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PROPERTIES OF MATRIX SOLUTIONS OF NONLINEAR WAVE EQUATIONS

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Abstract: Various properties of matrix solutions of nonlinear wave equations are considered. The one-to-one correspondence between a set of matrix solutions and a unit sphere in the many-dimensional space are proved. The helical strings described by matrix solutions are used for modeling the particle collisions and also for demonstration of some geometrical properties of superstrings. It is shown that great number of triplex forms can be defined using matrix solutions.

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1. Introduction

In the present paper we consider various properties of matrix solutions for nonlinear wave equations, constructed in our papers [5] and [6]. We show that the matrix solutions describe a set of points that is equivalent to a unit sphere in the many-dimensional vector space, and also show how matrix solutions can be extend beyond this vector space. The application of matrix solutions is emphasized for the describing some geometrical properties of the superstrings, represented by Green et al [3], Greene [4], and Escultura [2]. Matrix solutions are also used for demonstration some results of Atsukovsky [1]; in particular, we propose the existence of 128 elements in the periodic system of chemical

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elements. Another application concerns the modeling of triplex forms described by Seklitova and Strelnikova [7], and modeling of the passage of these triplex forms from one level to a higher one. We also describe pulsating transmission along a chain of unit spheres.

In our paper [5] the nonlinear Klein-Gordon equation

$$\frac{\partial^2 u}{\partial t^2} - \Delta u + \frac{dQ}{du} = 0, \qquad Q(u) = \frac{\mu^2}{4}(u^2-1)^2 \tag{1}$$

was considered with matrix solutions for equation (1) constructed as

$$u_n(\phi, \mathbf{a}) = \cos(\phi)E_n + \mathbf{a}\sin(\phi) = \exp(\phi\mathbf{a}), \quad \mathbf{a} = \sum_{j=1}^{2k+1} a_j M_j, \tag{2}$$

where $\sum_{j=1}^{2k+1} a_j^2 = 1$. We also use functions $u_n(m\phi, \mathbf{a})$ which satisfy equation (1) with more complicated potentials Q_m (see [5]).

Angular parameter ϕ is defined as

$$\phi \equiv \phi(\alpha z) = \operatorname{arccot}(-\sinh(\alpha z)), \qquad \alpha = \mu\sqrt{2/(1-v^2)}, \tag{3}$$

where $z = x - vt$ is the moving frame of reference, x is an arbitrary point along the chosen direction in 3D-space of coordinates x_1, x_2, x_3. Note that angular parameter ϕ for the nonlinear wave equations behaves as a kink-function of z with $0 \le \phi \le \pi$. For a linear wave equation the angular parameter is a linear function; for example $\phi = \alpha z$ both for solutions $u_n(\phi, \mathbf{a})$ of equation (1) with potential $Q_0 = -\mu^2 u^2 + Const$ and for solutions $u_n(m\phi, \mathbf{a})$ of equation (1) with potential $Q_{0m} = -m^2\mu^2 u^2 + Const$.

Other notation: E_n is the unit $n \times n$-matrix, M_j are unitary, $M_j M_j^* = E_n$ ($*$ means transposition and conjugation), anti-Hermitian, $M_j^* = -M_j$, anti-commuting $n \times n$-matrices.

Definition 1. Matrices $M_j(n) \equiv M_j$ of the order $n = 2^k$ $(k = 1, 2, \ldots)$ are sequentially constructed for $k = 2, 3, \ldots$ as follows:

$$M_j(n) = \begin{pmatrix} M_j(n/2) & 0 \\ 0 & -M_j(n/2) \end{pmatrix}, \qquad j = 1, 2, \ldots, 2k-1,$$

$$M_{2k}(n) = \begin{pmatrix} 0 & -E_{n/2} \\ E_{n/2} & 0 \end{pmatrix}, \quad M_{2k+1}(n) = \begin{pmatrix} 0 & iE_{n/2} \\ iE_{n/2} & 0 \end{pmatrix}$$

and $M_j(2) = H_j$ $(j = 1, 2, 3)$ for $k = 1$, where H_j are unit quaternions.

Matrices $M_1, M_2, \ldots, M_{2k}$ as many other systems of $2k$ matrices $M_j' = u_n M_j u_n^*$ are the matrix representation of $2k$ generators of associative 2^{2k}-dimensional Clifford algebra $Cl_{0,2k}$. In this paper, Clifford algebra $Cl_{0,2k}$ over reals R is generated by matrices $M_1, \ldots, M_{2k}$ and is considered as a 2^{2k}-dimensional vector space with basis

$$E_n, M_j, M_i M_j, \ldots, M_1 M_2 \ldots M_{2k}, \qquad i < j, \quad i, j = 1, 2, \ldots, 2k. \qquad (4)$$

Note that matrix M_{2k+1} from Definition 1 is equal, up to $\pm i$, to the product $M_1 M_2 \ldots M_{2k}$. There is no other product $M_{i_1} \ldots M_{i_m}$ $(1 < m < 2k, i_1 < \ldots < i_m)$ that anti-commutes with each of matrices M_j $(j = 1, 2, \ldots, 2k)$. Hence, the construction of matrix solutions $u_n \in Cl_{0,2k}$ of nonlinear wave equations is based precisely on the complete system of $2k+1$ linearly independent, anti-commuting, unitary, anti-Hermitian matrices M_j.

2. Rotational Property

We denote by V_{2k+1} the linear vector space generated by basis vectors

$$E_n, M_1, M_2, \ldots, M_{2k+1}, \qquad n = 2^k, \quad k = 1, 2, \ldots. \qquad (5)$$

Matrix solution $u_n(\phi, \mathbf{a})$ as a linear combination of basis vectors (5) belongs to the space V_{2k+1}. Matrix $u_n(\phi, \mathbf{a})$, being applied to vector $\mathbf{b} \in V_{2k+1}$ (orthogonal to $\mathbf{a}$), accomplishes a transformation in the form of rotation

$$\mathbf{b} \to \mathbf{b}' = u_n(\phi, \mathbf{a})\mathbf{b} = \mathbf{b} \cos\phi + \mathbf{ab} \sin\phi \qquad (6)$$

of vector $\mathbf{b}$ about vector $\mathbf{a}$ by angle ϕ in the plane of vectors $\mathbf{b}$ and $\mathbf{ab}$. Assume for simplicity that $\mathbf{b} = \sum_{j=1}^{2k+1} b_j M_j$ and $\sum_{j=1}^{2k+1} b_j^2 = 1$. Then we find

$$\mathbf{ab} = \sum_{j=1}^{2k+1} a_j b_j (-E_n) + \sum_{i<j}^{2k+1} (a_i b_j - a_j b_i) M_i M_j.$$

It is clear that the equality $\sum_{j}^{2k+1} a_j b_j = 0$ (orthogonality condition) leads to the relation $\mathbf{ab} = -\mathbf{ba}$, which allows us to show that $(\mathbf{ab})\mathbf{a} = -\mathbf{a^2 b} = \mathbf{b}$ and $\mathbf{b}(\mathbf{ab}) = -\mathbf{ab^2} = \mathbf{a}$, i.e. vectors $\mathbf{a}, \mathbf{b}, \mathbf{ab}$ form an orthogonal frame. Consider 3D-space of vectors $\mathbf{b}, \mathbf{ab}$ and z-axis, then the rotation of vector $\mathbf{b}$ describes a trace around z-axis in the form of helical curve, which we name helical string.

We can see that transformation (6) of vector $\mathbf{b}$ includes the pairwise products of basis vectors $M_1, \ldots, M_{2k+1}$ and, hence, vector $\mathbf{b}'$ belongs to a more wide space than V_{2k+1}, namely, to the space $Cl_{0,2k}$. Let us use the similarity transformation of an arbitrary vector $\mathbf{b} \in Cl_{0,2k}$ by matrix $u_n(\phi, \mathbf{a})$. First, we will denote the products of basis vectors M_j $(j = 1, \ldots, 2k)$ by $M_\nu = M_{i_1} \ldots M_{i_m}$, where $\nu = \{i_1, i_2, \ldots, i_m\}$ with $1 \leq i_1 < i_2 < \ldots < i_m \leq 2k$. Let there be $\mathbf{b} = \sum_\nu b_\nu M_\nu$, then for fixed $\mathbf{a} = M_m$ $(m = 1, \ldots, 2k+1)$ we will denote by M_ν^- vector M_ν of basis (4) that anti-commutes with M_m, and by M_ν^+ – the vector M_ν that commutes with M_m.

Theorem 1. *For vector* $\mathbf{b} = \sum_\nu b_\nu M_\nu$ *and given vector* $\mathbf{a} = M_m$ $(m = 1, \dots, 2k+1)$ *the following transformation is valid:*

$$\mathbf{b} \to u_n \mathbf{b} u_n^* = u_n(2\phi, M_m) \sum_\nu b_\nu M_\nu^- + \sum_\nu b_\nu M_\nu^+, \tag{7}$$

where the number of elements M_ν^- *is equal to the number of elements* M_ν^+.

Proof. It follows from the definition that

$$M_\nu^- \to u_n(\phi, M_m) M_\nu^- u_n^*(\phi, M_m) = u_n(2\phi, M_m) M_\nu^-$$

and

$$M_\nu^+ \to u_n(\phi, M_m) M_\nu^+ u_n^*(\phi, M_m) = M_\nu^+.$$

Then, due to $\mathbf{b} = \sum b_\nu M_\nu^- + \sum b_\nu M_\nu^+$, the required formula is valid. Now let us write the formulas, which describe elements M_ν^- and M_ν^+. By sequential permutations the following equalities are easily verified: for $m = 1, 2, \dots, 2k$

$$M_m(M_{i_1} \dots M_{i_{2s}}) = (-1)^{2s-1}(M_{i_1} \dots M_{i_{2s}}) M_m \text{ with } m \in \{i_1, \dots, i_{2s}\}, \tag{8}$$

$$M_m(M_{i_1} \dots M_{i_{2s+1}}) = (-1)^{2s+1}(M_{i_1} \dots M_{i_{2s+1}}) M_m, \ m \notin \{i_1, \dots, i_{2s+1}\}. \tag{9}$$

Thus all M_ν in (8) and (9) are M_ν^-. It is evident that other M_ν, namely: $M_{i_1} \dots M_{i_{2s}}$ with $m \notin \{i_1, \dots, i_{2s}\}$ and $M_{i_1} \dots M_{i_{2s+1}}$ with $m \in \{i_1, \dots, i_{2s+1}\}$ commute with M_m and, hence, these M_ν are M_ν^+.

In the case $m = 2k+1$, since M_{2k+1} is equal, up to $\pm i$, to the product $M_1 \dots M_{2k}$, we have

$$M_{2k+1}(M_{i_1} \dots M_{i_s}) = (-1)^s (M_{i_1} \dots M_{i_s}) M_{2k+1}. \tag{10}$$

The sign in the right side of (10) depends only on the number of factors in M_ν. Thus product M_ν with the odd number s of factors in (10) is M_ν^-, while product M_ν with the even number s is M_ν^+.

The number of the products M_ν with M_m $(m = 1, \dots, 2k)$ appearing as factor is equal to 2^{2k-1}, the number of the products M_ν without M_m is also equal to 2^{2k-1}. The same quantities concern also to the products with the odd only or the even only numbers of factors. Thus, taking into account equalities (8-10), we state that the number of elements M_ν^- as well as the number of elements M_ν^+ is equal to 2^{2k-1}. The theorem is proved. □

3. Properties of Matrix Solutions

In this section we show the correspondence between a set of matrix solutions u_n and the unit sphere S^{2k+1} in the vector space V_{2k+1}. Then we will show how the corresponding conditions can be found for construction matrix solutions outside of the space V_{2k+1} in more wide space $Cl_{0,2k}$.

Theorem 2. *There is one-to-one correspondence between unit sphere S^{2k+1} in the vector space V_{2k+1} and a set of matrix solutions $u_n(\phi, \mathbf{a})$ for all parameters $\phi \in [0, \pi]$ and all unit vectors $\mathbf{a}$.*

Proof. An arbitrary point $(x_0, x_1, \ldots, x_{2k+1}) \in S^{2k+1}$ satisfies an equality

$$x_0^2 + x_1^2 + \ldots + x_{2k+1}^2 = 1. \tag{11}$$

Let us compose the following relations

$$x_0 = \cos\phi, \quad x_j = a_j \sin\phi, \quad j = 1, 2, \ldots, 2k+1. \tag{12}$$

It is clear that for each $\phi \in [0, \pi]$ and each unit vector $\mathbf{a}$ there exists only one point $(x_0, x_1, \ldots, x_{2k+1})$ defined by relations (12) and satisfying relation (11), i.e. the point on the sphere S^{2k+1}. On the other hand for an arbitrary point $(x_0, x_1, \ldots, x_{2k+1}) \in S^{2k+1}$ with $x_0 \neq \pm 1$ there exists only one pair $(\phi, \mathbf{a})$ with $\phi = \arccos x_0$ and $a_j = x_j / \sin\phi$, which is fit for the matrix solution $u_n(\phi, \mathbf{a})$. The case $x_0 = 1$ or $x_0 = -1$ gives $\phi = 0$ or $\phi = \pi$, respectively, and, hence, the matrix solution should be taken as $u_n = E_n$ or $u_n = -E_n$, respectively. The theorem is proved. □

Thus, if we speak about sphere S^{2k+1}, we mention that this sphere is filled with the set of matrix solution $u_n(\phi, \mathbf{a})$. Since any point in the space V_{2k+1} can be taken as the center of a unit sphere, we can state that matrix solutions act in the whole vector space V_{2k+1}.

The set of all $n \times n$-matrices forms n^2-dimensional matrix space, which is one of the representations of Clifford algebra $Cl_{0,2k}$. Vector space V_{2k+1}, is a subspace of $Cl_{0,2k}$. Now we can show that in the remaining larger part of $Cl_{0,2k}$ basis vectors, which satisfy necessary conditions for the construction matrix solutions, can also be chosen. The required conditions are the following:

$$M_\nu^2 = -E_n, \quad M_\mu^2 = -E_n, \quad M_\nu M_\mu = -M_\mu M_\nu. \tag{13}$$

In the consequent theorems we use positive integers l and m for the numbering the factors in M_ν and M_μ.

Theorem 3. *If matrices M_ν and M_μ are products of $4m+1$ and $4l+1$ factors, respectively, without common factors, then required conditions (13) are valid.*

Proof. Taking into account that all generators M_j $(j = 1, 2, \dots, 2k)$ are anti-commuting and assuming that $s = 4m + 1$ and $\nu = \{i_1, i_2, \dots, i_s\}$ we find that

$$M_\nu M_\nu = (-1)^{s(s-1)/2}(-E_n)^s = (-1)^{(4m+1)2m}(-E_n) = -E_n.$$

In the same way can be proved the equality $M_\mu^2 = -E_n$, provided $\mu = \{j_1, j_2, \dots, j_t\}$, where $t = 4l + 1$. Now by sequential permutation we find

$$M_\nu M_\mu = (-1)^{st} M_\mu M_\nu = -M_\mu M_\nu. \tag{14}$$

Thus M_ν and M_μ are anti-commuting matrices. The theorem is proved. □

Theorem 4. *If matrices M_ν and M_μ are products of $4m + 1$ and $4l + 1$ factors, respectively, and they contain even number of common factors, then M_ν and M_μ are anti-commuting matrices.*

Proof. In the process of permutation it does not need to transpose common factor, say, M_j from M_ν over the same factor M_j in the M_μ, due to the own factor M_j of M_μ accomplishes the further movement. It looks as relay-race. Thus, one common factor decreases the number of (-1)-st on one unit in the process (14) of permutation. Two common factors in this process decreases the number of (-1)-st on two units, i.e. property of being odd is preserved for two common factors and is violated for one common factor. It is clear that even number of common factors also preserves the property of being odd for number of multipliers (-1) in the process (14) of permutation. The theorem is proved. □

Theorem 5. *If matrices M_ν and M_μ are products of $4m + 2$ (including $m = 0$) and $4l+1$ factors, respectively, and they contain odd number of common factors, then M_ν and M_μ are anti-commuting matrices and $M_\nu^2 = -E_n$.*

Proof. First, we show that $M_\nu^2 = -E_n$ for the number $4m + 2$ of factors. Indeed, by sequential permutation as in (14) we find

$$M_\nu M_\nu = (-1)^{(2m+1)(4m+1)}(-E_n)^{4m+2} = -E_n.$$

Now, let $2r + 1$ be an odd number of common factors, $2s = 4m + 2$, and $t = 4m + 1$. Then, due to $2s > 2r + 1$, we find

$$M_\nu M_\mu = (-1)^{t(2s-2r-1)} M_\mu M_\nu = -M_\mu M_\nu.$$

The theorem is proved. □

Corollary 1. *The following chain of basis vectors*

$$M_1 M_2 \dots M_{2j}, \quad M_1 M_\nu, \quad M_2 M_\mu, \dots, M_{2j} M_\kappa, \tag{15}$$

where $j = 2m + 1$ and all matrices $M_\nu, M_\mu, \dots, M_\kappa$ are the products of even *number of factors without common factors and without any of the factors $M_1, M_2, \ldots, M_{2j}$, satisfies the conditions required for construction matrix solutions u_n.*

It is clear that the number of basis vectors in the chain (15) is equal to $2j+1$. For example, if each matrix $M_1 M_\nu$, $M_2 M_\mu$ and so up to $M_{2j} M_\kappa$ is a product of $4m+1$ factors, then since these matrices are without common factors, the number of used factors here is equal to $2j(4m+1)$. This number is less than or equal to the number $2k$ of generators. Thus the chain (15) contains not more than $2k/(4m+1)+1$ terms.

4. Matrix Solution in the Form of Triplex

Let us take an arbitrary point in the space V_{2k+1} and construct unit sphere S^{2k+1} around this point as about the origin of local frame of reference. As follows from Definition 1, matrices M_j for $j = 1, 2, \ldots, 2k-1$ are diagonal ones, and the corresponding solution u_n also has diagonal form $u_n = \mathrm{diag}(u_{n/2}, u^*_{n/2})$, which can be represented as the right sum $u_n = u_{n/2} \oplus u^*_{n/2}$.

Note that matrix $u_{n/2}$ of order $n/2$ is generated based on matrices $M_1(n/2)$, $\ldots, M_{2k-1}(n/2)$, which are not commensurable with basis vectors $M_j(n)$ of n-th order. This means that vector space V_{2k-1} is not a subspace of V_{2k+1}, but is in the domain of other dimensions. For obviousness, we can represent vector space V_{2k-1} as a branch of a big tree V_{2k+1}. If we consider $u_{n/2}$ with components $a_j \geq 0$ and, hence, $u^*_{n/2}$ with components $-a_j$, then $u_{n/2}$ and $u^*_{n/2}$ in our tree-like model belong to the opposite branches. In other words, matrices $u_n, u_{n/2}, u^*_{n/2}$ can be considered as matrix representation of triplex form of all creatures described by Seklitova and Strelnikova [7]. They write that all creatures (all forms) in the Universe are constructed in one and the same triplex form, namely: one part is positive, the second part is negative, and the third is the governing part of the higher consequent order.

The governing part occupies one-half of the whole volume of the triplex form. The process of evolution suggests a constant increasing in a power of each form and a passage of these forms to the next level of a higher order. This process can be modeled by matrix solutions. Indeed, the increasing in the power depends on that in angular velocity $d\phi/dt$, which can be achieved by composition of similar rotations. According to the theorem of composition and decomposition (see our paper [6]) two rotations described by matrices $u_{n/2}(p\phi, \mathbf{a})$ and $u_{n/2}(q\phi, \mathbf{b})$ give a composite rotation $u_{n/2}(r\phi, \mathbf{c})$, where vectors of rotation **a**, **b**, and **c** are unit vectors with $a_j \geq 0$, $b_j \geq 0$ and, hence, $c_j \geq 0$. Let θ be the angle between vectors **a** and **b**; then $r^2 = p^2 + q^2 + 2pq\cos\theta$, $\cos\theta > 0$, and, hence, $r^2 > p^2 + q^2$, therefore $r\phi > p\phi$ and $r\phi > q\phi$. Thus, choosing two close to one another radial rotation vectors in the same sphere S^{2k-1} we obtain a composite rotation with the larger angular velocity, i.e. with higher power.

The existence of $u_{n/2}(r\phi, \mathbf{c})$ suggests the existence of its opposite $u^*_{n/2}$, both they form matrix $u_n(r\phi, \mathbf{c})$ of the n-th order. The latter matrix u_n with its opposite u^*_n forms new matrix $u_{2n}(r\phi, \mathbf{c})$ as the $u_n \oplus u^*_n$ coupling. Matrix u_{2n} of $2n$-th order with angular parameter $r\phi$ belongs to the higher level. A set of matrix u_{2n} forms unit sphere S^{2k+3} in the vector space V_{2k+3} generated by matrices $M_1(2n), \ldots, M_{2k+3}(2n)$, which are not commensurable with the basis vectors of space V_{2k+1}. Thus, all spaces V_{2k-1}, V_{2k+1}, and V_{2k+3} are in different dimensions.

The composition of rotations given above provides that vectors of rotation **a** and **b** are radial vectors in one and the same unit sphere. Now, consider another representation. Let there be a chain of unit spheres arranged along z-axis, which passes across its centers and its tangent points. Let angular parameter be $\phi = \alpha z$, which corresponds to the linear wave equation; and let a step along z-axis be chosen in such a way that ϕ increases by value π along the diameter of each sphere. We assume $z = 0$ at the left end of the diameter of the first sphere. Therefore, coordinate $x_0 = \cos\phi$ in relations (12) varies from (+1) to (-1) along the diameter of the first sphere S^{2k+1} and from (-1) to (+1) along the diameter of the adjacent second sphere S^{2k+1}, and so on, with period 2π. A cross-section of a unit sphere by hyperplane $x_0 = \cos\phi$ we denote by S_ϕ. As follows from (12) that $\sum_{j=1}^{2k+1} x_j^2 = (\sin\phi)^2$, i.e. the radius of this sphere is equal to $|\sin\phi|$. It is evident that this sphere has pulsating behavior along the z-axis.

5. Some Other Applications

The most interesting application of matrix solutions is the modeling of particle collision. Green et al [3] propose a scheme of particle collision in terms of superstrings. In paper [6] we realize a model of particle collision in terms of matrix solutions. Remind the main points of the model. Each particle is associated with the corresponding helical string described by its own matrix solution. Let two or more helical strings correspond to the same number of incoming particles which simultaneously meet each other in a definite unit sphere. The composition of simultaneous rotations associated with these helical strings gives one whole rotation in this sphere. Further, the decomposition of the whole rotation onto two or more simultaneous rotations in this sphere leads to forming the same number of helical strings that correspond to outgoing particles.

The representation of matrix solutions by helical string allows us to model some other geometrical properties of superstrings (considered, for example, by Greene [4] and Escultura [2]). In particular, an open oriented string is represented by complex solution $u_1 = \exp(i\phi)$ in the form of a semicircle with ends $u_1 = 1$ at $\phi = 0$ and $u_1 = -1$ at $\phi = \pi$ in the complex plane. These ends can be associated with positive and negative, respectively, states of an elementary particle. A closed string is represented by complex solution $u_1 = \exp(2i\phi)$ as well as by matrix solution $u_2(2\phi, \mathbf{a})$.

In the linear case $\phi = \alpha z$ the solution $u_1 = \exp(i\phi)$ gives infinitely many turns in the complex plane. In the spatial representation this solution represents a helical curve (helical string) with infinitely many coils along the z-axis. The notion "partners", which appear in the theory of superstrigs, we describe by two solutions u_1 and u_2 of the linear wave equation. While solution $u_1 = \exp(i\phi)$ describes a circle, solution $u_2(\phi, \mathbf{a})$ does this for the rotation about vector $\mathbf{a}$. Choosing vector $\mathbf{a}$ as the tangent vector to circle $\exp(i\phi)$ with the moving origin along the circle, we obtain the rotation in the form of toroidal motion about this circle. The behavior of these partners can be represented both as stream line $\exp(i\phi)$ and as winded lines around the former stream line – an analog of the magnetic field around the circular current.

The cyclic dimensions can be simulated by the chain of rotations, described by matrix solutions of linear wave equation, namely:

$$u_n(\phi, M_1)M_2, \quad u_n(\phi, M_2)M_3, \dots, u_n(\phi, M_{2k+1})M_1,$$

which represents $2k+1$ rotating linked circles. Note, for example, that the value $M_j' = u_n(\phi, M_{j-1})M_j$ starts from M_j at $\phi = 0$ and returns to M_j at $\phi = 2\pi$ taking intermediate values of the form $M_j' = M_{j-1}\cos\phi + M_{j-1}M_j \sin\phi$.

In the nonlinear case rotation $u_2(2N\phi, \mathbf{a})$ accomplishes N turns about vector $\mathbf{a}$ and therefore is suitable for description of a superstring whose solitonic state is described by number N (number of circuits of cyclic dimensions). A helical string described by $u_n(2N\phi, \mathbf{a})$ as a helical trace with N coils is proposed as an independent space object whose mass is equal to $2\pi RN$, where N is the number of coils and R is the radius of coils. Taking the quadratic norm of matrix $M_j(n)$ as $\| M_j(n) \| = \sqrt{n}$, we find that the mass of the helical string described by $u_n(2N\phi, \mathbf{a})M_j(n)$ is equal to $2\pi\sqrt{n}N$.

A nonstandard theory of superstrings is given by Escultura [2], where a superstring is defined as thin extended, helical and infinitesimal, consisting of a nested fractal configuration of matter without a smallest constituent. Matrix solutions in the form of helical strings can completely describe this definition. It is sufficient to construct a configuration where each helix is winded around the coils of another helix, and the coils of each next helix are smaller than the coils of the previous one and so on to the infinity.

The next application is related to the toroidal form of helical strings described by partners $u_1 = \exp(i\phi)$ and $u_2(\phi, \mathbf{a})$ in the linear case. Here u_1 describes a circle while u_2 accomplishes a rotation about this circle in the form of toroidal helical string. A set of these helices with the circular axis $\exp(i\phi)$ completely cover a torus surface and can be used for modeling the vortex rings arising in the fluid flow as well as in the gas-like ether. Note that Atsukovsky [1] describes the vortex ring in gas-like ether based on his own theory of ether-dynamics and proposes that the basic stable elementary particles have the form of vortex rings, in particular, proton and neutron. He constructs a model of stable alpha-particle (nucleus of helium) and on its basis constructs numerous nuclei with even number of protons. This model can be developed using symmetric helical construction in the form of the spiral chain of alpha-particles, where each coil of the chain contains eight alpha-particles and the maximum of such coils is also the number eight. Since each alpha-particle contains two protons the maximum number of protons 128 in the chain corresponds to the maximum number of elements in the periodic system of chemical elements.

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