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SWITCHING OF PREDATION ON PREY SPECIES IN THE PRESENCE OF PREDATOR INTERFERENCE-II

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Abstract: The effect of predator interference on a predator-prey system where the predator can switch to the most abundant prey species which lives in two different habitats, is considered. This work extends a previous work, by the authors, by using more general predatory interacting functions which are modified Hassel-Varley type functions. Applying the theory to three hypothetical systems, we find Hopf bifurcation points as well as regions of stability and instability.

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1. Introduction

In a fairly recent paper Skalski and Gillian [6] presented statistical evidence

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that predator interference appears to occur in many predator-prey systems and should be taken into account when treating such systems. They suggested that in a variety of such systems the behaviour of the predator can be better understood if predator interference is allowed to take place. They used as comparison models those of Beddington [1], DeAngelis, Goldstein and O'Neil [4], Crowley and Martin [3], and Hassel and Varley [5].

In a more recent work [2], the present authors, being encouraged by the work of [6], examined the effect of predator interference on the switching of predators on two prey species living in different habitats. They found that contrary to previous work, by them, on systems where the predators can switch to the most abundant prey species without interfering with each other, there were now multiple bifurcation points and also the non-existence of bifurcation points in some systems. In [2] the authors considered a single special case which described a particular system with predator interference similar to that of [5].

In the present work we extend the particular system considered in [2] by using more general interaction functions which possess three general properties. We apply our theory to three systems: one, with exponential predatory functions, where the predators may interact with the two prey species and themselves in a similar manner and the other two, using a combination of both exponential and multiplicative predatory functions, allowing the predators to interact with the two prey species and themselves in different ways. We found several systems with multiple bifurcation points, which were actually Hopf bifurcations, and as in [2] systems which had no bifurcation points.

In Section 2 the equations defining the model are given. In Section 3 the stability of the general equilibrium states is examined while in Section 4 we give some numerical results and figures.

2. The Model

The equations we shall use to define the predator-prey system which allows the predators to interfere with each other while switching to the most abundant prey species are:

$$\begin{aligned}\frac{dx_1}{dt} &= (\alpha_1-\epsilon_1)x_1+\epsilon_2 p_{21}x_2-\beta_1 x_1 y k_1\Big(\frac{x_2+cy}{x_1}\Big),\\ \frac{dx_2}{dt} &= (\alpha_2-\epsilon_2)x_2+\epsilon_1 p_{12}x_1-\beta_2 x_2 y k_2\Big(\frac{x_1+cy}{x_2}\Big),\end{aligned}\qquad (1)$$

$$\frac{dy}{dt} = \left[-\mu + c_1\beta_1 x_1 k_1\Big(\frac{x_2+cy}{x_1}\Big) + c_2\beta_2 x_2 k_2\Big(\frac{x_1+cy}{x_2}\Big)\right] y,$$

where

x_i: represents the prey population in the two different habitats,

y: represents the abundance of predator species,

β_i: the predator response rates towards the prey x_i,

c_i: the rate of conversion of prey to predator,

ϵ_i: inversion barrier strength in going out of the habitat,

p_{ij}: the probability of successful transition from the i-th habitat to the j-th habitat,

α_i: specific growth rate of the prey in the absence of predation,

μ: per capita death rate of the predator, and

$k_1\Big(\frac{x_2+cy}{x_1}\Big)$ and $k_2\Big(\frac{x_1+cy}{x_2}\Big)$ are general functions of $\frac{x_2+cy}{x_1}$ and $\frac{x_1+cy}{x_2}$ respectively.

The forms of these predatory functions are suggested in [1]. These functions are required to satisfy Assumptions 1 to 3, given in the next section. The functions k_i together with the parameters $\alpha_i, \beta_i, \epsilon_i, c_i, p_{ij}, i = 1,2$ and μ are all positive.

3. Stability of the Equilibria

The Equilibrium Equations. Denoting the equilibrium point of equations (1) by (X_1, X_2, Y), $X1 \neq 0$, $X2 \neq 0$, $Y \neq 0$, we can write the equilibrium equations as follows:

$$(\alpha_1 - \epsilon_1)X_1 + \epsilon_2 p_{21} X_2 - \beta_1 X_1 Y k_1\Big(\frac{X_2+cY}{X_1}\Big) = 0, \quad (2)$$

$$(\alpha_2 - \epsilon_2)X_2 + \epsilon_1 p_{12} X_1 - \beta_2 X_2 Y k_2\Big(\frac{X_1+cY}{X_2}\Big) = 0, \quad (3)$$

and

$$\Big(-\mu + c_1\beta_1 X_1 k_1\Big(\frac{X_2+cY}{X_1}\Big) + c_2\beta_2 X_2 k_2\Big(\frac{X_1+cY}{X_2}\Big)\Big) Y = 0. \quad (4)$$

Let $\bar{X} = \dfrac{X_1}{X_2} \neq 0$ and set

$$u_{12} = (\alpha_1 - \epsilon_1)X_1 + \epsilon_2 p_{21} X_2 \equiv M_1(\bar{X}) X_2$$

and

$$v_{12} = (\alpha_2 - \epsilon_2)X_2 + \epsilon_1 p_{12} X_1 \equiv M_2(\bar{X})X_2,$$

where $M_1(\bar{X}) = (\alpha_1 - \epsilon_1)\bar{X} + \epsilon_2 p_{21}$ and $M_2(\bar{X}) = (\alpha_2 - \epsilon_2) + \epsilon_1 p_{12}\bar{X}$, then eqns (2) to (4) can be written as

$$M_1(\bar{X}) - \beta_1 \bar{X} M(\bar{X}) X_2 k_1\Big(\frac{1 + cM(\bar{X})}{\bar{X}}\Big) = 0, \tag{5}$$

$$M_2(\bar{X}) - \beta_2 M(\bar{X}) X_2 k_2\Big(\bar{X} + cM(\bar{X})\Big) = 0, \tag{6}$$

and

$$Y = \frac{1}{\mu}(c_1 M_1(\bar{X}) + c_2 M_2(\bar{X}))X_2 \equiv M(\bar{X})X_2, \tag{7}$$

where

$$M(\bar{X}) = \frac{1}{\mu}(c_1 M_1(\bar{X}) + c_2 M_2(\bar{X}). \tag{8}$$

Equation (7) comes from equation (4) with the help of equations (2)-(3).

From equations (5)-(6) we see that we get two expressions for X_2 namely:

$$X_2 = \frac{M_1(\bar{X})}{\beta_1 \bar{X} M(\bar{X}) k_1\Big(\frac{1 + cM(\bar{X})}{\bar{X}}\Big)} \tag{9}$$

and

$$X_2 = \frac{M_2(\bar{X})}{\beta_2 M(\bar{X}) k_2\big(\bar{X} + cM(\bar{X})\big)}, \tag{10}$$

where $\bar{X}$ and $M(\bar{X})$ are non-zero. Y is obtained from $Y = M(\bar{X})X_2$ and X_1 from $X_1 = \bar{X}X_2$.

Since we have two expressions for X_2 this means that the equilibrium point (X_1, X_2, Y) may not exist. Also, even if it does exist it may not represent real populations since we are not sure that X_1, X_2, Y are positive. For existence we must have $\bar{X}$ satisfying the equation

$$\beta_2 M_1(\bar{X}) k_2(\bar{X} + cM(\bar{X})) = \beta_1 \bar{X} M_2(\bar{X}) k_1\Big(\frac{1 + cM(\bar{X})}{\bar{X}}\Big), \tag{11}$$

where $\bar{X} \neq 0,\ M(\bar{X}) \neq 0$.

Now for X_1, X_2 and Y to represent real populations, they must all be positive and hence we choose

$$M_1(\bar{X}) > 0, \quad M_2(\bar{X}) > 0, \tag{12}$$

thereby making $M(\bar{X}) > 0$ and thus $X_2 > 0$ and $X_1 > 0,\ Y > 0$.

We may use the above information to write the following lemma.

Lemma. *The equilibrium point of equations (1) exists and represents real populations if* $\bar{X}\Big(=\frac{X_1}{X_2}>0\Big)$ *satisfies the equation*

$$\beta_2 M_1(\bar{X})k_2(\bar{X}+cM(\bar{X}))=\beta_1\bar{X}M_2(\bar{X})k_1\Big(\frac{1+cM(\bar{X})}{\bar{X}}\Big), \tag{13}$$

where

$$M_1(\bar{X})>0, \quad M_2(\bar{X})>0. \tag{14}$$

Equation (13) provides the existence of the equilibrium point while equation (14) tells us that the populations are real.

Assumptions and Stability Conditions. Before we say anything about the stability of the equilibrium point (X_1, X_2, Y) we make the following assumptions on the predatory functions, k_i, $i=1,2$:

Assumption 1. Predatory functions, $k_1\Big(\frac{x_2+cy}{x_1}\Big)$, $k_2\Big(\frac{x_1+cy}{x_2}\Big)$ are positive, smooth and possess Taylor expansions about the equilibrium point, (X_1, X_2, Y), of equations (1).

Assumption 2. The predatory functions k_i , $i=1,2$ satisfy the inequalities

$$k_1(S)'<0, \quad k_2(\tilde{S})'<0, \tag{15}$$

where $S=\frac{x_2+cy}{x_1}$, $\tilde{S}=\frac{x_1+cy}{x_2}$ and $k_i(z)'$ is the derivative of $k_i(z)$ with respect to z.

Assumption 3. The predatory functions, k_i, $i\;=1,2$ for x_1, x_2, y all positive, satisfy:

1. For $\frac{x_1}{x_2}$, $\frac{x_1}{y}\gg 1$ i.e. as $\frac{x_1}{x_2}$, $\frac{x_1}{y}\to\infty$, $k_1\to 1$, $k_2\to 0$.
2. for $\frac{x_2}{x_1}$, $\frac{x_2}{y}\gg 1$ i.e. as $\frac{x_2}{x_1}$, $\frac{x_2}{y}\to\infty$, $k_1\to 0$, $k_2\to 1$.

This third assumption is used only when we apply the theory to a particular system. It describes the method the predators use to feed on their prey and each other. The method the predators use in the present work, to feed on their prey, is that of a switching mechanism.

Stability Analysis. To examine the stability of the equilibrium point we linearized equations (1) by allowing a small perturbation about the equilibrium point, that is, we substituted $x_1=X_1+u, x_2=X_2+v, y=Y+w$, where u, v, w are small, into the equations (1) and expanding all terms about the equilibrium point, (X_1, X_2, Y), using Taylor's Theorem while neglecting u^2, v^2, w^2 and higher order terms in u, v, w.

Defining $A, B, C_0, \tilde{C}_0, D, \tilde{D}, \tilde{E}, w_1, w_2$ by

$$\begin{aligned}
A &= c_1\beta_1\big(k_1(S) - Sk_1(S)'\big) + c_2\beta_2 k_2(\tilde{S}),\\
B &= c_1\beta_1 k_1(S)' + c_2\beta_2\big(k_2(\tilde{S}) - \tilde{S}k_2(\tilde{S})'\big),\\
C_0 &= \epsilon_1 p_{12} - \beta_2 Y k_2(\tilde{S})',\\
\tilde{C}_0 &= -\bar{X}C_0 + c\beta_2\frac{Y^2}{X_2}k_2(\tilde{S})',\\
D &= \epsilon_2 p_{21} - \beta_1 Y k_1(S)',\\
\tilde{D} &= -\frac{D}{\bar{X}} + c\beta_1\frac{Y^2}{X_1}k_1(S)',\\
\tilde{E} &= cY\big(c_1\beta_1 k_1(S)' + c_2\beta_2 k_2(\tilde{S})'\big),\\
w_1 &= X_1 k_1(S) + cYk_1(S)',\\
w_2 &= X_2 k_2(\tilde{S}) + cYk_2(\tilde{S})',
\end{aligned} \tag{16}$$

where $S = \frac{1+cM(\bar{X})}{\bar{X}}$, $\tilde{S} = \bar{X} + cM(\bar{X})$ and $k_1(S)'$, $k_2(\tilde{S})'$ are the derivatives of $k_1(S)$, $k_2(\tilde{S})$, with respect to S, $\tilde{S}$ respectively, we are able to write the linearized form of equations (1) as

$$\frac{dV}{dt} = JV,$$

where $V = \begin{pmatrix} u \\ v \\ w \end{pmatrix}$ and J is a 3×3 matrix.

With the aid of equation (16), the characteristic equation can be written as

$$|J - \lambda| = \begin{vmatrix} \tilde{D} - \lambda & D & -\beta_1 w_1 \\ C_0 & \tilde{C}_0 - \lambda & -\beta_2 w_2 \\ AY & BY & \tilde{E} - \lambda \end{vmatrix} = 0. \tag{17}$$

This equation has the form

$$\lambda^3 + b_1\lambda^2 + b_2\lambda + b_3 = 0, \tag{18}$$

where

$$\begin{aligned}
b_1 &= -(\tilde{C}_0 + \tilde{D} + \tilde{E}),\\
b_2 &= \tilde{D}\tilde{E} + \tilde{D}\tilde{C}_0 + \beta_1 AYw_1 + \beta_2 BYw_2 + \tilde{C}_0\tilde{E} - C_0 D,\\
b_3 &= -\tilde{C}_0\tilde{D}\tilde{E} + \beta_2 ADYw_2 + \beta_1 BC_0 Yw_1\\
&\quad - \beta_1 A\tilde{C}_0 Yw_1 - \beta_2 B\tilde{D}Yw_2 + C_0 D\tilde{E}.
\end{aligned} \tag{19}$$

Now the eigenvalue solutions, λ, must have negative real parts in order that the equilibrium points be stable. Conditions for this to hold are provided by the Routh-Hurwitz criteria. These criteria say that the eigenvalues will have negative real parts if and only if

$$b_1 > 0, \quad b_3 > 0 \quad \text{and} \quad b_1 b_2 - b_3 > 0.$$

However, from equation (16) and Assumption 2 we see that $b_1 > 0$, hence we have stability of the equilibrium point if and only if

$$b_3 > 0 \quad \text{and} \quad b_1 b_2 - b_3 > 0.$$

With the help of the above we can write the following theorem.

Theorem 1. *If the predatory functions, $k_i, i = 1, 2$, satisfy Assumptions 1 and 2, the conditions of the lemma hold and $A, B, C_0, \tilde{C}_0, D, \tilde{D}, \tilde{E}, w_1, w_2$ are defined by equations (16), then the stability of the equilibrium point, (X_1, X_2, Y), of equations (1) , is assured if and only if*

$$b_3 > 0 \quad \textit{and} \quad b_1 b_2 - b_3 > 0,$$

where b_1, b_2 and b_3 are given by equations (19).

4. Applications and Numerical Results

In [2] the authors looked at the special case where the predatory functions were given by

$$k_1 = \frac{1}{1 + \left(\dfrac{x_2 + cy}{x_1}\right)^n}\ , \quad k_2 = \frac{1}{1 + \left(\dfrac{x_1 + cy}{x_2}\right)^n}, \quad n = 1, 2, \ldots .$$

As an application of the present more general case we examined an exponential form for both predatory functions, k_1, k_2, as well as the case where $k_1, (k_2)$ may be exponential and $k_2, (k_1)$ multiplicative, respectively. We considered the following systems with $n = 1, 2, \ldots$:

System-a. $k_1 = e^{-\left(\frac{x_2 + cy}{x_1}\right)^n}$, $k_2 = e^{-\left(\frac{x_1 + cy}{x_2}\right)^n}$,

System-b. $k_1 = \dfrac{1}{1 + \left(\frac{x_2 + cy}{x_1}\right)^n}$, $k_2 = e^{-\left(\frac{x_1 + cy}{x_2}\right)^n}$ and

System-c. $k_1 = e^{-\left(\frac{x_2 + cy}{x_1}\right)^n}$, $k_2 = \dfrac{1}{1 + \left(\frac{x_1 + cy}{x_2}\right)^n}$.

Numerical Results. In our numerical investigations we determined the regions of stability/instability of systems defined by three pairs of predatory

n	c	STABLE	UNSTABLE	Bifurcation Point
		$\beta_1 = 0.01, \beta_2 = 0.02,$	$c_2 = 0.03$	
1	0.0316	$0 \leq c_1 \leq 0.059322$	$c_1 > 0.059322$	0.059322
1	0.2927	$0 \leq c_1 \leq 0.064497$	$c_1 > 0.064497$	0.064497
1	0.7296	$0 \leq c_1 \leq 0.076092$	$c_1 > 0.076092$	0.076092
		$c_1 \geq 0.747482$	$c_1 < 0.747482$ &	0.747482 (i)
2	0.0316	$0 \leq c_1 \leq 0.059088$	$c_1 > 0.059088$	0.059088
2	0.2927	$0 \leq c_1 \leq 0.067687$	$c_1 > 0.067687$	0.067687
2	0.7296	$0 \leq c_1 \leq 0.094020$	$c_1 > 0.094020$ &	0.094020 &
		$c_1 \geq 0.360922$	$c_1 < 0.360922$	0.360922 (ii)
		$\beta_1 = 0.02, \beta_2 = 0.01,$	$c_1 = 0.03$	
1	0.0316	$0 \leq c_2 \leq 0.059322$	$c_2 > 0.059322$	0.059322
1	0.2927	$0 \leq c_2 \leq 0.064497$	$c_2 > 0.064497$	0.064497
1	0.7296	$0 \leq c_2 \leq 0.076092$	$c_2 > 0.076092$ &	0.076092
		$c_2 \geq 0.747482$	$c_2 < 0.747482$	0.747482 (iii)
2	0.0316	$0 \leq c_2 \leq 0.059088$	$c_2 > 0.059088$	0.059088
2	0.2927	$0 \leq c_2 \leq 0.067687$	$c_2 > 0.067687$	0.067687
2	0.7296	$0 \leq c_2 \leq 0.094020$	$c_2 > 0.094020$ &	0.094020 &
		$c_2 \geq 0.360922$	$c_2 < 0.360922$	0.360922 (iv)

Table 1: k_1-expo, k_2-expo

functions. In these systems we used different β_1, β_2 and p_{12}, p_{21}. Actually, we took $\beta_1 > \beta_2$, and $\beta_1 < \beta_2$ and $p_{12} > p_{21}$ and $p_{12} < p_{21}$ with $n = 1,\ 2$ and c_1 or c_2 as the varying parameter.

System-a. We report on the results when both the predatory functions, k_1 and k_2 are of exponential form. Both cases, $\beta_1 < \beta_2$ ($\beta_1 = 0.01, \beta_2 = 0.02$) and $\beta_1 > \beta_2$ ($\beta_1 = 0.02, \beta_2 = 0.01$) are considered with $n = 1,\ 2$. The values of the other parameters used are: $\mu = 0.01,\ \alpha_1 = 0.015,\ \alpha_2 = 0.025,\ \epsilon_1 = 0.02,\ \epsilon_2 = 0.03,\ p_{12} = 0.3,\ p_{21} = 0.2$. We shall refer to these data as Set(a). We found four instances, labeled (i) to (iv) in Table 1, which were multiple bifurcation points.

The bifurcations at (i) to (iv), in Table 1, are all Hopf bifurcations from which stable orbits emanate. In determining the types of Hopf bifurcations we used *XPPAUT*, see [7].

All of the other cases examined, twelve in all, showed single bifurcation points. Actually, these other cases had:

	ϵ_1	ϵ_2	p_{12}	p_{21}	α_1	α_2
SET(1)	0.04	0.03	0.5	0.7	0.015	0.025
SET(2)	0.1	0.3	0.2	0.7	0.05	0.25
SET(3)	0.1	0.3	0.5	0.3	0.05	0.25

Table 2

n	c	STABLE	UNSTABLE	Bifurcation Point
		SET(3), $c_1 = 0.03$		
1	0.0316	$0.006535 \leq c_2 \leq 1$	$c_2 < 0.006535$	0.006535
1	0.2927	$0 \leq c_2 \leq 1$	NO WHERE	NONE
1	0.7296	$0 \leq c_2 \leq 1$	NO WHERE	NONE
2	0.0316	$0 \leq c_2 \leq 0.041828$ &	$c_2 > 0.041828$ &	0.041828
		$0.335056 \leq c_2 \leq 1$	$c_2 < 0.335056$	0.335056 **(v)**
2	0.2927	$0 \leq c_2 \leq 1$	NO WHERE	NONE
2	0.7296	$0 \leq c_2 \leq 1$	NO WHERE	NONE

Table 3

1. $\beta_1 = 0.01,\ \beta_2 = 0.02,\ c_1 = 0.03,\ n = 1,\ 2$ and $c = 0.0316,\ 0.2927,\ 0.7296$ with the other parameters as in Set(a). The parameter c_2 was allowed to vary. There were six cases here. We found only single bifurcation points.

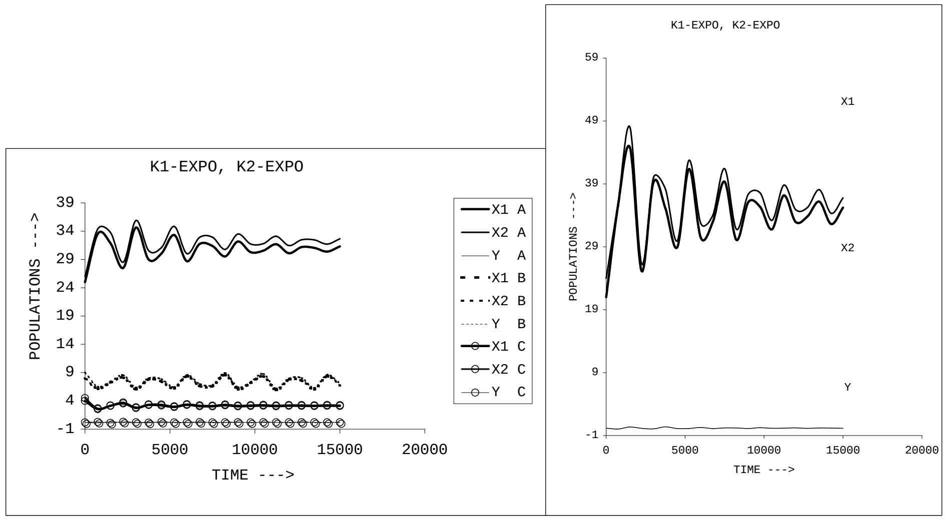
2. $\beta_1 = 0.02,\ \beta_2 = 0.01,\ c_2 = 0.03,\ n = 1,\ 2$ and $c = 0.0316,\ 0.2927,\ 0.7296$ with the other parameters as in Set(a). The parameter c_1 was allowed to vary. There were six cases here. We found only single bifurcation points.

We also considered cases where $p_{12} > p_{21}$ and $p_{12} < p_{21}$ and as above, we give the results for only two values of $p_{12},\ p_{21}$. We used $\mu = 0.01,\ \beta_1 = 0.01,\ \beta_2 = 0.02$ and $c = 0.0316,\ 0.2927,\ 0.7296,\ n = 1,\ 2$ together with the parameter values as SET(1), SET(2) and SET(3) in Table 2.

In the cases we examined we found one system with a multiple bifurcation point. This is shown in Table 3 and is labeled (v).

The bifurcations at (v) are Hopf bifurcations from which stable orbits emanate.

We notice from Table 3 that there are systems which are stable everywhere. Some of the other systems we investigated were found to be unstable everywhere.



(a) (b)

Figure 1

In further investigations we used the following data set: $\epsilon_1 = 0.02,\ \epsilon_2 = 0.03,\ p_{12} = 0.3,\ p_{21} = 0.2,\ \alpha_1 = 0.015,\ \alpha_2 = 0.025,\ \mu = 0.01,\ c = 0.0316$ and we shall refer to these data as Set(b) in what follows.

We support our numerical work by giving two figures, Figure 1(a) and Figure 1(b).

Figure 1 (a) corresponds to the case (iii) in Table 1 and the data used for Figure 1 (b) is Set(b) and $\beta_1 = 0.01, \beta_2 = 0.02, c_1 = 0.02, c_2 = 0.03$ and $n = 1$.

In the above figure and all of the following multi-bifurcation figures the letters A, B and C in the legends, refer to the left stable region, the center unstable region and the right stable region, respectively.

System-b. We next report on the results when k_1 is in multiplicative and k_2 is in exponential form. The same data set as for System-a, Set(a), is used.

We found two instances labeled (i), (ii) in Table 4, where there were multiple bifurcation points.

The bifurcations at (i), (ii) in Table 4 are Hopf bifurcations from which stable orbits emanate.

1. $\beta_1 = 0.01,\ \beta_2 = 0.02,\ c_1 = 0.03,\ n = 1,\ 2$ and $c = 0.0316, 0.2927, 0.7296$ with the other parameters as in Set(a). Here we allowed c_2 to vary. These were

n	c	STABLE	UNSTABLE	Bifurcation Point
		$\beta_1 = 0.01, \beta_2 = 0.02,$	$c_2 = 0.03$	
1	0.0316	$0 \leq c_1 \leq 0.068975$	$c_1 > 0.068975$	0.068975
1	0.2927	$0 \leq c_1 \leq 0.080022$	$c_1 > 0.080022$	0.080022
1	0.7296	$0 \leq c_1 \leq 0.118813$ &	$c_1 > 0.118813$ &	0.118813 &
		$c_1 \geq 0.343295$	$c_1 < 0.343295$	0.343295 **(i)**
2	0.0316	$0 \leq c_1 \leq 0.077297$	$c_1 > 0.077297$	0.077297
2	0.2927	$0 \leq c_1 \leq 0.103301$ &	$c_1 > 0.103301$ &	0.103301 &
		$c_1 \geq 0.600604$	$c_1 < 0.600604$	0.600604 **(ii)**
2	0.7296	$0 \leq c_1 \leq 1$	NO WHERE	NONE

Table 4: k_1-mult, k_2-expo

six cases here. We found only single bifurcation points.

2. $\beta_1 = 0.02,\ \beta_2 = 0.01,\ c_2 = 0.03,\ n = 1,\ 2$ and $c = 0.0316, 0.2927, 0.7296$ with the other parameters as in Set(a). Here we allow c_1 to vary. There were six case here. We found only single bifurcation points.

3. $\beta_1 = 0.02, \beta_2 = 0.01, c_1 = 0.03, n = 1, 2,$ and $c = 0.0316, 0.2927, 0.7296$ with the other parameters as in Set (a). Here we allowed c_2 to vary. There were six cases here. We found only single bifurcation points.

Again we examined cases with different p_{ij}, with $p_{12} < p_{21}$ and $p_{12} > p_{21}$, and we give the results for only two values of $p_{12},\ p_{21}$. We used $\mu = 0.01,\ \beta_1 = 0.01,\ \beta_2 = 0.02,\ c = 0.0316, 0.2927, 0.7196,\ n = 1,\ 2$ together with data SET(1), SET(2) and SET(3). In the cases we examined, we found three systems with multiple bifurcation points, labeled (iii), (iv) and (v) in Table 5.

The bifurcations at (iii), (v) are Hopf bifurcations from which stable orbits emanate. The points 0.164213 and 0.589995 of (iv) are Hopf bifurcations from which unstable and stable orbits emanate, respectively.

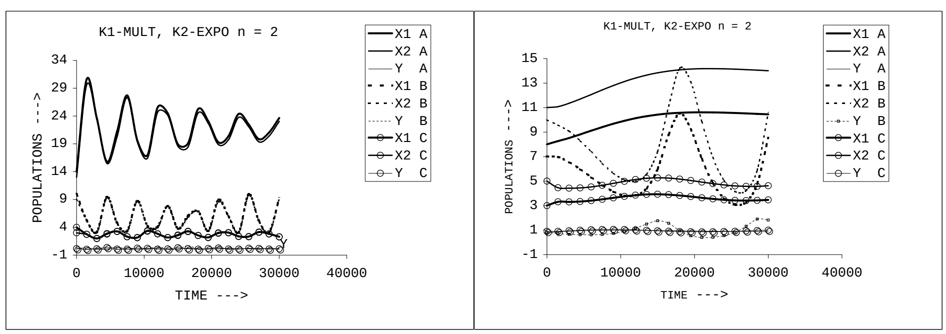
We support our numerical work by giving two figures, Figure 2(a) and Figure 2(b).

These figures correspond to cases (ii) in Table 4 and (iv) in Table 5.

System-c. Our third set of numerical investigations is when k_1 is in exponential form and k_2 is in multiplicative form. Again we used the Set(a) values and $\beta_1 < \beta_2$ and $\beta_1 > \beta_2$ as well as $p_{12} < p_{21}$ and $p_{12} > p_{21}$. We give the results for only two values of β_1, β_2 as before. We found six cases with multiple

n	c	STABLE	UNSTABLE	Bifurcation Point
		SET(3), $c_1 = 0.03$		
1	0.0316	$0 \leq c_2 \leq 1$	NO WHERE	NONE
1	0.2927	$0 \leq c_2 \leq 1$	NO WHERE	NONE
1	0.7296	$0 \leq c_2 \leq 1$	NO WHERE	NONE
2	0.0316	$0 \leq c_2 \leq 0.027130$ & $0.668201 \leq c_2 \leq 1$	$c_2 > 0.027130$ & $c_2 < 0.668201$	0.027130 & 0.668201 **(iii)**
2	0.2927	$0 \leq c_2 \leq 1$	NO WHERE	NONE
2	0.7296	$0 \leq c_2 \leq 1$	NO WHERE	NONE
		SET(1), $c_2 = 0.03$		
1	0.0316	$0 \leq c_1 \leq 0.096294$	$c_1 > 0.096294$	0.096294
1	0.2927	$0 \leq c_1 \leq 0.164213$ & $c_1 \geq 0.589995$	$c_1 > 0.164213$ & $c_1 < 0.589995$	0.164213 & 0.589995 **(iv)**
1	0.7296	$0 \leq c_1 \leq 1$	NO WHERE	NONE
2	0.0316	$0 \leq c_1 \leq 0.114493$	$c_1 > 0.114493$	0.114493
2	0.2927	$0 \leq c_1 \leq 0.198586$ & $c_1 \geq 0.675426$	$c_1 > 0.198586$ & $c_1 < 0.675426$	0.198586 & 0.675426 **(v)**
2	0.7296	$0 \leq c_1 \leq 1$	NO WHERE	NONE

Table 5: k_1-mult, k_2-expo



(a) (b)

Figure 2

bifurcation points. These are labeled (i) to (vi) in Table 6.

The bifurcation points at (**i**), (**ii**) and (**iv**) are Hopf bifurcation points

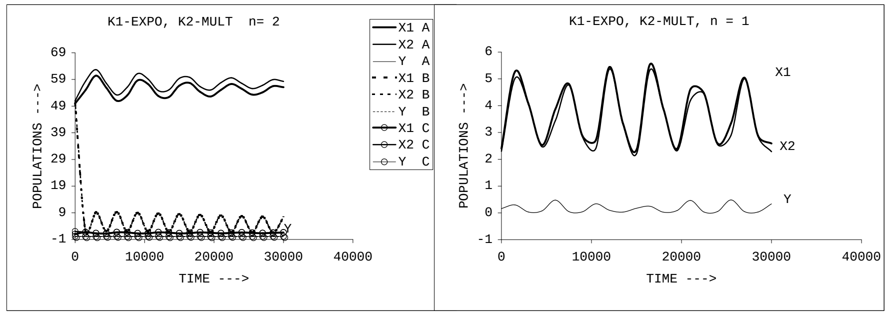
n	c	STABLE	UNSTABLE	Bifurcation Point
		$\beta_1 = 0.01, \beta_2 = 0.02,$	$c_1 = 0.03$	
1	0.0316	$0 \leq c_2 \leq 1$	NO WHERE	NONE
1	0.2927	$0 \leq c_2 \leq 1$	NO WHERE	NONE
1	0.7296	$0 \leq c_2 \leq 1$	NO WHERE	NONE
2	0.0316	$0 \leq c_2 \leq 0.049429$	$c_2 > 0.049429$	0.049429
2	0.2927	$0 \leq c_2 \leq 0.054471$ &	$c_2 > 0.054471$ &	0.054471 &
		$c_2 \geq 0.798504$	$c_2 < 0.798504$	0.798504 **(i)**
2	0.7296	$0 \leq c_2 \leq 0.069179$ &	$c_2 > 0.069179$ &	0.069179 &
		$c_2 \geq 0.254597$	$c_2 < 0.254597$	0.254597 **(ii)**
		$\beta_1 = 0.02, \beta_2 = 0.01,$	$c_2 = 0.03$	
1	0.0316	NO WHERE	$0 \leq c_1 \leq 1$	NONE
1	0.2927	$0 \leq c_1 \leq 0.000147$ &	$c_1 > 0.000147$ &	0.000147 &
		$c_1 \geq 0.448227$	$c_1 < 0.448227$	0.448227 **(iii)**
1	0.7296	$0 \leq c_1 \leq 0.000960$ &	$c_1 > 0.000960$ &	0.000960 &
		$c_1 \geq 0.160453$	$c_1 < 0.160453$	0.160453 **(iv)**
2	0.0316	$0 \leq c_1 \leq 0.006566$	$c_1 > 0.006566$	0.006566
2	0.2927	$0 \leq c_1 \leq 0.007000$ &	$c_1 > 0.007000$ &	0.007000 &
		$c_1 \geq 0.815574$	$c_1 < 0.815574$	0.815574 **(v)**
2	0.7296	$0 \leq c_1 \leq 0.007826$ &	$c_1 > 0.007826$ &	0.007826 &
		$c_1 \geq 0.305297$	$c_1 < 0.305297$	0.305297 **(vi)**

Table 6: k_1-expo, k_2-mult

from which stable orbits emanate. At (**iii**) the point 0.000147 is a Hopf bifurcation from which stable orbits emanate, while the point 0.448227, also a Hopf bifurcation, has unstable orbits emanating from it. At (**v**) both points are Hopf bifurcations but 0.007000 has stable orbits emanating while 0.815574 has unstable orbits emanating from it. At (**vi**) both points are Hopf bifurcations but 0.007826 has stable orbits emanating while 0.305297 has unstable orbits emanating from it.

For all of the other cases we investigated, we found either only one bifurcation point or stability everywhere. These other cases were

1. $\beta_1 = 0.01,\ \beta_2 = 0.02,\ c_2 = 0.03,\ n = 1,\ 2$ and $c = 0.0316, 0.2927, 0.7296$ and the other parameters as in Set(a). The parameter c_1 was allowed to vary. There were six cases here. We found either single bifurcation points or complete



(a) (b)

Figure 3

stability.

2. $\beta_1 = 0.02,\ \beta_2 = 0.01,\ c_1 = 0.03,\ n = 1,\ 2$ and $c = 0.0316, 0.2927, 0.7296$ and the other parameters as in Set(a). The parameter c_2 was allowed to vary. There were six cases here. We found only single bifurcation points.

In further investigations with $p_{12} < p_{21}$ and $p_{12} > p_{21}$ we used: $\mu = 0.01,\ \beta_1 = 0.01,\ \beta_2 = 0.02,\ c_1 = 0.03,\ c = 0.0316, 0.2927, 0.7296$ and the other parameters from SET(1) to SET(3) in Table 2. We found no multiple bifurcation points, only single bifurcation points, one system unstable everywhere and the rest of systems stable everywhere.

We support our numerical work by giving two figures, Figure 3(a) and Figure 3(b).

Figure 3(a) corresponds to the case (i) in Table 6. For Figure 3(b) Set(b) values and $\beta_1 = 0.02, \beta_2 = 0.01, c_1 = 0.03, c_2 = 0.50, n = 1$ were used.

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