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A GENERALIZATION OF KNAR'S FORMULA

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Abstract: We produce two derivations for an infinite double product functional equation for the Gamma function, a generalization of Knar's formula. Series are derived for the Polygamma function using the double product, as well as certain symmetric product identities.

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The Gamma function

$$\Gamma(z) = \int_0^{+\infty} x^{z-1} e^{-x} \mathrm{d}x$$

has many identities – we cite the relevant literature (see [2], [1]) for its study and development. Knar's formula (see [2]), a seemingly little-known product formula, is one of them:

$$\Gamma(1+z) = 2^{2z} \prod_{n=1}^{\infty} \frac{1}{\sqrt{\pi}} \Gamma\left(\frac{1}{2} + \frac{z}{2^n}\right). \tag{1}$$

Using the well-known value $\Gamma(1/2) = \sqrt{\pi}$, letting $z = 1$ and taking a square root throughout the given expression yields the highly symmetric equation

$$\prod_{n=1}^{\infty} \frac{1}{\Gamma^{1/2}(\frac{1}{2})} \Gamma^{1/2}\left(\frac{1}{2} + \frac{1}{2^n}\right) = \frac{1}{2}. \tag{2}$$

The purpose of this paper is to generalize (1) and derive results analogous to

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the above. We will use the following formulas:

$$\Gamma(z) = \lim_{k\to\infty} \frac{k!k^z}{(z+1)\cdots(z+k)}, \tag{3}$$

$$(2\pi)^{(n-1)/2} n^{1/2-nz}\Gamma(nz) = \prod_{k=0}^{n-1} \Gamma\left(z+\frac{k}{n}\right) \qquad (n\in\mathbb{Z}^+), \tag{4}$$

which are known as Euler's limit form and the Gauss multiplication formula, respectively, both shown and proven in [1].

Our first derivation of the titular identity comes from considering the finite product

$$\prod_{i=1}^{p} \frac{n\Gamma(n^{1-i}z)}{\Gamma(n^{-i}z)} = n^p \cdot \frac{\Gamma(z)}{\Gamma(z/n)} \cdot \frac{\Gamma(z/n)}{\Gamma(z/n^2)} \cdots \frac{\Gamma(n^{1-p}z)}{\Gamma(z/n^p)} = \frac{n^p}{\Gamma(z/n^p)}\Gamma(z). \tag{5}$$

From (3), $n^p/\Gamma(z/n^p)$ may be given by

$$\frac{n^p}{\Gamma(z/n^p)} = \left(\lim_{k\to\infty} \frac{k!k^{n^{-p}z}}{z(n^{-p}z+1)\cdots(n^{-p}z+k)}\right)^{-1};$$

and as $p\to\infty$ we see easily that $n^p/\Gamma(z/n^p)\to z$. Substituting the above into (5),

$$\prod_{i=1}^{\infty} \frac{n\Gamma(n^{1-i}z)}{\Gamma(n^{-i}z)} = z\Gamma(z) = \Gamma(1+z). \tag{6}$$

Now, the Gauss multiplication formula (4) can be used to give (8) by taking out the $k=0$ term to give the $\prod_{k=1}^{n-1}$ term; solving for $\Gamma(nz)/\Gamma(z)$, multiplying through by n and letting $z\to n^{-i}z$:

$$\frac{n\Gamma(n^{1-i}z)}{\Gamma(n^{-i}z)} = \frac{n^{n^{1-i}z}\sqrt{n}}{(2\pi)^{(n-1)/2}} \prod_{k=1}^{n-1} \Gamma\left(\frac{k}{n}+\frac{z}{n^i}\right).$$

Substituting this into (6) yields

$$\prod_{i=1}^{\infty}\left[\frac{\sqrt{n}}{(2\pi)^{(n-1)/2}} \prod_{k=1}^{n-1} \Gamma\left(\frac{k}{n}+\frac{z}{n^i}\right)\right] = \Gamma(1+z)\prod_{i=1}^{\infty} n^{-n^{1-i}z}. \tag{7}$$

But the product on the right-hand side of the above is just the exponential form of a geometric series, i.e.,

$$\prod_{i=1}^{\infty} n^{-n^{1-i}z} = \exp\left[-z\log(n)\sum_{i=0}^{\infty}\frac{1}{n^i}\right] = n^{-nz/(n-1)}$$

which gives

$$\prod_{i=1}^{\infty}\left[\frac{\sqrt{n}}{(2\pi)^{(n-1)/2}}\prod_{k=1}^{n-1}\Gamma\left(\frac{k}{n}+\frac{z}{n^i}\right)\right]=n^{-nz/(n-1)}\Gamma(1+z)$$

when substituted into (7). Note that by the Gauss multiplication formula

$$\frac{\sqrt{n}}{(2\pi)^{(n-1)/2}}=\prod_{k=1}^{n-1}\frac{1}{\Gamma(k/n)},$$

which gives the desired result

$$\prod_{i=1}^{\infty}\prod_{k=1}^{n-1}\frac{1}{\Gamma(k/n)}\Gamma\left(\frac{k}{n}+\frac{z}{n^i}\right)=n^{-nz/(n-1)}\Gamma(1+z)\quad (n\in\mathbb{Z}^+,\ z\in\mathbb{C}\backslash\mathbb{Z}^-). \tag{8}$$

A second derivation involves continually resubstituting directly from the Gauss multiplication formula (4). As before, we take out the $k=0$ term of the formula to give the slightly modified version

$$(2\pi)^{(n-1)/2}n^{1/2-nz}\Gamma(nz)=\Gamma(z)\prod_{k=1}^{n-1}\Gamma\left(z+\frac{k}{n}\right);$$

first we solve for $\Gamma(z)$ and, let $z\to z/n$, and then solve back for $\Gamma(z)$ on the left. This yields the two formulas

$$\Gamma(z)=(2\pi)^{(1-n)/2}n^{z-1/2}\Gamma(z/n)\prod_{k=1}^{n-1}\Gamma\left(\frac{z}{n}+\frac{k}{n}\right),$$

$$\Gamma(z)=\frac{(2\pi)^{(n-1)/2}n^{1/2-nz}\Gamma(nz)}{\prod_{k=1}^{n-1}\Gamma(z+k/n)}.$$

Equating the right-hand sides of the prior equations and solving for $\Gamma(nz)$ we find

$$\Gamma(nz)=(2\pi)^{2\cdot(1-n)/2}n^{nz+z-2\cdot 1/2}\Gamma(z/n)\prod_{k=1}^{n-1}\Gamma\left(z+\frac{k}{n}\right)\Gamma\left(\frac{z}{n}+\frac{k}{n}\right);$$

we can see that if we perform this process a times,

$$\Gamma(nz)=$$

$$(2\pi)^{(a+1)(1-n)/2}\exp\left(\sum_{r=0}^{a}[n^{1-r}z-1/2]\log n\right)\Gamma(z/n^a)\prod_{i=0}^{a}\left[\prod_{k=1}^{n-1}\Gamma\left(\frac{k}{n}+\frac{z}{n^i}\right)\right]$$

$$=(2\pi)^{(a+1)(1-n)/2}n^{nz(n-n^{-a})/(n-1)-(a+1)/2}\Gamma(z/n^a)\prod_{i=0}^{a}\left[\prod_{k=1}^{n-1}\Gamma\left(\frac{k}{n}+\frac{z}{n^i}\right)\right].$$

We may include the $(2\pi)^{(a+1)(1-n)/2}$ term in the double product such that we have the reworked form

$$\Gamma(nz) = n^{nz(n-n^{-a})/(n-1)-(a+1)/2}\Gamma(z/n^a)\prod_{i=0}^{a}\left[\frac{1}{(2\pi)^{(n+1)/2}}\prod_{k=1}^{n-1}\Gamma\left(\frac{k}{n}+\frac{z}{n^i}\right)\right].$$

By applying (3) to $\Gamma(z/n^a)$,

$$n^{nz(n-n^{-a})/(n-1)-(a+1)/2}\Gamma(z/n^a) = n^{n^{1-a}z/(n-1)}\lim_{k\to\infty}\frac{k!n^{(a-1)/2}k^{n^{-a}z}}{z(n^{-a}z+1)\cdots(n^{-a}z+k)}$$

and as $a\to\infty$,

$$\lim_{a\to\infty} n^{nz(n-n^{-a})/(n-1)-(a+1)/2}\Gamma(z/n^a) = \lim_{a\to\infty} n^{(a+1)/2}n^{n^2z/(n-1)-1};$$

and we can before taking the limit formally put the $n^{(a+1)/2}$ term into the double product as well to yield the infinite product

$$\Gamma(nz) = n^{n^2z/(n-1)-1}\prod_{i=0}^{\infty}\left[\frac{\sqrt{n}}{(2\pi)^{(n-1)/2}}\prod_{k=1}^{n-1}\Gamma\left(\frac{k}{n}+\frac{z}{n^i}\right)\right].$$

Removing the $i=0$ term of the product and substituting from (4) on $\Gamma(nz)$, we have after simplification

$$n^{nz-n^2z/(n-1)}\Gamma(z) = \prod_{i=1}^{\infty}\left[\frac{\sqrt{n}}{(2\pi)^{(n-1)/2}}\prod_{k=1}^{n-1}\Gamma\left(\frac{k}{n}+\frac{z}{n^i}\right)\right].$$

This is just (8), which may be finalized by combining fractions on the exponent of n on the left-hand side.

Logarithmic derivatives of (8) give the relations

$$\sum_{i=1}^{\infty}\sum_{k=1}^{n-1}\frac{1}{n^i}\Psi\left(\frac{k}{n}+\frac{z}{n^i}\right) = \frac{n\log n}{1-n}+\Psi(1+z),$$

$$\sum_{i=1}^{\infty}\sum_{k=1}^{n-1}\frac{1}{n^{i(g+1)}}\Psi_g\left(\frac{k}{n}+\frac{z}{n^i}\right) = \Psi_g(1+z) \quad (g\ge 1),$$

where

$$\Psi(z) = \frac{\mathrm{d}}{\mathrm{d}z}\log\Gamma(z),$$

$$\Psi_g(z) = \frac{\mathrm{d}^{g+1}}{\mathrm{d}z^{g+1}}\log\Gamma(z)$$

are the Digamma and Polygamma functions, respectively. The previous two double-sum identities have the analogous single-sum expansion

$$\sum_{i=1}^{\infty}\frac{1}{n^{ig}}\left[n^g\Psi_g(n^{1-i}z)-\Psi_g(n^{-i}z)\right] = \Psi_g(1+z) \quad (g\ge 0),$$

which may be derived by logarithmically differentiating (6).

An analog of the symmetric identity (2) is given by setting $z = 1$ and taking both sides to the $([n-1]/n)$-th power:

$$\prod_{i=1}^{\infty}\left[\frac{1}{\Gamma(1/n)}\Gamma\left(\frac{1}{n}+\frac{1}{n^i}\right)\cdots\frac{1}{\Gamma(1-1/n)}\Gamma\left(\frac{n-1}{n}+\frac{1}{n^i}\right)\right]^{(n-1)/n}=\frac{1}{n};$$

alternatively, setting $z = n - 1$ and taking both sides of (8) to the $(1/n)$-th power:

$$\prod_{i=1}^{\infty}\left[\frac{1}{\Gamma(1/n)}\Gamma\left(\frac{1}{n}+\frac{n-1}{n^i}\right)\cdots\frac{1}{\Gamma(1-1/n)}\Gamma\left(\frac{n-1}{n}+\frac{n-1}{n^i}\right)\right]^{1/n}=\frac{\Gamma(n)}{n}.$$

References

[1] G. Andrews, R. Askey, R. Roy, *Special Functions*, Cambridge University Press, Cambridge (1999).

[2] A. Erdélyi, W. Magnus, F. Oberhettinger, F. Tricomi, *Higher Trascendental Functions*, Volume 1, McGraw-Hill, New York (1953).