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AN X-RANK CHARACTERIZATION OF RATIONAL NORMAL CURVES

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Abstract: Here we characterize the rational normal curves of $\mathbb{P}^n$, $n \geq 5$ and odd, among the smooth curves $X \subset \mathbb{P}^n$ in terms of the dimension of all $P \in \mathbb{P}^n$ with X-rank $\geq (n+3)/2$ and with X-border rank maximal, $(n+1)/2$. We also prove that this set is large if X is a linearly normal elliptic curve.

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Let $X \subseteq \mathbb{P}^n$ be a smooth, connected and non-degenerate curve defined over an algebraically closed field $\mathbb{K}$ such that $\mathrm{chark}(\mathbb{K}) = 0$. For any $P \in \mathbb{P}^n$ the X-rank $r_X(P)$ of P is the minimal cardinality of a finite set $S \subset X$ such that $P \in \langle S \rangle$, where $\langle \ \ \rangle$ denote the linear span. For any integer $t \geq 1$ let $\sigma_t(X)$ denote the closure in $\mathbb{P}^n$ of the union of all $(t-1)$-dimensional linear subspaces of $\mathbb{P}^n$ spanned by t points of X. Hence $\sigma_1(X) = X$. Set $\sigma_0(X) := \emptyset$. We say that a point $P \in \sigma_t(X) \setminus \sigma_{t-1}(X)$ has border rank t and write $b_X(P) = t$. If X is a smooth curve and $P \in \sigma_t(X) \setminus \sigma_{t-1}(X)$, then there is a degree t zero-dimensional scheme $Z \subset X$ such that $P \in \langle Z \rangle$ and $P \notin \langle Z' \rangle$ for any $Z' \subsetneq Z$. Set $\sigma_{t,x}(X) := \{P \in \mathbb{P}^n : b_X(P) = t, r_X(P) = x\}$. Fix an integer t such that $1 \leq t \leq n$. Let $A(X,t)$ be the set of all zero-dimensional schemes $Z \subset X$ such that $\deg(Z) = t$ and Z is linearly independent, i.e. $\dim(\langle Z \rangle) = t-1$. Since

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$t \le n$, the algebraic set $A(X,t)$ is smooth, irreducible, of dimension t and non-empty. Let $B(X,t)$ the be set of all unreduced $Z \in A(X,t)$. Thus $B(X,1) = \emptyset$, while $B(X,t)$ is non-empty, irreducible and of dimension $t-1$ if $2 \le t \le n$; a general element of $B(X,t)$ has a degre 2 unreduced connected component and $t-2$ reduced connected components. For a description of the ranks of a rational normal curve, see [2] or [3], Theorem 4.1. For a description of $\sigma_{2,t}(X)$ when X is a smooth and linearly normal elliptic curve, see [1], Theorem 3.13. Theorem 2 shows that Theorem 1 is sharp.

Theorem 1. *Fix an odd integer $n \ge 5$. Let $X \subset \mathbb{P}^n$ be a smooth, connected and non-degenerate curve. X is a rational normal curve if and only if* $\dim(\cup_{t\ge(n+3)/2}\sigma_{(n+1)/2,t}(X)) = n-1$.

Theorem 2. *Fix an odd integer $n \ge 5$. Let $X \subset \mathbb{P}^n$ be a smooth and linearly normal elliptic curve. Then $\sigma_{(n+1)/2,(n+3)/2}(X) \ne \emptyset$ and*

$$\dim(\cup_{t\ge(n+3)/2}\sigma_{(n+1)/2,t}(X)) = n-2.$$

Proof of Theorem 1. See [2] or [3], Theorem 4.1, for the "if" part. From now on we assume that X is not a rational normal curve, i.e. $d := \deg(X) > n$. Notice that $B(X,(n+1)/2)$ is integral, non-empty and of dimension $n-1$. Thus to prove the theorem it is sufficient to prove $r_X(P) \le (n+1)/2$ for a general $P \in B(X,(n+1)/2)$. Fix a general $P \in B(X,(n+1)/2)$ and assume $r_X(P) \ge (n+3)/2$. Let $Z \subset X$ be a degree $(n+1)/2$ subscheme such that $P \in \langle Z\rangle$. For general P the scheme Z is unique and it is of the form $2Q_1 + Q_2 + \cdots + Q_{(n-1)/2}$ for a general $(Q_1,\dots,Q_{(n-1)/2}) \in X^{(n-1)/2}$. To get a contradiction it is sufficient to prove $r_X(Q) \le (n+1)/2$ for a general $Q \in \langle Z\rangle$. Let H_Z be the set of all hyperplanes $H \subset \mathbb{P}^n$ containing $|Z|$. The set H_Z is a projective space of dimension $(n+1)/2$. Since $n \ge 5$, we have $(n+1)/2 \le n-1$. Hence an easy monodromy argument gives $\langle Z\rangle \cap X = Z$ (scheme-theoretically) for a general Z. Hence for a general $H \in H_Z$ the scheme $X \cap H$ is the disjoint union of H and of a set X_H formed by $d-(n+1)/2$ distinct points, none of them in Z_{red}; here we use the assumption $d > n$. Moreover (again monodromy) any $S \subset X_H$ formed by $(n+1)/2$ points spans a linear subspace of dimension $(n-1)/2$. Here we only need the existence of an $(n+1)/2$)-dimensional family Γ of reduced $S \subset H_Z$ such that $\sharp(S) = (n+1)/2$ and $\dim(\langle S\rangle) = (n-1)/2$. Fix $S \in \Gamma$. We have $\langle Z\rangle \cap \langle S\rangle \ne \emptyset$. Notice that $r_X(P) \le (n+1)/2$ for all $P \in \langle Z\rangle \cap \langle S\rangle$ and all $S \in \Gamma$. Hence to get a contradiction it is sufficient to prove that varying S in Γ the sets $\langle Z\rangle \cap \langle S\rangle$ cover a non-empty open subset of $\langle Z\rangle$. Fix $S \in \Gamma$ and move independently the points of S to nearby points. $\square$

Proof of Theorem 2. Since X is a linearly normal elliptic curve, $\deg(X) = n+1$. Let $\mathcal{W}$ be the non-zero open subset of all $Z \in B(X,(n+1)/2)$ such that $\mathcal{O}_X(1) \neq \mathcal{O}_X(2Z)$, i.e. such that $\mathcal{O}_X(2A) \ncong \mathcal{O}_X(1)$, where $A \in |\mathcal{O}_X(1)(-Z)|$. Fix $Z \in \mathcal{W}$ and $A \in |\mathcal{O}_X(1)(-Z)|$. Since $Z+A \in |\mathcal{O}_X(1)|$, there is a hyperplane $H_A \subset \mathbb{P}^n$ such that $X \cap H_A = Z + A$. Since any degree n effective divisor of X is linearly independent, $\langle Z\rangle$ and $\langle A\rangle$ are linear subspaces of dimension $(n-1)/2$. Since both are contained in H_A, we have $\langle Z\rangle \cap \langle A\rangle \neq \emptyset$. Let $\mathcal{Z}$ be the set of all unreduced A. Notice that $\mathcal{Z}$ is non-empty and of codimension 1 in $|\mathcal{O}_X(1)(-Z)|$. Since $Z + A \in |\mathcal{O}_X(1)|$, $\langle Z\rangle \cap \langle A\rangle$ is a point, $P_{Z,A}$. If A is reduced, then obviously $r_X(P_{Z,A}) \leq (n+1)/2$. Now assume that A is not reduced, i.e. $A \in \mathcal{Z}$.

Claim. $r_X(P_{Z,A}) \geq (n+3)/2$.

Proof of the Claim. Assume $r_X(P_{Z,A}) \leq (n+1)/2$ and fix $S \subset X$ computing $r_X(P_{Z,A})$. Since $\deg(Z) + \deg(S) \leq n+1$ and $P \in \langle Z\rangle \cap \langle S\rangle$, there is $H \in H_Z$ such that $H \cap X$ contains the scheme $Z+S$. Since $b_X(P) = (n+1)/2$, we have $\sharp(S) = (n+1)/2$. Hence $S \in |\mathcal{O}_X(1)(-Z)|$. Similarly, since $P \in \langle A\rangle \cap \langle S\rangle$ and $A \neq S$, we get $A + S \in |\mathcal{O}_X(1)|$. Since $\mathcal{O}_X(S) \cong \mathcal{O}_X(1)(-Z) \cong \mathcal{O}_X(A)$, we get $\mathcal{O}_X(2A) \cong \mathcal{O}_X(1)$, contradicting the definition of $\mathcal{W}$.

By the claim, each point of $\langle Z\rangle \cap \langle A\rangle$ is an element of $\cup_{t \geq (n+3)/2} \sigma_{(n+1),2,t}(X)$. Moreover, $\langle Z\rangle \cap \langle A\rangle$ is a unique point for a general $Z \in B(X,(n+1)/2)$. The theorem follows by counting the dimension of all possible $Z \in B(X,(n+1)/2)$ and all $A \in \mathcal{Z}$. □

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