denotes the averaging time. The sampled realizations $v_1, \ldots, v_N$ of the random variables $V_1, \ldots, V_N$ can be evaluated by the weakly consistent estimator $\sqrt{\overline{V}_N}$ yielding an estimate $\widehat{D}$ of the dispersion of the thermal voltage; cf. Remark 2.2.

Table 1 shows the obtained estimates depending on dc voltage u imposed

dc voltage u [kV]	0	1.8	4.5	8.0	10.0
estimate $\widehat{D}$ [mV]	0.5475	0.4411	18.86	31.64	48.85

Table 1

on the capacitor. The presented values suggest that the amplification of the thermal voltage tends to increase with increasing dc voltage u, which complies qualitatively with formula (1.2) because dc voltage u has influence on the extend of the compression of the electronic gas in the rod in the proximity of the positively charged plate.

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