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POSTULATION OF GENERAL UNIONS OF LINES, A LINEAR SPACE AND PLANAR LENGTH 3 SUBSCHEMES

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Abstract: Here we handle the postulation of a general disjoint union $X \subset \mathbb{P}^n$ of an m-dimensional linear space and a prescribed number of lines and degree 3 planar connected zero-dimensional subschemes.

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1. Introduction

Several papers considered the postulation of general disjoint unions inside $\mathbb{P}^n$ of a prescribe number of linear subspaces with prescribed dimension (see [10], [7], [8], [6]). As a technical tool it was often useful to handle the postulation of general disjoint unions of linear spaces and the following type of zero-dimensional subschemes (see [10], Assertion $H''_{n,N}$, [6]). For any integral scheme M and any $P \in M_{reg}$ let $\chi_M(P)$ denote the first infinitesimal neighborhood of P in M, i.e. the closed subscheme of M with $(\mathcal{I}_{P,M})^2$ as its ideal sheaf. Thus $\chi_M(P)_{red} = \{P\}$, length$(\chi_M(P)) = \dim_P(M) + 1$ and $\chi_M(P) = \chi_{T_PM}(P)$. If $\dim_P(M) = 1$ (resp. $\dim_P(M) = 2$) and $M \subseteq \mathbb{P}^n$, $n \geq 2$, we often call $\chi_M(P)$ a *tangent vector* (resp. a *planar length* 3 *subscheme*) of $\mathbb{P}^n$. The set of all tangent vectors (resp. planar length 3 subschemes) of $\mathbb{P}^n$ is parametrized by an integral quasi-projective variety. Hence for all non-negative integers α ,

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β we may look at the cohomological properties of a general union $Z \subset \mathbb{P}^m$ of α tangent vectors and β planar length 3 subschemes of $\mathbb{P}^m$. Here (as in [6]) we work over an algebraically closed field $\mathbb{K}$ such that $\mathrm{char}(\mathbb{K}) = 0$. This assumption makes trivial to handle the addition of general tangent vectors (see Lemma 1 lifted from [6]). Thus here we only consider planar length 3 subschemes. A closed subscheme $X \subseteq \mathbb{P}^n$ is said to have *maximal rank* if for every integer t the restriction map $\rho_{X,t} : H^0(\mathcal{O}_{\mathbb{P}^n}(t)) \to H^0(X, \mathcal{O}_X(t))$ has maximal rank, i.e. it is either injective or surjective. If (as in our case) $h^i(X, \mathcal{O}_X(t)) = 0$ for all integers $t > 0$ and $i \geq 1$, X has maximal rank if and only if for every integer $t > 0$ either $h^0(\mathcal{I}_X(t)) = 0$ or $h^1(\mathcal{I}_X(t)) = 0$. Here we prove the following result.

Theorem 1. *Fix integers n, m, d, z such that $n-2 \geq m > 0$, $d \geq 0$, $z \geq 0$. Let $X \subset \mathbb{P}^n$ be a general disjoint union of an m-dimensional linear space, d lines and z planar length 3 schemes. Then X has maximal rank.*

Take n, m, d, z as in the statement of Theorem 1 and let $Y \subset \mathbb{P}^n$ be any disjoint union of an m-dimensional linear subspace, d lines and z planar length 3 schemes. We have $h^0(Y, \mathcal{O}_Y(t)) = \binom{m+t}{m} + (t+1)d + 3z$ for every integer $t \geq 0$. Let k be the first positive integer such that $\binom{m+k}{m} + (k+1)d + 3z \leq \binom{n+k}{n}$. Castelnuovo-Mumford's Lemma implies that Y has maximal rank if and only if $h^0(\mathcal{I}_Y(k-1)) = 0$ and $h^1(\mathcal{I}_Y(k)) = 0$.

2. Preliminaries

Remark 1. Fix a reducible conic $D \subset \mathbb{P}^n$, $n \geq 3$, and let P be its singular point. Let $M \subseteq \mathbb{P}^n$ be any 3-dimensional linear space containing D. The scheme $D \cup \chi_M(P)$ is a flat degeneration inside M and hence inside $\mathbb{P}^n$ of a flat family whose general element is the disjoint union of two lines (see [10]). Let $H \subset \mathbb{P}^n$ be any hyperplane containing P, but no irreducible component of D. The scheme $D \cap H$ is a tangent vector of H with P as its reduction. Fix P and H, but take as D a general reducible conic with P as its singular point. Then $D \cap H$ is the general tangent vector of H with P as its support. Now take M general. Then $(D \cup \chi_M(P)) \cap H = \chi_{M \cap H}(P)$. Thus $(D \cup \chi_M(P)) \cap H$ is a general planar length 3 subscheme of H with P as its reduction.

The first part of the following remark is often called "The Horace Lemma". The last part of the following remark is a particular case of the so-called "Differential Horace Lemma".

Remark 2. Let X be any projective scheme and D any effective Cartier divisor of X. For any closed subscheme Z of X let $\mathrm{Res}_D(Z)$ denote the residual scheme of Z with respect to D, i.e. the closed subscheme of X with $\mathcal{I}_Z : \mathcal{I}_D$ as its ideal sheaf. For every $L \in \mathrm{Pic}(X)$ we have the exact sequence

$$0 \to \mathcal{I}_{\mathrm{Res}_D(Z)} \otimes L(-D) \to \mathcal{I}_Z \otimes L \to \mathcal{I}_{Z\cap D,D} \otimes (L|D) \to 0. \qquad (1)$$

From (1) we get

$$h^i(X, \mathcal{I}_Z \otimes L) \leq h^i(X, \mathcal{I}_{\mathrm{Res}_D(Z)} \otimes L(-D)) + h^i(D, \mathcal{I}_{Z\cap D,D} \otimes (L|D))$$

for every integer $i \geq 0$. Now assume $\dim(X) \geq 3$ and both X and D integral. Hence it make sense to speak about the general tangent vector and the general planar length 3 subscheme of D_{reg}. Fix an integer $i \in \{0, 1\}$. A particular case of [5], Lemma 2.1, says that to prove $h^i(X, \mathcal{I}_{Z\cup A} \otimes L) = 0$, where A is a general length 3 planar subscheme, it is sufficient to prove

$$h^i(X, \mathcal{I}_{\mathrm{Res}_D(Z)\cup v} \otimes L(-D)) = h^i(D, \mathcal{I}_{(Z\cap D)\cup\{P\}} \otimes L|D)) = 0,$$

where v is a general tangent vector of D_{reg} and P is a general point of D.

Lemma 1. *Fix any scheme $W \subset \mathbb{P}^n$ and any integer $t \geq 0$. Then (after fixing W and t) fix a general $P \in \mathbb{P}^n$ and a general tangent vector v such that $v_{red} = \{P\}$. If $h^0(\mathcal{I}_W(t)) \leq 2$, then $h^0(\mathcal{I}_{W\cup v}(t)) = 0$. If $h^0(\mathcal{I}_W(t)) \geq 2$, then $h^0(\mathcal{I}_{W\cup v}(t)) = h^0(\mathcal{I}_W(t)) - 2$ and $h^1(\mathcal{I}_{W\cup v}(t)) = h^1(\mathcal{I}_W(t))$.*

Lemma 2. *Fix any scheme $W \subset \mathbb{P}^n$, $n \geq 2$, and any integer $t > 0$. Assume $h^0(\mathcal{I}_W(t)) \geq 3$. Let Ψ denote the rational map induced by the linear system $|\mathcal{I}_W(t)|$. Then (after fixing W and t) fix a general planar length 3 subscheme Z of $\mathbb{P}^n$.*

(a) We have $h^0(\mathcal{I}_{W\cup Z}(t)) \leq h^0(\mathcal{I}_W(t)) - 2$ and equality holds if and only if the image of Ψ has dimension 1.

(b) If $h^0(\mathcal{I}_W(t)) \geq t + 3$, then $h^0(\mathcal{I}_{W\cup Z}(t)) = h^0(\mathcal{I}_W(t)) - 3$.

Lemma 3. *Take the set-up and the assumptions of Lemma 2. Fix an integer $s \geq 2$. Let $E = \sqcup_{i=1}^s Z_i \subset \mathbb{P}^n$ be a general disjoint union of s planar length 3 subschemes. Assume $h^0(\mathcal{I}_W(t)) \geq 3(s-1)+t+3$. Then $h^0(\mathcal{I}_{W\cup E}(t)) = h^0(\mathcal{I}_W(t)) - 3s$.*

We need the following well-known lemmas.

Lemma 4. *Fix integers $n \geq 2$, $d \geq 3$ and $s > 0$. Let $Z \subset \mathbb{P}^n$ be a general union of s planar length 3 schemes. Assume $(n, d, s) \neq (2, 4, 5)$. Then either $h^0(\mathcal{I}_Z(d)) = 0$ (case $3s \geq \binom{n+d}{n}$) or $h^1(\mathcal{I}_Z(d)) = 0$ (case $3s \leq \binom{n+d}{n}$).*

Proof. In the case $n = 2$, this is a well-known particular case of a theorem of Alexander-Hirschowitz (see [1], [2], [3], [4], [9]). Hence we may assume $n \geq 3$

and use induction on n. Let $H \subset \mathbb{P}^n$ be a hyperplane. Let k be the minimal positive integer such that $3s \le \binom{n+k}{n}$. It is sufficient to prove $h^1(\mathcal{I}_Z(k)) = 0$ and $h^0(\mathcal{I}_Z(k-1)) = 0$. We only write down the first vanishing, because the second one requires only notational modifications. For fixed n, k it is sufficient to prove the h^1-vanishing when $s = \sigma := \lfloor \binom{n+k}{n}/3 \rfloor$. Since the case $k = 1$ is trivial (linear algebra), we may assume $k \ge 2$ and work (for the fixed integer n) by induction on k. Set $a := \lfloor \binom{n+k-1}{n}/3 \rfloor$ and $b := \binom{n+k-1}{n} - 3a$. Hence $0 \le b \le 2$. Let $A \subset \mathbb{P}^n$ be a general union of a planar length 3 suchemes. By the inductive assumption on k we have $h^1(\mathcal{I}_A(k-1)) = 0$ and hence $h^0(\mathcal{I}_A(k-1)) = b$. By the generality of A we may assume $A \cap H = \emptyset$. Let $B \subset \mathbb{P}^n$ be a general zero-dimensional scheme with degree b of H. Hence $B = \emptyset$ if $b = 0$, B is a general point of H if $b = 1$, while B is a general tangent vector of H if $b = 2$.

Claim. *We claim that* $h^0(\mathcal{I}_{A\cup B}(k)) = 0$ *and* $h^1(\mathcal{I}_{A\cup B}(k-1)) = 0$.

Proof of the Claim. Since $\deg(A \cup B) = \binom{n+k-1}{n}$, the two statements of the Claim are equivalent. Hence it is sufficient to prove the first one. If $b = 0$, then we have $h^0(\mathcal{I}_A(k-1)) = 0$ by the choice of A. Now assume $b = 1$. Since A is general and $\deg(A) \ge \binom{n+k-2}{n}$, the inductive assumption gives $h^0(\mathcal{I}_A(k-2)) = 0$. Since $A \cap H = \emptyset$, $h^0(\mathcal{I}_A(k-2)) = 0$ implies $h^0(\mathcal{I}_{A\cup H}(k-1)) = 0$. Since B is a general point of H and $h^0(\mathcal{I}_A(k-1)) = 1$, the vanishing of $h^0(\mathcal{I}_{A\cup H}(k-1))$ gives the Claim. Now assume $b = 2$. Since $h^0(\mathcal{I}_{A\cup H}(k-1)) = 2$ and we are in characteristic zero, it is sufficient to prove that the restriction to H of the rational map induced by the linear system $|\mathcal{I}_A(k-1)|$ is not constant. This is true for fixed A for general H, because $h^0(\mathcal{I}_A(k-1)) \ge 2$. This implies that for fixed H, this is true for general H, concluding the proof of the claim. □

(a) Here we assume $b = 0$. Let $D \subset H$ be a general union of $\sigma - a$ planar length 3 subschemes. Since $\binom{n+k}{n} - \binom{n+k-1}{n} = \binom{n+k-1}{n-1}$, the definition of the integers a and b, the inductive assumption on n and the generality of D gives $h^1(H, \mathcal{I}_D(k)) = 0$. The Horace Lemma (Remark 2) gives $h^1(\mathcal{I}_{A\cup D}(k)) = 0$, concluding the proof of the lemma in this case.

(b) Now assume $b > 0$. Let $D_1 \subset H$ be a general union of $\sigma - a - 1$ planar length 3 subschemes. The inductive assumption gives $h^1(H, \mathcal{I}_{D_1}(k)) = 0$. Hence $h^0(H, \mathcal{I}_{D_1}(k)) \ge 3 - b$.

(b1) Assume $b = 1$. Let v be a general tangent vector of H such that v_{red} is the point B. Since B is general in H independently of the choice of D_1, $D_1 \cup v$ may be seen as a general union of D_1 and a general tangent vector of H. Hence $h^0(H, \mathcal{I}_{D_1\cup B_{red}}(k)) = h^0(H, \mathcal{I}_{D_1}(k)) - 2$ Lemma 1). Thus $h^1(H, \mathcal{I}_{D_1\cup B_{red}}(k)) = 0$. Set $Z := A \cup D_1 \cup V$, where V is a planar length 3 subscheme of $\mathbb{P}^n$ such that $v = V \cap HV$. Notice that $Z \cap H = D_1 \cup v$ and $\mathrm{Res}_H(Z) = A \cup B$. Thus the Horace Lemma (Remark 2) gives $h^1(\mathcal{I}_Z(k)) = 0$, proving the lemma in this case.

(b2) Here we assume $b = 2$. Notice that B may be seen as a general tangent vector of H. Let $E \subset \mathbb{P}^n$ be a general planar length 3 subscheme of $\mathbb{P}^n$. To prove the lemma in this case it is sufficient to prove $h^1(\mathcal{I}_{A\cup E\cup D_1}(k)) = 0$. This is true by the Differential Horace Lemma (last part of Remark 2), because $h^1(\mathcal{I}_{A\cup B}(k)) = 0$ and $h^1(H, \mathcal{I}_{D_1\cup B_{red}}(k)) = 0$. □

In the proof of Lemma 4 we proved the following result.

Lemma 5. *Fix integers $n \ge 2$ and $s \ge 3$ such that $(n, s) \ne (2, 5)$. Let $Z \subset \mathbb{P}^n$ be a general union of s planar length 3 schemes. Then Z has maximal rank.*

Lemma 6. *Fix integer $n \ge m + 2 \ge 3$ and $z > 0$. Fix an m-dimensional linear subspace $V \subset \mathbb{P}^n$. Let $Z \subset \mathbb{P}^n$ be a general union of z planar length 3 subschemes of $\mathbb{P}^n \setminus V$. Then $V \cup Z$ has maximal rank.*

Proof. Let k be the critical value of $V \cup Z$, i.e. the first positive integer k such that $\binom{m+k}{m} + 3z \le \binom{n+k}{n}$. It is sufficient to prove $h^0(\mathcal{I}_{V\cup Z}(k-1)) = 0$ and $h^1(\mathcal{I}_{V\cup Z}(k-1)) = 0$. We use induction on m and for fixed m induction on n starting from the case $n' := m + 2$. In each inductive step we take as H a hyperplane not containing V. Hence in each inductive step $\dim(V \cap H) = m-1$ (even in the case $m = 1$) and we may use the inductive assumption on m (the case $m = 0$ being easier than Lemma 4). With this caveat we copy the proof of Lemma 4. the case $n = m + 2$ □

3. The Case $z = 0$

Here we prove the case $z = 0$ of Theorem 1, i.e. we prove the following result.

Proposition 1. *Fix integers $n \ge m + 2 \ge 3$ and $d \ge 0$. Let $X \subset \mathbb{P}^n$ be a general union of an m-dimensional linear space and d lines. Then X has maximal rank.*

For all positive integes n, m, k such that $n \ge m+2$ define the integers $u_{n,m,k}$ and $v_{n,m,k}$ by the relations

$$\binom{m+k}{m} + (k+1)u_{n,m,k} - v_{n,m,k} = \binom{n+k}{n}, \quad 0 \le v_{n,m,k} \le k. \tag{2}$$

Taking the difference of the equation in (2) with the same equation for the triple $(n', m', k') := (n, m, k-1)$ we get

$$\binom{m+k-1}{m-1} + k(u_{n,m,k} - u_{n,m,k-1}) + u_{n,m,k} + v_{n,m,k-1} - v_{n,m,k} = \binom{n+k-1}{n-1}. \quad (3)$$

Fix integers n, m, k such that $u_{n,m,k}$ and $v_{n,m,k}$ are defined and $u_{n,m,k} \geq 2v_{n,m,k}$. Let $Y \subset \mathbb{P}^n$ be a disjoint union of an m-dimensional linear space, $u_{n,m,k} - 2v_{n,m,k}$ lines and $v_{n,m,k}$ reducible conics. Notice that $h^i(Y, \mathcal{O}_Y(t)) = 0$ for all $i > 0$ and $t \geq 0$ and that $h^0(Y, \mathcal{O}_Y(k)) = \binom{n+k}{n}$. Hence Y has critical value k. Y has maximal rank if and only if $h^i(\mathcal{I}_Y(k)) = 0$ for one $i \in \{0, 1\}$.

We fix a hyperplane $H \subset \mathbb{P}^n$. Fix positive integers n, m, k such that $n \geq m+2$. We say that Assertion $M_{n,m,k}$ is defined if $u_{n,m,k} \geq 2v_{n,m,k}$. We say that Assertion $M_{n,m,k}$ is true if it is defined and a general disjoint union of an m-dimensional linear space, $u_{n,m,k} - 2v_{n,m,k}$ lines and $v_{n,m,k}$ reducible conics with singular point contained in H has maximal rank.

Lemma 7. *We have $u_{n,m,k} \geq 4k-3$ for all positive integers n, m, k such that either $(n, m) = (4, 2)$ and $k \geq 5$ or $n \geq m+3 \geq 5$ and $k \geq 2$ or $n \geq m+2 \geq 5$ and $k \geq 3$.*

Proof. Since $u_{n,m,k} := \lceil (\binom{n+k}{n} - \binom{m+k}{m})/(k+1) \rceil$, it is sufficient to prove

$$\binom{n+k}{n} \geq \binom{m+k}{m} + (k+1)(4k-3). \quad (4)$$

First assume $(n, m) = (4, 2)$. In this case (4) is true if $k \geq 5$. If $(n, m) = (5, 2)$, then (4) is true for all $k \geq 2$. Since the right hand side of (4) is an increasing function of n and (for fixed n of $n-m$), (4) is true if $k \geq 2$ and $n \geq m+3 \geq 5$. If $(n, m) = (5, 3)$, then (4) is true if $k \geq 3$. Hence (4) is true if $n \geq m+2 \geq 5$ and $k \geq 2$. □

Lemma 8. *We have $u_{n,m,k} \geq 2v_{n,m,k}$ for all positive integers n, m, k such that $n \geq m+2$.*

Proof. We have $u_{n,m,1} = \lceil (n-m)/2 \rceil$, $v_{n,m,1} = 0$ if $n \equiv m \pmod 2$ and $v_{n,m,1} = 1$ if $n \equiv m+1 \pmod 2$. Hence the lemma is true if $k = 1$. If $k \geq 3$ and $n \geq m+3$, then we may use Lemma 8. If $n = m+2 \geq 5$ and $k = 2$, then it is sufficient to prove $u_{m+2,m,2} \geq 4$, which is true, because it is true if $m = 3$ (use $u_{n,m,k} := \lceil (\binom{n+k}{n} - \binom{m+k}{m})/(k+1) \rceil$ and hence $u_{5,3,2} = 4$). Now assume

$(n, m) = (4, 2)$. By Lemma 7 we only need to do the cases $k = 2, 3, 4$. We have $(u_{4,2,2}, v_{4,2,2}) = (3, 0)$, $(u_{4,2,3}, v_{4,2,3}) = (7, 3)$ and $(u_{4,2,4}, v_{4,2,4}) = (11, 0)$. □

Lemma 9. *Fix positive integers n, m, k such that $n \geq m + 2$ and $n \geq 4$. Then $u_{n,m,k} - u_{n,m,k-1} \geq k$.*

Proof. By (3) it is sufficient to prove

$$\binom{m+k-1}{m-1} + k^2 + u_{n,m,k} + k - 1 \leq \binom{n+k-1}{n-1}. \tag{5}$$

We reduce to the case $(n, m) = (4, 2)$. In this case it is sufficient to prove

$$u_{4,2,k-1} \leq \binom{k+3}{3} - k^2 - k - 1. \tag{6}$$

Since $ku_{4,2,k-1} \leq \binom{k+3}{4} - \binom{k+2}{2} + k - 1$ and $u_{4,2,1} = 1$, (6) is always true. □

Remark 3. The assertion $M_{n,1,k}$ is true for all $n \geq 4$ (Assertion $H'_{k,n}$ of [10]). Here we prove the following weaker form $M'_{3,1,k}$ of $M_{3,1,k}$.

Lemma 10. *Fix positive integers n, m, k such that $n \geq m + 2$ and $n \geq 4$. Assume that Proposition 1 for the integers $m' := m - 1$ and $n' := n - 1$. Then $M_{n,m,k}$ is true.*

Proof. By Lemma 8 the assertion $M_{n,m,k}$ is defined. If $k = 1$, then the result is obvious. Hence we may assume $k \geq 2$. We write $V \sqcup B \sqcup D$ in which V is an m-dimensional linear subspace, B is a disjoint union of lines and D is a disjoint union of reducible conics (we allow the cases $B = \emptyset$ and/or $D = \emptyset$).

(a) Here we assume $n = m+2$. We use induction on k. Let $Y = V \sqcup B \sqcup D$ be a general disjoint union union of an m-dimensional linear subpace, $u_{m+2,m,k-1} - 2v_{m+2,m,k-1}$ lines and $v_{m+2,m,k-1}$ reducible conics with singular point contained in H. Since $M_{m+2,m,k-1}$ is assumed to be true, $h^i(\mathcal{I}_Y(k-1)) = 0$, $i = 0, 1$. Since V is general, $H \cap V$ is an $(m-1)$-dimensional linear subspace of H. Since Y is general, the scheme $Y \cap H$ is a general union of the $(m-1)$-dimensional linear space, $u_{m+2,m,k-1} - v_{m+2,m,k-1}$ points and $v_{m+2,m,k-1}$ tangent vectors.

(a1) Here we assume $v_{m+2,m,k-1} \leq v_{m+2,m,k}$. Let $A \subset H$ be a general union of $u_{m+2,m,k} - u_{m+1,m,k-1}$ lines, with the only restriction that $v_{m+2,m,k} - v_{m+2,m,k-1}$ of them contain a different point of $B \cap H$; here we use $u_{m+2,m,k} - u_{m+2,m,k-1} \geq v_{m+2,m,k} - v_{m+2,m,k-1}$, which is true by Lemma 8. Thus $Y \cup A$ is a disjoint union of V, $u_{m+2,m,k} - 2v_{m+2,m,k}$ lines and $v_{m+2,m,k}$ reducible conics whose singular points are contained in H. The scheme $(Y \cup A) \cap H$ is a disjoint union of $V \cap H$, and a general disjoint union of $u_{m+2,m,k} - u_{m+1,m,k-1}$ lines,

$u_{m+2,m,k-1} - v_{m+2,m,k} + v_{m+2,m,k-1}$ points and $v_{m+2,m,k-1}$ tangent vectors. By (3) with $n := m+2$ we have $h^0((Y \cup E) \cap H, \mathcal{O}_{(Y\cup E)\cap H}(k)) = \binom{m+k+1}{m+1}$. Since $h^i(\mathcal{I}_Y(k-1)) = 0$, $i = 0,1$, and $\mathrm{Res}_H(Y \cup E) = Y$, to prove $M_{m+2,m,k}$ it is sufficient to prove $h^i(H, \mathcal{I}_{(Y\cup E)\cap H}(k)) = 0$, $i = 0,1$ (Remark 2). By (3) it is sufficient to prove $h^1(H, \mathcal{I}_{(Y\cup E)\cap H}(k)) = 0$. By Lemma 1 it is sufficient to prove $h^1(H, \mathcal{I}_U(k-1)) = 0$, where U is the positive-dimensional part of $(Y \cup E) \cap H$. Since $V \cap H$ is an $(m-1)$-dimensional linear space and $h^0(U, \mathcal{O}_U(k)) \le \binom{m+k+1}{m+1}$, the assumption for the integer $m' := m-1$ gives $h^1(H, \mathcal{I}_U(k-1)) = 0$.

(a2) Here we assume $v_{m+2,m,k-1} > v_{m+2,m,k}$. Fix $S \subseteq \mathrm{Sing}(D)$ such that $\sharp(S) = v_{m+2,m,k-1} - v_{m+2,m,k}$. For each $P \in S$ let H_P be a general 3-dimensional linear subspace containing the connected component of D containing P. Set $\chi := \chi_{H_P}(P)$. Let $E \subset H$ be a general union of $u_{m+2,m,k} - u_{m+2,m,k-1}$ lines. By Remark 1 the scheme $Y \cup E \cup \chi$ is a flat degeneration of a flat family of disjoint unions of V, $u_{m+2,m,k} - 2v_{m+2,m,k}$ lines and $v_{m+2,m,k}$ reducible conics with their singular points contained in H. As above it is sufficient to prove $h^i(H, \mathcal{I}_{H\cap(Y\cup E\cup\chi)}(k)) = 0$, $i = 0,1$. The scheme $(Y \cup E \cup \chi) \cap H$ is a general disjoint union in H of $V \cap H$, $u_{m+2,m,k} - u_{m+2,m,k-1}$ lines, $v_{m+2,m,k-1} - v_{m+2,m,k}$ planar length 3 subschemes, $v_{m+2,m,k-1}$ tangent vectors and $u_{m+2,m,k-1} - 2v_{m+2,m,k-1}$ points. Let U' be the positive-dimensional part of $(Y \cup E \cup \chi) \cap H$. Since Proposition 1 is true for the integer $m' := m-1$, we have $h^1(_{U'}(k)) = 0$. Since $v_{m+2,m,k-1} - v_{m+2,m,k} \le k-1$ and $u_{m+2,m,k-1} - 2v_{m+2,m,k-1} \ge 3(k-2) + k + 3$ (Lemma 8) we may first apply Lemma 3

(b) Here we assume $n \ge m+3$ and use induction on the integer n. For the fixed integer n we use induction on the critical value k. We make the same proof using the integer n instead of the integer $m+2$. □

Proof of Proposition 1. Let k be the critical value of the datum (n, m, d), i.e. the first integer $k > 0$ such that $\binom{m+k}{m} + (k+1)d \le \binom{n+k}{n}$. The algebraic subset $\Sigma(n, m, d)$ of $\mathrm{Hilb}(\mathbb{P}^n)$ parametrizing the set of all disjoint unions in $\mathbb{P}^n$ of an m-dimensional linear space and d lines is irreducible. Thus the semicontinuity theorem for cohomology shows that it is sufficient to find $X_1, X_2 \in \Sigma(n, m, d)$ such that $h^0(\mathcal{I}_{X_1}(k-1)) = 0$ and $h^1(\mathcal{I}_{X_2}(k)) = 0$. Proposition 1 is true if $m = 1$ (see [10]) and if $m = 2$ (see [8], Theorem 5.1). Hence from now on we assume $m \ge 3$. By induction on m we may also assume Proposition 1 for all $m' < m$. Hence $M_{n,m,k-1}$ and $M_{n,m,k}$ are true (Lemma 10). We first prove the existence of $X_1 \in \Sigma(n, m, d)$ such that $h^0(\mathcal{I}_{X_1}(k-1)) = 0$. Since k is the critical value of (n, m, d), we have $d \ge u_{n,m,k-1}$. Let $V \sqcup B \sqcup D$ be a solution of $M_{n,m,k-1}$. Hence

$h^i(\mathcal{I}_{V\cup B\cup D}(k-1)) = 0$, $i = 0, 1$. Let $E \subset \mathbb{P}^n$ be a disjoint union of $d - u_{n,m,k-1}$ lines such that $E \cap (V \cup B \cup D) = \emptyset$. For each $P \in \mathrm{Sing}(D)$ take a 3-dimensional linear subspace H_P of $\mathbb{P}^n$ containing the connected component of D containing P. Set $Y := V \cup B \cup D \cup E \cup (\bigcup_{P\in\mathrm{Sing}(D)} \chi_{H_P}(P))$. The scheme Y is a flat degeneration of a flat family of disjoint unions inside $\mathbb{P}^n$ of V and d lines. Since $V \cup B \cup D \subseteq Y$ and $h^0(\mathcal{I}_{V\cup B\cup D}(k-1)) = 0$, we have $h^0(\mathcal{I}_Y(k-1)) = 0$. By semicontinuity we get the existence of X_1. Now we prove the existence of X_2. Since k is the critical value of the datum (n, m, k), we have $d \le u_{n,m,k}$ and if equality holds we have $v_{n,m,k-1} = 0$. Hence if $d = u_{n,m,k}$, then we may take as X_2 any solution of $M_{n,m,k}$. Hence we may assume $d < u_{n,m,k}$. Adding $u_{n,m,k} - d - 1$ disjoint lines we reduce to the case $d = u_{n,m,k} - 1$. Let $V \sqcup B \sqcup D$ be a general solution of $M_{n,m,k-1}$ with respect to the hyperplane H. Since $u_{n,m,k} > u_{n,m,k-1}$ (Lemma 7), we have $u_{n,m,k} - 1 \ge u_{n,m,k-1}$. Let $F \subset H$ be a general disjoint union of $u_{n,m,k} - 1 - u_{n,m,k-1}$ lines. For each $P \in \mathrm{Sing}(D)$ fix a general hyperplane H_P containing the connected component of D containing P. Set $Y_2 := V \cup B \cup D \cup E \cup (\bigcup_{P\in\mathrm{Sing}(D)} \chi_{H_P}(P))$. The scheme Y_2 is a flat degeneration inside $\mathbb{P}^n$ of a flat family of elements of $\Sigma(n, m, u_{n,m,k} - 1)$. Thus by semicontinuity to prove the existence of X_2 it is sufficient to prove $h^1(\mathcal{I}_{Y_2}(k)) = 0$. This is true as in step (a2) of the proof of Lemma 10, because $u_{n,m,k-1} - 2v_{n,m,k-1} \ge k + 3 + 3(k-2)$ by Lemma 8. □

4. Proof of Theorem 1

The inductive proof of Theorem 1 must start somewhere. The following lemma allows us to start it with the case $m = 1$.

Lemma 11. *Fix integers $n \ge 3$, $d \ge 0$ and $z \ge 0$. Let $X \subset \mathbb{P}^n$ be a general disjoint union of d lines and z planar length 3 subschemes. Then X has maximal rank.*

Proof. Let $\Theta(n, d, z)$ denote the set of all disjoint unions in $\mathbb{P}^n$ of d lines and z planar length 3 subschemes. Let k be the minimal integer > 0 such that $(d+1)k + 3z \le \binom{n+k}{k}$ and x the minimal integer > 0 such that $(x+1)d \le \binom{n+x}{n}$, except that we take $x = 0$ if $d = 0$. By semicontinuity, Castelnuovo-Mumford's Lemma and the irreducibility of $\Theta(n, d, z)$ it is sufficient to find $X_i \in \Theta(n, d, z)$, $i = 1, 2$, such that $h^0(\mathcal{I}_{X_1}(k-1)) = 0$ and $h^1(\mathcal{I}_{X_2}(k)) = 0$.

(a) Here we assume $n = 3$. We assume $k \ge 3$, because the case $k \le 1$ is trivial. We just prove the existence of X_2, since the existence of X_1 is similar (no numerical lemma is needed).

(a1) Assume $x \geq 2$. Set $a := \lfloor \binom{x+1}{3}/(x-1) \rfloor$ and $b := \binom{x+1}{3} - (x-1)a$. Fix a smooth quadric surface Q. By [10] we may assume $z > 0$. We use this assumption to avoid any use of H'_{3k-1} as in [10], §2 (the approach of [10] would work very well here, too). Let $Y \subset \mathbb{P}^3$ be a general union of a disjoint lines. Hence $Q \cap Y$ is a general union of $2a$ points. Fix a general $P \in Q$. Set $\Pi(P,Q)_0 = \emptyset$ and $\Pi(P,Q)_1 := \{P\}$. Let $\Pi(P,Q)_2$ be a general tangent vector of Q with P as its reduction. For a general Y and (case $b = 2$) a general tangent vector, we have $h^i(\mathcal{I}_{Y \cup \Pi(P,Q)_b}(x-2)) = 0$, $i = 0,1$. Then we add in Q $d-a$ general lines of type $(1,0)$. We need to control the postulation of the linear system $|\mathcal{O}_Q(x-d+a,x)|$ with respect to some planar length 3 schemes. To apply [11], Proposition 4.1 and Theorem 7.2, we only need to avoid one case; it is sufficient to have either $d-a \leq x-2$ or $b \neq 0$. Then (if $x < k$) we continue as in part (b) of the proof of Lemma 4 adding a scheme Π_u, $u \in \{0,1,2\}$ at each step and using a plane in each inductive step.

(a2) Assume $x \leq 1$. If $x = 0$, then $d = 0$ and hence Lemma 11 is true by Lemma 5. If $x = 1$, then $d \leq 2$. If $d = 1$, then use Lemma 6. If $d = 2$, then add the lines in a smooth quadric Q at the last step of the inductive proof $k-2 \Leftrightarrow k$.

(b) Now assume $n \geq 4$. We use induction on n. The details are as in step (a), applying Lemma 5 in a hyperplane H $=\mathbb{P}^{n-1}$ instead of quoting [11]. □

Proof of Theorem 1. Let k be the minimal positive integer such that $\binom{m+k}{m} + (k+1)d + 3z \leq \binom{n+k}{k}$. The algebraic subset $\Sigma(n,m,d,z)$ of Hilb$(\mathbb{P}^n)$ parametrizing the set of all disjoint unions in $\mathbb{P}^n$ of an m-dimensional linear space, d lines and z planar length 3 schemes is irreducible. Thus the semicontinuity theorem for cohomology and Castelnuovo-Mumford's Lemma show that it is sufficient to find $X_1, X_2 \in \Sigma(n,m,d)$ such that $h^0(\mathcal{I}_{X_1}(k-1)) = 0$ and $h^1(\mathcal{I}_{X_2}(k)) = 0$. We only prove the existence of X_2. Fix a hyperplane $H \subset \mathbb{P}^n$. Set $\Pi_0 := \emptyset$. Let Π_1 be a general point of H. Let Π_2 be a general tangent vector of H. Here "general" means "general with respect to some previous choice (e.g. general after fixing a scheme $Y_1 \subset \mathbb{P}^n$ and a scheme $Y_2 \subset H$)". Let x be the minimal positive integer such that $\binom{m+x}{m} + (x+1)d \leq \binom{n+x}{n}$. Obviously $1 \leq x \leq k$. For each integer y such that $x \leq y \leq k$ define the integers z_y and e_y by the relations

$$\binom{m+y}{m} + d(y+1) + 3z_y + e_y = \binom{n+y}{n}, \quad 0 \leq e_y \leq 2. \tag{7}$$

For each integer y such that $x \leq y \leq k$ we define the following Assertion A_y:

Assertion A_y. *Let $Y \subset \mathbb{P}^n$ be a general disjoint union of V, d lines, z_y planar length 3 subschemes and a general Π_{e_y}. Then $h^i(\mathcal{I}_Y(y)) = 0$, $i = 0, 1$.*

Notice that if Y is a disjoint union of V, d lines, z_y planar length 3 subschemes and a scheme Π_{e_y}, then $h^0(Y, \mathcal{O}_Y(y)) =$ fo.r all $i > 0$ and $h^0(Y, \mathcal{O}_Y(y)) = \binom{n+y}{n}$. Thus $h^1(\mathcal{I}_Y(y)) = h^0(\mathcal{I}_Y(y))$. Thus to prove Assertion A_y it is sufficient to find Y satisfying either $h^1(\mathcal{I}_Y(y)) = 0$ or $h^0(\mathcal{I}_Y(y)) = 0$. Notice that if A_k is true, then Theorem 1 is true. Notice that $u_{n,m,x-1} \le d$.

(a) Here we assume $x \ge 2$ and $u_{n,m,x-1} - 2v_{n,m,x-1} \ge x + 3$ and $d \ge k + 3$. Take a general solution $V \sqcup B \sqcup D$ of $M_{n,m,x-1}$ (Lemma 10). For each $P \in \mathrm{Sing}(D)$ let H_P be a general 3-dimensional linear subspace containing the connected component of D containing P. Set $\chi := \cup_{P \in \mathrm{Sing}(D)} \chi_{H_P}(P)$. Let $E \subset H$ be a general union of $d - u_{n,m,x-1}$ lines, z_x planar length 3 subschemes and Π_{e_x}. The scheme $Y_x := V \cup B \cup D \cup \chi \cup E$ is a flat degeneration of a family of disjoint unions of V, d lines, z_y planar length 3 subschemes and a schemel Π_{e_x}. Hence to prove the Assertion A_x it is sufficient to prove $h^1(\mathcal{I}_{Y_x}(x)) = 0$. Since $\mathrm{Res}_H(Y_x) = V \cup B \cup D$, it is suficient to prove $h^1(H, \mathcal{I}_{Y_x \cap H}(x)) = 0$. In H we have the $(m-1)$-dimensional linear subspace, some general lines, some general planar length 3 subschemes, Π_{e_x} and $u_{n,m,x-1} - 2v_{n,m,x-1}$. The union T of these lines has maximal rank and hence $h^1(H, \mathcal{I}_T(x)) = 0$. Since $u_{n,m,x-1} - 2v_{n,m,x-1} \ge k + 3 \ge x + 3$, we may apply Lemma 3 and get that the union T' of T and these planar length 3 subschemes satisfies $h^1(\mathcal{I}_{T'}(x))$. Then we apply Lemma 1 to handle Π_{e_x} if $e_x = 2$ and get $h^1(H, \mathcal{I}_{Y_x \cap H}(x)) = 0$. Then we conclude proving $A_y \Longrightarrow A_{y+1}$ for all $x \le y \le k - 1$ as in step (b) of the proof of Lemma 4 to handle Π_{e_y} if $e_y > 0$; in this step we may again apply Lemma 3, because here we have d general points and $d \ge k + 3 \ge (y + 1) + 3$. Taking away Π_{e_k} and $z_k - z$ planar length 3 subschemes from a solution of A_k we get $X_2 \in \Sigma(n, m, d, z)$ such that $h^1(\mathcal{I}_{X_2}(k)) = 0$. The existence of X_1 is proved starting from A_{k-1} in the same way.

(b) Here we assume $x \ge 2$, $u_{n,m,x-1} - 2v_{n,m,x-1} \ge x + 3$ and $d \le k + 2$. As in step (a) we get A_y for all $x \le y \le d - 3$. Take $V \cup B \cup Z_{d-2} \cup \Pi_{e_{d-2}}$ satisfying A_{d-2}. From this point on we use Lemma 6 In H with respect to the integer $n' := n - 1$, $m' := m - 1$ and the $(m-1)$-dimensional linear space $V \cap H$.

(c) Here we assume $x \ge 2$ and $u_{n,m,x-1} - 2v_{n,m,x-1} \le x + 2$. By [10] either $(n, m) = (4, 2)$ and $x \le 5$ or $(n, m) = (5, 3)$ and $x \le 3$ or $x \ne 2$. In all cases it would be sufficient to prove A_x, because all the implications $A_y \Longrightarrow A_{y+1}$, $x \le y \le k - 1$, are proved as in step (b). First assume $x = 2$. We have $v_{n,m,1} = 0$ if $n \equiv m \pmod 2$ and $v_{n,m,1} = 1$ if $n \equiv m + 1 \pmod 2$. Thus inequality $u_{n,m,1} - 2v_{n,m,1} \le 4$ is true if and only if either $n \le m + 8$ and $n - m$

is even or $n \le m+7$ and $n-m$ is odd. In all these cases and in the remaining cases, we have $d \le n_{n,m,k} - n_{n,m,k-1}$. Hence we may first apply Lemma 8 for the critical value $k' := k-1$ and only the last step $k-1 \Longrightarrow k$ we add the lines. Indeed, to prove $h^1(\mathcal{I}_{X_2}(k)) = 0$ we may increase z to a new integer $z' \ge z$ such that

$$\binom{n+k}{n} - 2 \le \binom{m+k}{m} + (k+1)d + 3z' \le \binom{n+k}{n}.$$

With this new z' all lines may be inserted at the last step by Lemma 8.

(d) Here we assume $x = 1$. Since A_1 is obviously true, we use Lemma 6 $k-1$ times as in step (b). □

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References

[1] J. Alexander, Singularités imposables en position générale aux hypersurfaces de $\mathbb{P}^n$, *Compositio Math.*, **68** (1988), 305-354.

[2] J. Alexander, A. Hirschowitz, Un lemme d'Horace différentiel: application aux singularité hyperquartiques de $\mathbb{P}^5$, *J. Alg. Geom.*, **1** (1992), 411-426.

[3] J. Alexander, A. Hirschowitz, La méthode d'Horace éclaté: application à l'interpolation en degré quatre, *Invent. Math.*, **107** (1992), 585-602.

[4] J. Alexander, A. Hirschowitz, Polynomial interpolation in several variables, *J. Alg. Geom.*, **4** (1995), 201-222.

[5] J. Alexander, A. Hirschowitz, An asymptotic vanishing theorem for generic unions of multiple points, *Invent. Math.*, **140** (2000), 303-325.

[6] E. Ballico, Postulation of disjoint unions of lines and a few planes, *Preprint.*

[7] E. Carlini, M.V. Catalisano, A.V. Geramita, Subspace arrangements, configurations of linear spaces and quadrics containing them, *ArXiv:* 0909.3821 v2[math.AG].

[8] E. Carlini, M.V. Catalisano, A.V. Geramita, Bipolynomial Hilbert functions, *ArXiv:* 0910.3569 [math.AG].

[9] K. Chandler, A brief proof of a maximal rank theorem for generic double points in projective space, *Trans. Amer. Math. Soc.*, **353**, No. 5 (2000), 1907-1920.

[10] R. Hartshorne, A. Hirschowitz, Droites en position générale dans $\mathbb{P}^n$, *Algebraic Geometry, Proceedings, La Rábida* (1981), 169-188; *Lect. Notes in Math.*, **961**, Springer, Berlin (1982).

[11] A. Laface, On linear systems of curves on rational scrolls, *Geom. Dedicata*, **90** (2002), 127-144; *ArXiv:* 025271v2 [math.AG].