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ON WEAKLY CONTRA-CONTINUOUS FUNCTIONS

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Abstract: New properties of weakly contra-continuous functions are developed. It is shown that weak contra-continuity is closely related to connectedness and to set-connected functions. Additional properties are investigated. In particular, conditions are established under which weakly contra-continuous functions have a closed graph.

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1. Introduction

Contra-continuous functions were introduced by Dontchev [2] in 1996. Since then many variations of contra-continuous have been investigated. For example, in 2002 Jafari and Noiri [4] developed the notion of contra-precontinuity and almost contra-precontinuity was introduced by Ekici [3] in 2004. Recently the concept of a weakly contra-continuous function was developed by Baker [1]. The purpose of this note is to develop additional properties of weakly contra-continuous functions. We show that these functions are closely related to connectedness and to the notion of a set-connected mapping developed by Kwak [7] in 1971. For example, we show that a function is weakly contra-continuous if and only if inverse images of disjoint closed sets are separated. It then follows that the image of a connected space under a surjective weakly contra-continuous function is connected. Weakly contra-continuous functions and set connected functions share some interesting properties. For example, both types functions are strictly between continuity and slight continuity. In

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particular, both types of functions inversely preserve clopen sets. Surjective weakly contra-continuous functions are set-connected. However, in general the two concepts are independent.

2. Preliminaries

The symbols X and Y represent topological spaces with no separation properties assumed unless explicitly stated. All sets are considered to be subsets of topological spaces. The closure and interior of a set A are signified by $\mathrm{Cl}(A)$ and $\mathrm{Int}(A)$, respectively.

Definition 2.1. A function $f : X \to Y$ is said to be contra-continuous (see [2]) if $f^{-1}(V)$ is closed for every open subset V of Y.

Definition 2.2. A function $f : X \to Y$ is said to be weakly contra-continuous (see [1]) provided that, whenever $A \subseteq V \subseteq Y$, A is closed in Y, and V is open in Y, then $\mathrm{Cl}(f^{-1}(A)) \subseteq f^{-1}(V)$.

The following characterization of weak contra-continuity is easily proved by use of the definition and complements.

Theorem 2.3. *A function $f : X \to Y$ is weakly contra-continuous if and only if $f^{-1}(A) \subseteq \mathrm{Int}(f^{-1}(V))$ whenever $A \subseteq V \subseteq Y$, A is closed in Y, and V is open in Y.*

Definition 2.4. A function $f : X \to Y$ is said to be slightly-continuous (see [9]), if for each point $x \in X$ and each clopen neighborhood V of $f(x)$, there exists a neighborhood U of x such that $f(U) \subseteq V$.

The following characterizations for slight continuity will be useful.

Theorem 2.5. (see [9]) *For a function $f : X \to Y$, the following statements are equivalent.*

(a) *f is slightly continuous.*

(b) *For every clopen subset V of Y, $f^{-1}(V)$ is open.*

(c) *For every clopen subset V of Y, $f^{-1}(V)$ is clopen.*

3. Relationships to Connectedness

Recall that two subsets A and B of a space are called separated (see [5]) provided that $\mathrm{Cl}(A) \cap B = A \cap \mathrm{Cl}(B) = \emptyset$. Also a space is connected if and only if it is not the union of two nonempty separated sets.

Theorem 3.1. *A function $f : X \to Y$ is weakly contra-continuous if and only if, for every pair of disjoint closed sets, F_1 and F_2 in Y, $f^{-1}(F_1)$ and $f^{-1}(F_2)$ are separated.*

Proof. Assume $f : X \to Y$ is weakly contra-continuous. Let F_1 and F_2 be disjoint closed subsets of Y. Then, since, $F_1 \subseteq Y - F_2$ with F_1 closed and $Y - F_2$ open, it follows from the definition of weak contra-continuity that $\mathrm{Cl}(f^{-1}(F_1)) \subseteq f^{-1}(Y - F_2) = X - f^{-1}(F_2)$. Therefore $\mathrm{Cl}(f^{-1}(F_1)) \cap f^{-1}(F_2) = \emptyset$. Similarly $\mathrm{Cl}(f^{-1}(F_2)) \cap f^{-1}(F_1) = \emptyset$. Therefore F_1 and F_2 are separated.

Assume that for every pair of disjoint closed sets, F_1 and F_2, $f^{-1}(F_1)$ and $f^{-1}(F_2)$ are separated. Let $A \subseteq V \subseteq Y$, where A is closed in Y and V is open in Y. Then, since A and $Y - V$ are disjoint closed sets, $f^{-1}(A)$ and $f^{-1}(Y - V)$ are separated. Then $\mathrm{Cl}(f^{-1}(A) \cap (X - f^{-1}(V)) = \emptyset$ and hence $\mathrm{Cl}(f^{-1}(A)) \subseteq f^{-1}(V)$, which proves that f is weakly contra-continuous. □

The next result is an immediate consequence of Theorem 3.1.

Corollary 3.2. *If the function $f : X \to Y$ is weakly contra-continuous and surjective and X is connected, then Y is connected.*

As we see in the next example, it is not true that, if $f : X \to Y$ is weakly contra-continuous and X is connected, then $f(X)$ is connected in Y.

Example 3.3. Let $X = \{a, b\}$ have the indiscrete topology and let $Y = \{a, b, c\}$ have the topology $\{Y, \emptyset, \{a\}, \{b\}, \{a, b\}\}$. Since there are no nonempty disjoint closed sets in Y, the inclusion function $f : X \to Y$ is weakly contra-continuous. However, X is connected and $f(X)$ is disconnected.

Theorem 3.4. *If $f : X \to Y$ is weakly contra-continuous and Y is T_1, then any two fibers of f are separated.*

Proof. Suppose $f^{-1}(f(x_1))$ and $f^{-1}(f(x_2))$ are two fibers of f. Since Y is T_1, $\{f(x_1)\}$ and $\{f(x_2)\}$ are disjoint closed sets in Y. By Theorem 3.1 $f^{-1}(f(x_1))$ and $f^{-1}(f(x_1))$ are separated. □

For an injective function the nonempty fibers are singleton sets. Therefore we have the following result.

Corollary 3.5. *If $f : X \to Y$ is a weakly contra-continuous injection and Y is T_1, then X is T_1.*

Corollary 3.6. *If $f : X \to Y$ is weakly contra-continuous and Y is T_1, then the set $E = \{(x_1, x_2) : f(x_1) = f(x_2)\}$ is a union of mutually separated sets.*

Proof. Note that $E = \cup_{y \in f(X)}(f^{-1}(y) \times f^{-1}(y))$. For $y_1 \neq y_2$, it follows from Theorem 3.4 that $f^{-1}(y_1)$ and $f^{-1}(y_2)$ are separated in X. Therefore

$f^{-1}(y_1) \times f^{-1}(y_1)$ and $f^{-1}(y_2) \times f^{-1}(y_2)$ are separated in $X \times Y$. □

4. Relationships to Set-Connected Functions

Since weak contra-continuity is strictly stronger than slight continuity (see [1]), it follows from Theorem 2.5 that weakly contra-continuous functions inversely preserve clopen sets. This property leads to a close relationship between weakly contra-continuous functions and the set-connected functions developed by Kwak [7].

Definition 4.1. A space is said to be connected between A and B (see [6]), if there is no clopen set F such that $A \subseteq F$ and $F \cap B = \emptyset$.

Definition 4.2. A function $f : X \to Y$ is said to be set-connected (see [7]), provided that, if X is connected between A and B, then $f(X)$ is connected between $f(A)$ and $f(B)$, with respect to the relative topology on $f(X)$.

The following theorem gives a useful characterization of set-connectedness proved by Kwak [7].

Theorem 4.3. (see [7]) *A function $f : X \to Y$ is set-connected if and only if $f^{-1}(F)$ is clopen in X for every clopen subset F of $f(X)$ (with respect to the relative topology).*

Corollary 4.4. *A surjective weakly contra-continuous function is set-connected.*

The function in Example 3.3 is weakly contra-continuous (and hence also slightly continuous) but not set-connected. Kwak [7] proved that the range of a set-connected function with a connected domain is connected and, as we observed in Example 3.3, X is connected and $f(X)$ is not connected. Therefore a non-surjective weakly contra-continuous function can fail to be set-connected.

In the next example we see that even a surjective set-connected function can fail to be weakly contra-continuous. Thus weak contra-continuity and set-connectedness are independent concepts.

Example 4.5. Let X denote the real numbers, $\sigma = \{X, \emptyset, \{0\}\}$, and let τ be the usual topology on X. Since (X, τ) is connected, the identity mapping $f : (X, \sigma) \to (X, \tau)$ is set-connected. However, f is not weakly contra-continuous. To see this, note that $\{0\} \subseteq (-1, 1) \subseteq (X, \tau)$, but $\mathrm{Cl}(f^{-1}(0)) \nsubseteq f^{-1}((-1, 1))$

Therefore the converse of Corollary 4.4 does not hold.

Kwak [7] showed that continuity is strictly stronger than set-connectedness and Single and Jain [9] showed that set-connectedness is strictly stronger than slight continuity. Also from [1] we have that continuity is also strictly stronger than weak contra-continuity. Thus we now have the following implications, none of which are reversible.

$$\begin{array}{ccccc} & \nearrow & \text{weak contra-cont.} & \searrow & \\ \text{cont.} & & & & \text{slightly cont.} \\ & \searrow & \text{set-connected} & \nearrow & \end{array}$$

We see from the following result that contra-continuity, developed by Dontchev [2], implies set-connectedness.

Theorem 4.6. *If the function $f : X \to Y$ is contra-continuous, then f is set-connected.*

Proof. Assume the function $f : X \to Y$ is contra-continuous and let $F \subseteq f(X)$, where F is clopen with respect to the relative topology on $f(X)$. Then there exist a set V open in Y and a set A closed in Y for which $F = V \cap f(X) = A \cap f(X)$. Then we see that $f^{-1}(F) = f^{-1}(V \cap f(X)) = f^{-1}(V)$ and hence $f^{-1}(F)$ is closed. Similarly, $f^{-1}(F) = f^{-1}(A \cap f(X)) = f^{-1}(A)$ and hence $f^{-1}(F)$ is open. Therefore $f^{-1}(F)$ is clopen and hence by Theorem 4.3 f is set-connected. □

Since continuity implies set-connectedness (see [7]), obviously set-connectedness does not imply contra-continuity. Also from [1] contra-continuity is strictly stronger than weak contra-continuity. Therefore we have the following implications, none of which are reversible.

$$\begin{array}{ccccc} & \nearrow & \text{weak contra-cont.} & \searrow & \\ \text{contra-cont.} & & & & \text{slightly cont.} \\ & \searrow & \text{set-connected} & \nearrow & \end{array}$$

Theorem 4.7. *If the function $f : X \to Y$ is weakly contra-continuous and $f(X)$ is clopen in Y, then f is set-connected.*

Proof. Let $A \subseteq f(X)$ such that A is clopen in $f(X)$. Since $f(X)$ is clopen in Y, A is clopen in Y. By Theorem 2.5 $f^{-1}(A)$ is clopen in X and therefore by Theorem 4.3 f is set-connected. □

Recall that a space is call 0-dimensional provided it has a base consisting of clopen sets.

Theorem 4.8. *If the function $f : X \to Y$ is weakly contra-continuous and*

Y is 0-dimensional, then f is continuous (and hence also set-connected).

5. Additional Properties

Kwak [7] proved that a set-connected function with a extremally disconnected Hausdorff codomain has a closed graph. We show that a weakly contra-continuous function has a closed graph under a weaker hypothesis. Finally properties involving restriction and composition are investigated.

Recall that a space X is extremally disconnected if the closures of open sets are open. The graph of a function $f : X \to Y$ is the subset $G(f) = \{(x, f(x)) : x \in X\}$ of the product space $X \times Y$.

Theorem 5.1. *If $f : X \to Y$ is weakly contra-continuous and Y is Urysohn, then $G(f)$ is closed.*

Proof. Let $(x, y) \in X \times Y - G(f)$. Then $y \neq f(x)$. Since Y is Urysohn, there exist open sets V and W in Y such that $y \in V$, $f(x) \in W$, and $\mathrm{Cl}(V) \cap \mathrm{Cl}(W) = \emptyset$. By Theorem 3.1 $f^{-1}(\mathrm{Cl}(V))$ and $f^{-1}(\mathrm{Cl}(W))$ are separated. Thus $\mathrm{Cl}(f^{-1}(\mathrm{Cl}(V))) \cap f^{-1}(\mathrm{Cl}(W)) = \emptyset$. Then we see that $(x, y) \in (X - \mathrm{Cl}(f^{-1}(\mathrm{Cl}(V)))) \times V \subseteq X \times Y - G(f)$, which proves that $G(f)$ is closed. □

In the above theorem the requirement that f be weakly contra-continuous cannot be replaced by set-connectedness. Since the real numbers with the usual topology are connected, any surjective function onto the real numbers is set-connected and the real numbers are obviously Urysohn.

Since extremally disconnected Hausdorff implies Urysohn, we have the following corollary to Theorem 5.1.

Corollary 5.2. *If $f : X \to Y$ is weakly contra-continuous and Y is extremally disconnected and Hausdorff, then $G(f)$ is closed.*

Long [8] proved that any function from a first countable space into a countably compact space having a closed graph is continuous. Therefore we have the following result.

Corollary 5.3. *Let $f : X \to Y$ be weakly contra-continuous. Let X be first countable and let Y be countably compact and Urysohn. Then f is continuous.*

Kwak [7] showed that the restriction of a set-connected function to a set having the property that the image of the set is a clopen subset of the range is set-connected. We see in the next theorem that the restriction of a weakly contra-continuous function to any set is weakly contra-continuous.

Theorem 5.4. *If $f : X \to Y$ is weakly contra-continuous and $B \subseteq X$, then*

$f|_B : B \to Y$ *is weakly contra-continuous.*

Proof. Let $A \subseteq V \subseteq Y$ where A is closed in Y and V is open in Y. Then $\mathrm{Cl}_B(f|_B^{-1}(A)) = \mathrm{Cl}_B(f^{-1}(A) \cap B) = \mathrm{Cl}_X(f^{-1}(A) \cap B) \cap B \subseteq \mathrm{Cl}_X(f^{-1}(A)) \cap B \subseteq f^{-1}(V) \cap B = f|_B^{-1}(V))$, which proves that $f|_B : B \to Y$ is weakly contra-continuous. □

If the codomain of a weakly contra-continuous function is restricted to a proper subset containing the range, then the function can fail to be weakly contra-continuous. To see this, consider the function in Example 3.3. If the codomain is restricted to the range, then the function is not weakly contra-continuous because X is connected whereas $f(X)$ is disconnected.

Finally some results concerning composition are proved. In particular, we show that the composition of a weakly contra-continuous function with a continuous function is weakly contra-continuous.

Theorem 5.5. *If $f : X \to Y$ is continuous and $g : Y \to Z$ is weakly contra-continuous, then $g \circ f : X \to Z$ is weakly contra-continuous.*

Proof. Let $A \subseteq V \subseteq Z$, where A is closed in Z and V is open in Z. Since g is weakly contra-continuous, $\mathrm{Cl}(g^{-1}(A)) \subseteq g^{-1}(V)$. Then $f^{-1}(\mathrm{Cl}(g^{-1}(A))) \subseteq f^{-1}(g^{-1}(V))$. Since f is continuous, $\mathrm{Cl}(f^{-1}(g^{-1}(A))) \subseteq f^{-1}(\mathrm{Cl}(g^{-1}(A))) \subseteq f^{-1}(g^{-1}(V))$, which proves that $g \circ f : X \to Z$ is weakly contra-continuous. □

The proof of the following theorem is straightforward and is omitted.

Theorem 5.6. *If $f : X \to Y$ is weakly contra-continuous and $g : Y \to Z$ is continuous, then $g \circ f : X \to Z$ is weakly contra-continuous.*

Theorem 5.7. *Let $f : X \to Y$ and $g : Y \to Z$ be functions. If $g \circ f$ is weakly contra-continuous and g is injective and clopen, then f is weakly contra-continuous.*

Proof. Let $A \subseteq V \subseteq Y$, where A is closed in Y and V is open in Y. Then $g(A) \subseteq g(V)$ with $g(A)$ closed in Z and $g(V)$ open in Z. Therefore $\mathrm{Cl}(f^{-1}(A)) = \mathrm{Cl}(f^{-1}(g^{-1}(g(A)))) \subseteq f^{-1}(g^{-1}(g(V))) = f^{-1}(V)$, which proves that f is weakly contra-continuous. □

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