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THE DIRICHLET PROBLEM FOR THE STOKES SYSTEM AND THE INTEGRAL EQUATIONS' METHOD

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Abstract: A boundary value problem for the Stokes system is studied in a cracked domain in R^n, $n > 2$, where the Dirichlet condition is specified on the boundary of the domain. The jump of the velocity and the jump of the stress tensor in the normal direction are prescribed on the crack. We construct a solution of this problem in the form of appropriate potentials and determine the unknown source densities via Fredholm integral equations' systems of the second kind on the boundary of the domain. Here the boundary value problem

$$-\Delta\vec{u} + \nabla p = 0 \quad \text{in} \quad G, \qquad \nabla \cdot \vec{u} = 0 \quad \text{in} \quad G, \qquad \vec{u} = \vec{b} \quad \text{on} \quad \partial G,$$

for the Stokes system plays an important role. The solution of which is given explicitly in the form of a series. As a consequence, also a maximum modulus estimate for the Stokes system can be proved.

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1. The Dirichlet Problem for the Stokes System

Let $G \subset R^n$ $(n \geq 3)$ be a domain with bounded boundary ∂G of class $C^{1,\alpha}$

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with $\alpha > 0$. The boundary of G might be disconnected. We study the Dirichlet problem for the Stokes system in G with prescribed boundary value $\vec{b}$:

$$-\Delta \vec{u} + \nabla p = 0 \quad \text{in} \quad G, \qquad \nabla \cdot \vec{u} = 0 \quad \text{in} \quad G, \qquad \vec{u} = \vec{b} \quad \text{on} \quad \partial G.$$

If the domain G is unbounded we require, moreover, the following decay conditions as $|x| \to \infty$:

$$\vec{u}(x) = O(|x|^{2-n}), \qquad |\nabla \vec{u}(x)| = O(|x|^{1-n}),$$
$$p(x) = O(|x|^{1-n}), \qquad |\nabla p(x)| = O(|x|^{-n}).$$

We study classical solutions of the above problem for a prescribed continuous boundary value $\vec{b}$ as well as L^q-solutions if the boundary value only satisfies $\vec{b} \in L^q(\partial G)$, $2 \le q < \infty$.

Let us suppose that for each $x \in \partial G$ the open cones $\Gamma(x)$, $\Gamma'(x)$ with vertex x are given such that

$$\overline{\Gamma(x)} \setminus \{x\} \subset \Gamma'(x) \subset G.$$

Suppose, moreover, that $\Gamma(x)$ is the displacement of a fixed cone Γ, and that $\Gamma'(x)$ is the displacement of a fixed cone Γ'. If now $\vec{u}$ is a vector function defined in G and $x \in \partial G$, then we call

$$x \to u^*(x) = \sup_{y \in \Gamma(x)} |\vec{u}(y)|$$

the non-tangential maximal function u^* of $\vec{u}$ and

$$x \to \vec{u}(x) = \lim_{y \to x,\, y \in \Gamma(x)} \vec{u}(y)$$

the non-tangential limit of $\vec{u}$ (see Medková, Varnhorn [3]).

Now let $2 \le q < \infty$ and $\vec{b} \in L^q(\partial G)$. Then we call $(\vec{u}, p)$ an L^q-solution of the Dirichlet problem for the Stokes system in G with prescribed boundary value $\vec{b}$ if $\vec{u} \in C^2(G)$, $p \in C^1(G)$, and

$$-\Delta \vec{u} + \nabla p = 0 \quad \text{in} \quad G, \quad \nabla \cdot \vec{u} = 0 \quad \text{in} \quad G,$$
$$u^* \in L^q(\partial G),$$
$$\vec{b}(x) = \lim_{y \to x, y \in \Gamma(x)} \vec{u}(y) \qquad \text{for a. a. } x \in \partial G.$$

If G is unbounded we require, moreover, the above mentioned decay conditions

$$\vec{u}(x) = O(|x|^{2-n}),\ |\nabla \vec{u}(x)| = O(|x|^{1-n}),$$
$$p(x) = O(|x|^{1-n}),\ |\nabla p(x)| = O(|x|^{-n})$$

as $|x| \to \infty$.

Theorem 1. *Let $2 \leq q < \infty$. If G is unbounded, then there is a unique L^q-solution of the Dirichlet problem for each boundary value $\vec{b} \in L^q(\partial G)$. If G is bounded, then there is an L^q-solution $(\vec{u}, p)$ of the Dirichlet problem with prescribed boundary value $\vec{b}$ if and only if*

$$\int_{\partial G} \vec{b} \; do = 0. \tag{1}$$

The velocity vector $\vec{u}$ is unique, while the pressure function p is determined uniquely up to an additive constant, only. If $\vec{b} \in C(\partial G)$, then any L^q-solution is also a classical solution.

We can prove this result using the method of integral equations (see, e.g. Varnhorn [4], [5]). If G is a bounded domain with a connected boundary, then this result is well-known. In this case the solution can be represented in the form of a hydrodynamical double layer potential, and the correct boundary condition is proved to be satisfied by the integral equations' method. If G is not a bounded domain with connected boundary, then the solution of the Dirichlet problem for the Stokes system has not the form of a pure hydrodynamical double layer potential. So we need to look for a modified representation of the solution such that the corresponding integral equations can be solved. In this way we are able not only to prove the existence of the solution of the integral equations but also to construct the solution explicitly.

To do so, let us introduce the Stokes fundamental tensor

$$E_{jk}(x) = \frac{1}{2\omega_n}\Big\{\delta_{jk}\,\frac{|x|^{2-n}}{n-2} + \frac{x_j x_k}{|x|^n}\Big\} \qquad j,k = 1,\dots,n,$$

$$E_{n+1,k}(x) = \frac{x_k}{\omega_n |x|^n}, \qquad k = 1,\dots,n.$$

Here ω_n stands for the surface area of the $(n-1)$-dimensional unit sphere in R^n. Then we call

$$E_{\partial G}\vec{\varphi}(x) = (E^{\bullet}_{\partial G}\vec{\varphi}(x), [E_{\partial G}\vec{\varphi}(x)]_{n+1})^T = \int_{\partial G} E(x-y)\vec{\varphi}(y)\; do_y$$

the hydrodynamical single layer potential with density $\vec{\varphi}$. Set

$$D_{ki}(x,y) = -\frac{n}{\omega_n}\,\frac{z_k z_i z \cdot \vec{N}}{|z|^{n+2}}, \qquad k,i = 1,\dots,n,$$

$$D_{n+1,i}(x,y) = -\frac{2}{\omega_n}\left\{ n\,\frac{z_i z\cdot\vec{N}}{|z|^{n+2}} - \frac{N_i}{|z|^n}\right\}, \qquad i = 1,\dots,n,$$

where $z = x - y$ and $\vec{N} = \vec{N}(y)$ is the outward (with respect to G) unit normal vector in $y \in \partial G$. Then we call

$$D_{\partial G}\vec{\varphi}(x) = (D^{\bullet}_{\partial G}\vec{\varphi}(x), [D_{\partial G}\vec{\varphi}(x)]_{n+1})^T = \int_{\partial G} D(x,y)\vec{\varphi}(y)\, do_y$$

the hydrodynamical double layer potential with density $\vec{\varphi}$.

Now let η be some nonnegative constant. If G is a bounded domain with a connected boundary, then η might be equal to zero, in all other cases η must be positive. Define

$$(\vec{u}, p)^T = D_{\partial G}\vec{\varphi} + \eta E_{\partial G}\vec{\varphi}.$$

Then $(\vec{u}, p)$ satisfy the Stokes differential equations and, moreover, represent a solution of the Dirichlet problem for the Stokes system in G with prescribed boundary value $\vec{b}$ if and only if the boundary integral equations

$$\frac{1}{2}\vec{\varphi} + D^{\bullet}_{\partial G}\vec{\varphi} + \eta E^{\bullet}_{\partial G}\vec{\varphi} = \vec{b} \tag{2}$$

are satisfied. Here we require $\vec{\varphi}$, $\vec{b} \in C(\partial G)$ for classical solutions and $\vec{\varphi}$, $\vec{b} \in L^q(\partial G)$ for L^q-solutions.

If G is unbounded, then the integral equations (2) have a unique solution for arbitrary $\vec{b}$. If $\vec{b} \in C(\partial G)$, then we have $\vec{\varphi} \in C(\partial G)$, and if $\vec{b} \in L^q(\partial G)$, $q \geq 2$, then we have $\vec{\varphi} \in L^q(\partial G)$. But if G is bounded, then the integral equation (2) is solvable if and only if the condition (1) holds. In this case, a solution of the integral equations (2) is not uniquely determined. Instead of the integral equations (2) we will study a uniquely solvable integral equations' system, the solution of which solves also the equations (2) provided only that the condition (1) holds true.

To do so, set

$$V\vec{\varphi} := \left(\frac{1}{|\partial G|}\int_{\partial G} \vec{N}\cdot\vec{\varphi}\, do\right)\vec{N},$$

$$L\vec{\varphi} := \frac{1}{2}\vec{\varphi} + D^{\bullet}_{\partial G}\vec{\varphi} + \eta E^{\bullet}_{\partial G}\vec{\varphi} + aV\vec{\varphi}.$$

Here we assume $a = 0$ for unbounded G and $a > 0$, if G is bounded. In this case the equation $L\vec{\varphi} = \vec{b}$ is uniquely solvable. If the condition (1) is satisfied, then the solution $\vec{\varphi}$ of $L\vec{\varphi} = \vec{b}$ is also a solution of the integral equations (2). Thus, if we can solve the integral equations $L\vec{\varphi} = \vec{b}$, we can also solve the Dirichlet problem for the Stokes system in G.

Define

$$A_0 := \max_{j=1,\ldots,n} \sup_{x\in\partial G} \int_{\partial G} \sum_{k=1}^{n} |E_{jk}(x-y)|\, do_y.$$

If M is some positive constant with

$$\int_{\{y\in\partial G; |x-y|<r\}} 1 \ do \ \le \ Mr^{n-1}$$

for each $x \in \partial G$ and $0 < r < \operatorname{diam} \partial G$, then

$$A_0 \le \frac{n^2}{2\omega_n} M \operatorname{diam} \partial G.$$

Here $\operatorname{diam} K = \sup\{|x-y|;\ x, y \in K\}$ is the diameter of K.

Theorem 2. *Fix $\gamma > (1 + \eta A_0 + a)/2$. Then the solution $\vec{\varphi}$ of $L\vec{\varphi} = \vec{b}$ is given by the series*

$$\vec{\varphi} \ = \ \gamma^{-1} \sum_{k=0}^{\infty} (I - \gamma^{-1} L)^k \, \vec{b}.$$

Moreover,

$$\|(I - \gamma^{-1} L)^k\| \ \le \ d\, t^k,$$

where the constants $d \in [1, \infty)$, $t \in (0,1)$ do not depend on $\vec{b}$. This holds true in $L^q(\partial G)$ and also in $C(\partial G)$, where the constants d, t depend on the choice of the space.

2. Successive Approximations

We can also calculate the solution $\vec{\varphi}$ of the equation $L\vec{\varphi} = \vec{b}$ using the method of successive approximations. To do so we fix $\gamma > (1 + \eta A_0 + a)/2$. Then we rewrite the equation $L\vec{\varphi} = \vec{b}$ in the form

$$\vec{\varphi} = (I - \gamma^{-1} L)\vec{\varphi} + \gamma^{-1}\vec{b}.$$

Fix an initial value $\vec{\varphi}_0$ as follows: If $\vec{b} \in C(\partial G)$, then choose $\vec{\varphi}_0 \in C(\partial G)$, and if $\vec{b} \in L^q(\partial G)$, $2 \le q < \infty$, then choose $\vec{\varphi}_0 \in L^q(\partial G)$.) Now define

$$\vec{\varphi}_{k+1} = (I - \gamma^{-1} L)\vec{\varphi}_k + \gamma^{-1}\vec{b}.$$

Then we obtain $\vec{\varphi}_k \to \vec{\varphi}$ as $k \to \infty$ in the corresponding space, where $\vec{\varphi}$ is the solution of $L\vec{\varphi} = \vec{b}$. Moreover,

$$\|\vec{\varphi} - \vec{\varphi}_k\| \ \le \ c\, t^k \|\vec{\varphi}_1 - \vec{\varphi}_0\|,$$

where the constants $c \in \langle 1, \infty)$, $t \in (0,1)$ do not depend on $\vec{b}$ and $\vec{\varphi}_0$.

3. Domains with Cracks

Besides the Dirichlet problem for the Stokes system we also study a crack problem as follows. Suppose that V^+ is a nonempty bounded open set with boundary of class $C^{2+\alpha}$, $\alpha > 0$. Let $S \subset \partial V^+ \cap \overline{G}$ be a nonempty compact set such that $\overline{G \cap S} = S$. Finally, define $V^- = R^n \setminus \overline{V^+}$.

Assume now $\vec{g} \in C^{1+\gamma}(\partial V^+)$ and $\vec{h} \in C^{\gamma}(\partial V^+)$ where $0 < \gamma < \alpha$ and $\vec{g} = \vec{h} = 0$ on $G \cap \partial V^+ \setminus S$. Suppose, moreover, $\vec{f} \in C^0(\partial G \setminus S)$ is continuously extendible to $\overline{V^+} \cap \partial G$ and $\overline{V^-} \cap \partial G$, and $\vec{f}^+ - \vec{f}^- = \vec{g}$ on $S \cap \partial G$. Here by the superscripts "+" and "-" we denote the limiting values of functions on S with respect to V^+ and V^-, respectively. We can suppose, moreover, that the supports of $\vec{g}$ and $\vec{h}$ are compact subsets of ∂V^+. Under theses assumptions we consider the following problem:

Find $(\vec{v}, q) \in C^\infty(G \setminus S)$ such that $\vec{v} \in C^0(\overline{G} \setminus S)$ is continuously extendible to $\overline{V^+ \cap G}$ and $\overline{V^- \cap G}$, and such that $\nabla \vec{v}$, q are continuously extendible to $\overline{V^+} \cap G$ and $\overline{V^-} \cap G$, with

$$-\Delta \vec{v} + \nabla q = 0 \quad \text{in} \quad G \setminus S, \quad \nabla \cdot \vec{v} = 0 \quad \text{in} \quad G \setminus S, \quad \vec{v} = \vec{f} \quad \text{on} \quad \partial G,$$

$$\vec{v}^+ - \vec{v}^- = \vec{g}, \quad (T^v_q N)^+ - (T^v_q N)^- = \vec{h} \qquad \text{on } S \cap G.$$

Here

$$T^v_q = -\nabla \vec{v} - (\nabla \vec{v})^T - qI$$

denotes the stress tensor. If G is unbounded we require, moreover,

$$\vec{v}(x) = O(|x|^{2-n}), \ |\nabla \vec{v}(x)| = O(|x|^{1-n}),$$

$$q(x) = O(|x|^{1-n}), \ |\nabla q(x)| = O(|x|^{-n})$$

as $|x| \to \infty$.

If we set $(\vec{v}, q) = (\vec{u}, p) + D_S g - S_S h$, then $(\vec{v}, q)$ is a solution of the above crack problem if and only if $(\vec{u}, p)$ is a solution of the Dirichlet problem for the Stokes system in G with the boundary condition $\vec{b} = f - D_S g + S_S h$. Here $D_S g$ and $S_S h$ denote the double layer potential and the single layer potential corresponding to the surface ∂V^+. To solve this problem we can use the results described above (see Krutitskii, Medková [1] and Medková [2]).

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