International Journal of Pure and Applied Mathematics

Volume 61 No. 3 2010, 297-300

A NOTE ON FIXED POINTS OF ITERATIONS OF REAL-VALUED FUNCTIONS

Talitha M. Washington

Department of Mathematics
University of Evansville
1800, Lincoln Avenue, Evansville, IN 47722, USA
e-mail: tw65@evansville.edu

Abstract: In this paper we consider the fixed points of both real- and complex-valued continuous functions. We provide clarification of a result by Mohammad K. Azarian.

AMS Subject Classification: 65-01
Key Words: fixed point, fixed point free, continuous function, composition of functions

1. Introduction

We say that a function f has a *fixed point* x_p if $f(x_p) = x_p$ (see [2]). Similarly, we say that a function is *fixed point free* if it has no fixed points. In [1], Mohammad K. Azarian studies the fixed points of a function, $f(x)$. In this paper, we considers how the fixed points of f are related to the fixed point of the iterates

$$f^n(x) = \underbrace{f(f(\dots f(x)\dots))}_{n\times},$$

where $n \geq 1$. The main result of [1] is the following:

Received: March 11, 2010

Proposition 2.1. *(i) If x_p is a fixed point of the function $f(x)$, then x_p is a fixed point of the composite function $f^n(x)$ $(n \geq 2)$, and*

(ii) if $f(x)$ is fixed point free, then $f^n(x)$ $(n \geq 2)$ is fixed point free as well.

The purpose of this exposition is to discuss the validity of the statement (ii) above. The issue is that there is no asserted domain for f. The main result of this paper is as follows:

Proposition 1. *(i) For a continuous function $f : \mathbb{R} \to \mathbb{R}$, $f(x)$ is fixed point free if and only if $f^n(x)(n \geq 2)$ is fixed point free as well.*

(ii) There exists a continuous function $f : \mathbb{C} \to \mathbb{C}$ such that $f(x)$ is fixed point free, yet there are infinitely many iterates $f^n(x)$ with each have infinitely many fixed points x_p.

2. Iterates of Real-Valued Functions

Proof of Proposition 1(i)

We shall prove the following:

(i) For a continuous function $f : \mathbb{R} \to \mathbb{R}$, $f(x)$ is fixed point free if and only if $f^n(x)(n \geq 2)$ is fixed point free as well.

Proof. First assume that each iterate $f^n(x)$ is fixed point free for all $n \geq 2$. We show that $f(x)$ is fixed point free by contradiction, so assume that $f(x)$ has a fixed point x_p. Then $f^2(x_p) = f\big(f(x_p)\big) = f(x_p) = x_p$, so that x_p is a fixed point for $f^n(x)$ for $n = 2$. In fact, Proposition 2.1 (i) of [1] shows that x_p is a fixed point $f^n(x)$ for all $n \geq 2$. This is clearly a contradiction.

Conversely, assume that $f(x)$ is fixed point free. We show that $f^n(x)$ is fixed point free for all $n \geq 2$. Consider the function $F : \mathbb{R} \to \mathbb{R}$ defined by $F(x) = f(x) - x$. Since f is continuous and fixed point free, F is continuous and nonzero. Hence F has the same sign for all real numbers x, say positive. We have the telescoping sum

$$f^n(x) - x = f^n(x) - f^0(x) = \sum_{k=0}^{n-1} \Big[f^{k+1}(x) - f^k(x) \Big] = \sum_{k=0}^{n-1} F(f^k(x)).$$

As this is a sum of positive numbers, say, it itself must be strictly positive. This shows that there cannot exist x_p such that $f^n(x_p) - x_p = 0$. $\square$

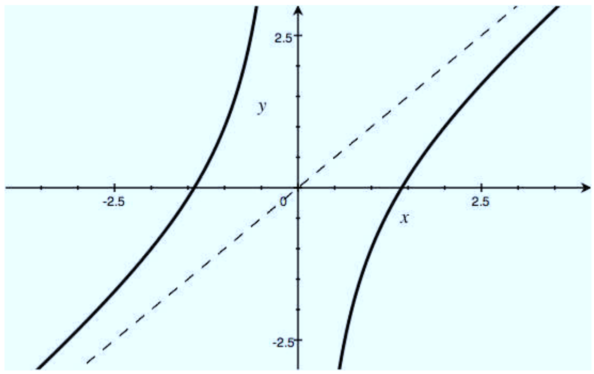

Figure 1: Graph of $f(x) = x - \frac{2}{x}$

Counterexample

The continuity of $\mathbb{R} \to \mathbb{R}$ seems necessary indeed. Consider the function

$$f(x) = x - \frac{2}{x} \qquad \Longrightarrow \qquad f\big(f(x)\big) = x - \frac{4\,(x^2-1)}{x^3 - 2\,x}.$$

Hence $f(x)$ has no fixed points yet $f^2(x)$ has two real-valued fixed points $x_p = \pm 1$. We shall use mathematical induction to show that

$$f^n(x_p) = \begin{cases} \pm 1 & \text{if } n \text{ is even,} \\ \mp 1 & \text{if } n \text{ is odd.} \end{cases}$$

We have already shown this to be true for $n = 1$ and $n = 2$. Assume that f^n is true. We show that the case f^{n+2} is true. Now $f^{n+2}(x) = (f^n \circ f^2)(x)$ and since $f^2(x_p) = x_p$, we have $f^{n+2}(x_p) = f^n(x_p)$. Note that n is odd (even, respectively) if and only if $n + 2$ is odd (even, respectively).

In particular, $f^n(x_p) = x_p$ for all nonnegative even integers n. However, we have to be a little careful with this function because the domain of definition of the iterates f^n is unclear. Note that $f\left(\sqrt{2}\right) = 0$ but $f^2\left(\sqrt{2}\right)$ is undefined.

3. Iterates of a Complex-Valued Function

In this section, we show the following:

(ii) There exists a continuous function $f : \mathbb{C} \to \mathbb{C}$ such that $f(x)$ is fixed point free, yet there are infinitely many iterates $f^n(x)$ with each have infinitely many fixed points x_p.

Proof. Consider the complex-valued function as well as the complex number

$$f(x) = x + \pi i e^{x} \qquad \text{and} \qquad x_p = 2\pi i p$$

for any integer p. Then

$$f^1(x_p) = f(x_p) = x_p + \pi i e^{x_p} = 2\pi i p + \pi i e^{2\pi i p} = \pi i\big(2p+1\big)$$

$$\implies f^2(x_p) = f\big(f(x_p)\big) = f(x_p) + \pi i e^{f(x_p)} = \pi i\big(2p+1\big) + \pi i e^{\pi i(2p+1)}$$
$$= \pi i\big(2p\big) = x_p.$$

We shall use mathematical induction to show that

$$f^n(x_p) = \pi i \cdot \begin{cases} 2p+1, & \text{if } n \text{ is odd,} \\ 2p & \text{if } n \text{ is even.} \end{cases}$$

We have already shown this to be true for $n = 1$ and $n = 2$. Assume that f^n is true. We show that the case f^{n+2} is true. Now $f^{n+2}(x) = (f^n \circ f^2)(x)$ and since $f^2(x_p) = x_p$, we have $f^{n+2}(x_p) = f^n(x_p)$ and n is odd (even, respectively) if and only if $n+2$ is odd (even, respectively).

In particular, $f^n(x_p) = x_p$ for all nonnegative even integers n. □

4. Moral of the Story

If a function f is fixed point free, f^n is fixed point free if f is a continuous, real-valued function. However, this does not necessarily hold true if the function f is not continuous or is a complex-valued function. That is, when one utilizes complex numbers, sometimes properties of functions can get a bit more *complex*.

Acknowledgments

The author would like to thank Mohammad K. Azarian for inspiration with this project.

References

[1] Mohammad K. Azarian, On the fixed points of a function and the fixed points of its composite functions, *International Journal of Pure and Applied Mathematics*, **46**, No. 1 (2008), 37-44.

[2] V. Hutson, J.S. Pym, *Applications of Functional Analysis and Operator Theory*, Academic Press, London (1980).