International Journal of Pure and Applied Mathematics
Volume 62 No. 1 2010, 79-128

A SURVEY OF TOPICS IN ANALYSIS AND PARTIAL DIFFERENTIAL EQUATIONS

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Abstract: This survey is the result of investigations suggested by recent publications. The topics, which are dealt with, are related to linear or nonlinear evolution equations, of first or second order in the time-variable, linear and nonlinear elliptic equations, spectral properties, singular perturbations, formation of layers, and spaces of functions, in particular of BMO, Sobolev, and UMD type. They are presented in the alphabetical order, so that the article can be used as a dictionary. The aim of this document is to provide guidance for (young) researchers in mathematics and applied sciences, biology, chemistry, mechanics and physics.

AMS Subject Classification: 35x02, 46x02, 47x02
Key Words: attractors, fractals, evolution equations and systems, elliptic equations, Schrödinger operator, spectral theory, singular perturbations, internal and boundary layers, spaces of functions

Introduction

For convenience the article is divided into two sections as follows.

Received: June 18, 2010

Contents

The development of each entry is, of course, non exhaustive. The main ones include history, definitions, an overview of main results, examples or applications; details and complements will be found in the references. Several entries already appeared in D. Huet [61], some of them have been completed. Because of the length of the paper, some entries, such as: Birman-Schwinger operator, equal area condition, Mathieu-Hill type equation, quasi-periodic Schrödinger operators, resonances, Trudinger condition, that were presented in [61], do not appear in the present article

1. From Attractor to Fokker-Planck Equation

1.1. Attractor

Attractor of an Iterated Function System

(see K. Falconer [48] and [49]). Let X be a closed subset of $\mathbb{R}^n$. An *iterated function system* is a family of contractions $\{F_1, ..., F_m\}$ on $X, m \geq 2$. Let (S, d) be the complete metric space of compact subsets of X equipped with the distance

$$d(A, B) = \inf\{\delta : A \subset B_\delta \text{ and } B \subset A_\delta\},$$

where A_δ is a δ-neigbourhood of A, and define the map

$$A \in S \to F(A) = \bigcup_{i=1}^{m} F_i(A) \in S.$$

Then, F is a contraction, and thanks to the fixed point theorem (cf. L. Schwartz [88]) there exists a unique compact set $E \in S$ such that

$$F(E) = \bigcup_{i=1}^{m} F_i(E) = E.$$

The compacet set E, which is often a fractal set, is called the *attractor* of the iterated function system $\{F_1, ..., F_m\}$. Moreover, for all $A \in S$, $F^k(A) \to E$ for the metric d, as $k \to \infty$, and

$$F(E) = \bigcap_{k=1}^{\infty} F^k(A)$$

for all $A \in S$, such that $F_i(A) \subset A$ for all i. The sets $E_k = F^k(A)$ are called *pre-fractals* for E.

Example 1. $X = \mathbb{R}, F_1(x) = \frac{1}{3}x, F_2(x) = \frac{1}{3}x + \frac{2}{3}$. Then, the attractor of the system $\{F_1, F_2\}$ is the Cantor set (cf. E.C. Titchmarsh [95] and Figure 1).

Figure 1: The Cantor set

Example 2. (cf. M.F. Barnsley [14]) $X = \mathbb{R}^2, F_1(x, y) = (\frac{x}{2}, \frac{y}{2}), F_2(x, y) = (\frac{x}{2}+\frac{1}{2}, \frac{y}{2}), F_3(x, y) = (\frac{x}{2}, \frac{y}{2}+\frac{1}{2})$, then, the attractor of the system is the Sierpinski triangle (cf. Section 1.6 and Figure 2, p. 91).

Global Attractor for an Autonomous Evolution Equation

Definition 1. Let

$$\partial_t u(x, t) = A(u(x, t)), \quad u(t, 0) = u_o(x) \tag{1}$$

be the abstract form of an autonomous evolution problem. It is assumed that the Cauchy problem (1) has a unique solution. Here u_o and $u(x, t), \forall t \geq 0$, belong, for example, to a Banach or metric space E. Let $\{S(t)\} = \{S(t), \forall t \geq 0\}$ be the corresponding (nonlinear) semigroup of operators such that $u(x, t) = S(t)u_o(x), \forall t \geq 0$. A set $\mathcal{A} \subset E$ is a *global attractor* of (1) or of $\{S(t)\}$ if $\mathcal{A}$ is compact, $S(t)\mathcal{A} = \mathcal{A}, \forall t \geq 0$, and for each bounded set $B \subset E$, $\mathcal{A}$ attracts B, i.e. $dist_E(S(t)B, \mathcal{A}) \to 0$ as $t \to \infty$, where dist_E denotes the Hausdorff semi-distance (cf. Section 1.8).

In fact, $\mathcal{A}$ attracts the solutions $u(x, t) = S(t)u_o(x)$ to (1) as $t \to \infty$, uniformly with respect to bounded initial data $u_o(x)$. The global attractor of equation (1) describes all the possible limits of its solutions (cf. V.V. Chepyzhov, M.I. Vishik [26])

Example 3. In [9], S. Amraoui and H. Labani study the following reaction-diffusion system:

$$\frac{\partial u_i}{\partial t} - d_i \Delta u_i = f_i(u) \;\text{ in }\; \Omega \times]O, T[,\; 1 \leq i \leq 3, \tag{2}$$

associated with the boundary condition

$$(1-\lambda)u_i(t, x) + \lambda \frac{\partial u_i}{\partial \nu}(x, t) = \alpha_i(x),\; x \in \gamma,\; t \in]O, T[, 1 \leq i \leq 3,\; 0 \leq \lambda < 1, \tag{3}$$

and the initial condition

$$u(x,0) = u_o(x), \quad x \in \Omega, \tag{4}$$

where Ω is a bounded subset of $\mathbb{R}^n, n \geq 1$, with a smooth boundary γ, $T > 0$, $u_i \geq 0, d_i > 0, f_1(u) = f_2(u) = -f_3(u) = u_3 - u_1 u_2, \alpha_i(x) > 0$.

Under suitable hypotheses on the data, they construct a global attractor $\mathcal{M} \subset E$ of the system (2)-(4), where $E = \{v = (v_1, v_2, v_3) \in (L^\infty(\Omega))^3, v_i \geq 0, 1 \leq i \leq 3)\}$ equipped with the norm $\sum\limits_{i=1}^{3} ||v_i||_\infty$

Exponential Attractors

The Hilbert Space Case

Definition 2. Let H be a *Hilbert space* , and $\mathcal{A}$ be the global attractor of a semigroup of operators $S(t)$ in H. Let $\chi \subset H$ be a compact invariant set containing $\mathcal{A}$. A compact set $\mathcal{E} \subset \chi$ is called *an exponential attractor* for $S(t)$ restricted to χ if: i) $\mathcal{E}$ is invariant for $S(t)$, ii) The fractal dimension (cf. Section 1.6) of $\mathcal{E}$ in H is finite, iii) There exist $K_\chi \geq 0$ and $\epsilon_\chi > 0$ such that $\mathrm{dist}_H(S(t)\chi, \mathcal{E}) \leq K_\chi e^{-\epsilon_\chi t}, t \geq 0$. The set χ is called the basin of attraction of $\mathcal{E}$ (see S. Gatti, M. Grasselli, V. Pata [51]).

Example 4. In [28], M. Conti, V. Pata and M. Squassina are interested in the convergence, in an appropriate sense, as $\epsilon \to 0$, of the solution $u_\epsilon(x,t)$, $x \in \Omega$, $t \in \mathbb{R}$ to problem (p_ϵ):

$$u_t - \omega \Delta u - (1-\omega) \int_0^\infty k_\epsilon(s) \Delta u(t-s) ds + \varphi(u) = f, \ t > 0,$$

with Dirichlet boundary conditions on the boundary of the smooth bounded domain Ω in $\mathbb{R}^3, \omega > 0, \varphi$ is a suitable nonlinearity, f is a time independent source term, and the memory kernel k_ϵ converges to the Dirac distribution at the origin, $u(x,t)$ is supposed to be a given datum for $t \leq 0$. Let $A = -\Delta$ on $L^2(\Omega)$ with domain $\mathcal{D}(A) = H_o^1(\Omega) \cap H^2(\Omega)$, $H^r = \mathcal{D}(A^{\frac{r}{2}}), r \in \mathbb{R}$. Following C. Dafermos [31] and M. Grasselli and V. Pata [55], they introduce, under additional assumptions on k, φ, f, $(\varphi(x) = x^3 - x$ is allowed), the auxiliary variable $\eta^t(x,s) = \int_o^s u(x, t-y) dy$, the functions $\mu(s) = -(1-\omega)k'(s), \mu_\epsilon(s) = \frac{1}{\epsilon^2}\mu(\frac{s}{\epsilon})$, the Hilbert spaces $\mathcal{M}_\epsilon^r = L^2_{\mu_\epsilon}(\mathbb{R}^+, H^{r+1})$ and $\mathcal{H}_\epsilon^r = H^r \times \mathcal{M}_\epsilon^r$ for $\epsilon > 0$, $\mathcal{H}_o^r = H^r$. The correct reformulation of p_ϵ, in the frame work of dynamical systems, is (P_ϵ): find $(u_\epsilon, \eta_\epsilon) \in C([0,\infty), \mathcal{H}_\epsilon^o)$ solution to

$$u_t + \omega A u + \int_o^\infty \mu_\epsilon(s) A\eta(s) ds + \varphi(u) = f, \quad \partial_t \eta = -\partial_s \eta + u$$

for $t > 0$, associated with the initial condition $(u_o, \eta_o) \in \mathcal{H}_\epsilon^o$. The existence, for $\epsilon > 0$, of a strongly continuous semi-group $S_\epsilon(t)$ on $\mathcal{H}_\epsilon^o$ corresponding to P_ϵ and of an exponential attractor $\mathcal{E}_\epsilon$ for $S_\epsilon(t)$ are proved. The convergence of $\mathcal{E}_\epsilon$ is also studied.

The Metric Space Case

Definition 3. *Let E be a metric space,* X be a compact subset of E and $\{S(t)\}_{t\geq 0}$ be a continuous semi-group on E. A compact set $\mathcal{M}$ is called an exponential attractor for $S(t)$ for the topology of E if $S(t)\mathcal{M} \subset \mathcal{M}, \forall t \geq 0$, the fractal dimension of $\mathcal{M}$ is finite, and there exists a constant $c > 0$ such that, for every bounded set $B \subset X$ there exists a constant $c_1(B)$ such that:

$$\mathrm{dist}_E(S(t)B, \mathcal{M}) \leq c_1 e^{-ct} \quad \forall t \geq 0$$

(see A. Bonfoh, M. Grasselli, A. Miranville [17]).

Example 5. (The Cahn-Hilliard-Gurtin Equation) In [17], the authors study the Cahn-Hilliard-Gurtin type problem

$$A\rho = \frac{\partial \rho}{\partial t} - \mathbf{d}\cdot\nabla\frac{\partial \rho}{\partial t} - \mathrm{div}\left(\tilde{B}\nabla\frac{\partial \rho}{\partial t} - \alpha B\nabla\Delta\rho + B\nabla f'(\rho)\right) = 0 \tag{5}$$

on $\Omega = \prod_{i=1}^n (0, L_i), L_i > 0, n \leq 3$ (resp. its singular perturbed problem

$$\epsilon\frac{\partial^2 \rho}{\partial t^2} + A\rho = 0), \tag{6}$$

associated with the conditions: ρ is Ω-periodic and

$$\rho|_{t=0} = \rho_o \tag{7}$$

(resp. (7) and $\frac{\partial \rho}{\partial t} = \rho_1$). Here $\mathbf{d}\in \mathbb{R}^n, \alpha > 0, B, \tilde{B}$, are two positive definite $n \times n$ matrices with constant coefficients and f is a double-well-like potential.

They prove the existence of a family of exponential attractors $\mathcal{M}_\epsilon$ in the Banach space $\mathcal{H}_\epsilon = H^1_{per} \times (H^1_{per})'$ equipped with a norm which depends on ϵ, for the semigroup $S_\epsilon(t)$ associated to the perturbed problem and study the convergence of $\mathcal{M}_\epsilon$ to an exponential attractor associated to the unperturbed problem (5), (7).

See, also, the examples in Section 1.5.

1.2. Aubry-Type Sets

History

Aubry-type sets are a part of the Aubry-Mather theory. A good introduction to the theory can be found in O. Knill [67]. An historic of the theory, which started in 1924 with M. Morse [81], and perspectives of new developments in different domains of mathematics and physics are presented in the review, by V. Bangert [13], of the lecture notes [67].

Applications to Elliptic Perturbations of Hamilton-Jacobi Equations

In [23], F. Camilli and A. Cesaroni consider the singular perturbation problem

$$\begin{cases} -\epsilon\Delta v_\epsilon + H(x,\nabla v_\epsilon) - \epsilon c(x) = 0 \text{ in } D, \\ v_\epsilon = g(x) \text{ on } \partial D, \end{cases} \tag{8}$$

under the following assumptions: D is a bounded set with Lipschitz boundary in $\mathbb{R}^n$, $H(x,p)$ is a continuous, Lipschitz continuous in x, convex and coercive in p Hamiltonian, $H(x,0) \leq 0$ and the critical value of H is 0, $c : \overline{D} \to \mathbb{R}$ is a continuous nonnegative function, g is a continuous function on $\partial\Omega$. The limit problem, i.e. the Hamilton-Jacobi problem:

$$\begin{cases} H(x,\nabla v) = 0 \text{ in } D, \\ v(x) = g(x) \text{ on } \partial D \end{cases} \tag{9}$$

has, in general, many viscosity solutions (cf. P.L. Lions [75]). As in several previous papers, in particular A. Eizenberg [43], A. Siconolfi [90], and A. Fathi, A. Siconolfi [50], the authors introduce the convex set $Z(x) = \{p \in \mathbb{R}^n | H(x,p) \leq 0\}$, $x \in \overline{D}$, the support fuction $\sigma(x,\cdot)$ of $Z(x)$ and the distance

$$S(x,y) = \inf \{ \int_0^1 \sigma(\Phi(s),\dot{\Phi}(s))ds, \quad \Phi \in \mathcal{C}_{x,y} \},$$

where $\mathcal{C}_{x,y}$ is the set of $W^{1,\infty}([0,1],D)$ curves such that $\Phi(0) = x, \Phi(1) = y$. They give the following *definition of the Aubry-type set* $\mathcal{A}$ *associated to problem* (9) a point $x \in \mathcal{A}$, if there exists a sequence $\Phi_n \in \mathcal{C}_{x,x}$ whose Euclidean length l satisfies $l(\Phi_n) \geq \delta > 0$ and whose length with respect to σ vanishes, i.e. $\inf_n\{\int_0^1 \sigma(\phi_n(s),\dot{\Phi}_n(s))ds\} = 0$. In fact, $\mathcal{A}$ *behaves as an hidden boundary on which a datum must be fixed to have existence and uniqueness for problem* (9). Namely if g is a real function, defined and continuous on $\partial D \cup \mathcal{A}$, such that $-S(y,x) \leq g(x) - g(y) \leq S(y,x)$ for $x,y \in \partial D \cup \mathcal{A}$, there exists a unique viscosity solution u to (9) such that $u(x) = g(x)$ on $\partial D \cup \mathcal{A}$. Moreover, if $\mathcal{A}$ satisfies additional conditions, and if ∇v_ϵ is bounded by a constant independent of ϵ, then v_ϵ, solution to (8), converges uniformly to the maximal viscosity solution $G(x) = \min_{y\in\partial D}\{g(y)+S(y,x)\}$ to (9), as $\epsilon \to 0$. Examples of hamiltonians of the forms $H(x,p) = \frac{|p|^2}{2} - b(x)\cdot p, H(x,p) = -b(x)\cdot p + K(x,p)$, and the Eikonal Hamiltonian $H(x,p) = F(p) - f(x)$, for which the above results hold, are investigated.

1.3. BMO and Related Spaces

Definitions

The definitions of the following spaces are recalled in J. Xiao [100].

***BMO* Space.** Let $\sup_I$ be the supremum over all cubes $I \subset \mathbb{R}^n$ with edges parallel to the coordinate axes, $\ell(I)$ be the sidelength of I and f_I the mean value of f over I. The square form of John-Nirenberg's $BMO = BMO(\mathbb{R}^n)$ space (cf. F. John, L. Nirenberg [63]) is the space of localy integrable complex-valued functions f defined on $\mathbb{R}^n, n \geq 2$, such that

$$||f||_{BMO} = \left(\sup_I (\ell(I))^{-n} \int_I |f(x) - f(_I|^2 dx\right)^{\frac{1}{2}} < \infty.$$

$\mathcal{Q}_\alpha$ Spaces. For $\alpha \in (-\infty, \infty)$, $\mathcal{Q}_\alpha(\mathbb{R}^n)$ is the Banach space of all measurable complex-valued functions on $\mathbb{R}^n$, modulo constants, such that

$$||f||_{\mathcal{Q}_\alpha} = \sup_I \left((\ell(I))^{2\alpha-n} \int_I\int_I \frac{|f(x) - f(y)|^2}{|x-y|^{n+2\alpha}} dxdy\right)^{\frac{1}{2}} < \infty$$

(cf. M. Essen, S. Janson, L. Peng, J. Xiao [46]).

Space $\dot{L}^2_\alpha$, $\alpha \in (0,1)$. The homogeneous Sobolev space $\dot{L}^2_\alpha(\mathbb{R}^n)$ is the space of complex-valued functions f such that

$$||f||_{\dot{L}^2_\alpha} = \left(\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|f(x) - f(y)|^2}{|x-y|^{n+2\alpha}} dxdy\right)^{\frac{1}{2}} < \infty.$$

Space $\mathcal{L}_{2,n-2\alpha}$, $\alpha \in (0,1)$. It is the space of measurable complex-valued functions f on $\mathbb{R}^n$ such that

$$||f||_{\mathcal{L}_{2,n-2\alpha}} = \sup_I \left(\ell(I))^{2\alpha-n} \int_I |f(x) - f(_I|^2 dx\right)^{\frac{1}{2}} < \infty.$$

Space $\mathcal{Q}^{-1}_{\alpha;T}$, $\alpha \in (0,1), T \in (0,\infty)$. A temperated distribution f on $\mathbb{R}^n$ belongs to this space provided

$$||f||_{\mathcal{Q}^{-1}_{\alpha;T}} = \sup_{x\in\mathbb{R}^n, r\in(0,T)} \left(r^{2\alpha-n} \int_0^{r^2} \int_{|y-x|<r} |e^{t\Delta} f(y)|^2 t^{-\alpha} dydt\right)^{\frac{1}{2}} < \infty,$$

where $e^{t\Delta}(x,y)$ is the heat kernel.

Relations between these Spaces

For $\alpha \in \mathbb{R}, f \in \mathcal{S}'(\mathbb{R}^n)$, modulo polynomials, $(-\Delta)^{-\frac{\alpha}{2}} f = \mathcal{F}^{-1}(\Big|\xi|^{-\alpha}\hat{f}(\xi)\Big) = I_{\alpha f}$ (cf. R. Strichartz [93]). In [100], when $\alpha \in (0,1)$, it is proved that:

1) $\mathcal{Q}_\alpha = (-\Delta)^{-\frac{\alpha}{2}} \mathcal{L}_{2,n-2\alpha}$,

2)$\dot{L}^2_\alpha = (-\Delta)^{-\frac{\alpha}{2}} L^2$,

3) $\mathcal{Q}^{-1}_{\alpha;\infty} = \nabla\cdot(\mathcal{Q}_\alpha)^n$ and sharp estimates for the norms of the embeddings $\mathcal{Q}_\alpha \hookrightarrow BMO$, and $\dot{L}^2_\alpha \hookrightarrow L^{\frac{2n}{n-2\alpha}}$, are obtained.

We point out that 3) means that $f \in \mathcal{Q}^{-1}_{\alpha;\infty}$ if and only if there are $f_j \in \mathcal{Q}_\alpha$ such that $f = \sum_{j=1}^n \partial_j f_j$.

1.4. Bounded Variation (Functions of)

Several equivalent definitions of functions of bounded variation are well known (see e.g., E.C. Titchmarsh [95] and H. Brezis [20]). Let $\Omega \subset \mathbb{R}^d$ be an open set with a smooth boundary.

Space $BV(\Omega, \mathbb{R}^q)$

A function $f \in L^1_{loc}(\Omega, \mathbb{R}^q)$ has a bounded variation, i.e. $f \in BV(\Omega, \mathbb{R}^q)$, if ∇f is a Radon measure of finite total mass. Let $|\mathbf{f}|_{\mathbf{BV}} = \int_{\mathbf{\Omega}} |\nabla \mathbf{f}|$ *be a BV-semi-norm.* In J. Davila [33], the following property of $|f|_{BV}$ is proved: there exists a positive constant K, which depends on d, such that, for every family of non negative radial mollifiers $\rho_\epsilon \in L^1_{loc}((0,\infty),\mathbb{R}_+)$ satisfying

$$\int_0^\infty \rho_\epsilon(r)r^{d-1}dr = 1 \text{ and } \lim_{\epsilon\to 0}\int_\delta^\infty \rho_\epsilon(r)r^{d-1}dr = 0, \quad \forall\delta > 0,$$

we have

$$\lim_{\epsilon\to 0}\int_{\Omega^2}\frac{|f(x)-f(y)|}{|x-y|}\rho_\epsilon(|x-y|)dxdy = K|f|_{BV}, \quad \forall f \in BV(\Omega,\mathbb{R}^q).$$

Application

In [79], B. Merlet shows, by means of the above property of $|f|_{BV}$, that, if $u \in BV(\Omega, S^1)$, there exists a lifting $\varphi \in BV(\Omega,\mathbb{R})$ of u (i.e. $u(x) = e^{i\varphi(x)}, \forall x \in \Omega$) such that $|\varphi|_{BV} \leq 2|u|_{BV}$.

1.5. Cahn-Hilliard-Gurtin Equation

After the initial Cahn-Hilliard equation:

$$\frac{\partial\rho}{\partial t} = \kappa\Delta[f'(\rho) - \alpha\Delta\rho]$$

$\alpha > 0, \kappa > 0$, which governs the evolution of the order parameter $\rho(x,t)$, where $f(\rho)$ is the coarse-grain free energy. (cf. J.W. Cahn [22]), generalized in M.E. Gurtin [56], several generalizations have been studied, mostly in relation with attractors and singular perturbations.

Example 6. In [34], A. Debussche studies the Cahn-Hilliard problem (P)

$$Au = \frac{\partial u}{\partial t} + \nu\Delta^2 u - \Delta f(u) = 0 \quad \text{in } \Omega\times\mathbb{R}^+,$$

where $u(x,t)$ is a real-valued function, $\Omega = [0,L]$ and f is a polynomial of even order whose leading coefficient is positive, supplemented with an initial condition u_o and Neumann or periodic boundary conditions.

He considers the perturbed problem (P_ϵ):

$$\epsilon\frac{\partial^2 u}{\partial t^2} + Au = 0$$

associated with the same boundary conditions, the initial condition u_o and $\frac{\partial u}{\partial t} = u_1(x)$. He studies the existence of a global attractor $\mathcal{A}_\epsilon$ for (P_ϵ) and its convergence to the global attractor of (P), in suitable functional spaces.

Example 7. In [52], S. Gatti, M. Grasselli, A. Miranville and V. Pata study the following initial and boundary value problem:

$$\omega u_{tt} + u_t - \Delta(-\Delta u + \phi(u) + \delta u) = 0, \quad \text{in } \Omega \times \mathbb{R}^+, \tag{10}$$

associated with the initial conditions

$$u(0) = u_o, \quad \omega u_t(0) = \omega u_1 \tag{11}$$

and the boundary condition

$$u|_{\partial\Omega} = \Delta u|_{\partial\Omega} = 0, \tag{12}$$

where Ω is a bounded domain in $\mathbb{R}^3$, with a smooth boundary $\partial\Omega$, $\omega, \delta \in [0,1]$, $\delta \geq \mu\omega$ with $\mu \in (0,1]$, and ϕ is a smooth function with cubic controlled growth.

If $\omega = \delta = 0$, equation (10) is the Cahn-Hilliard equation, if $\omega = 0$ and $\delta > 0$, it is known as the viscous Cahn-Hilliard equation. The authors construct a family of exponential attractors $\mathcal{E}_{\omega,\delta}$ for problem (10)-(12) which is a robust perturbation of an exponential attractor $\mathcal{E}_{0,0}$ of the usual Cahn-Hilliard equation. They show the existence of constants $C > 0$ and $\tau \in (0,1)$ such that dist-sym $(\mathcal{E}_{\omega,\delta}, \mathcal{E}_{0,0}) \leq C(\omega+\delta)^\tau$ in a suitable functional space.

Example 8. See Example 5, p. 84.

1.6. Dimensions in a Metric Space

Definitions

Let K be a compact set in a metric space E.

The Kolmogorov ϵ-Entropy. For $\epsilon > 0$, $N_\epsilon(K,E)$ denotes the minimum number of open balls in E, with radius ϵ, necessary to cover K. The number

$$\mathbb{H}_\epsilon(K,E) = \ln(N_\epsilon(K,E))$$

is called the ϵ-entropy of K in E.

The Fractal Dimension of K in E is

$$\dim_F(K,E) = \limsup_{\epsilon \to 0} \frac{\mathbb{H}_{\epsilon(K,E)}}{\ln 1/\epsilon}.$$

The Hausdorff dimension of K **in** E Given $d \in \mathbb{R}^+$ and $\epsilon > 0$ set

$$\mu(K,d,\epsilon) = \inf \sum r_i^d,$$

where the infimum is taken over all the coverings $\{B_{r_i}(x_i)\}$ of K by balls in E with radius $r_i \leq \epsilon$. The Hausdorff dimension of H in E is the number

$$\dim_H(K,E) = \inf_{d \in \mathbb{R}^+} [\lim_{\epsilon \to 0} \mu(K,d,\epsilon)].$$

Remarks and Examples

Remark 1. (cf. V.V. Chepyzhov, M.I. Vishik [26], p. 52) In general, the following inequality holds:

$$\dim_H(K, E) \leq \dim_F(K, E).$$

If K is a smooth m-dimensional compact submanifold in E, then

$$\dim_H(K, E) = \dim_F(K, E) = m.$$

Every countable set has Hausdorff dimension 0 while the fractal dimension can be arbitrarily large. For example, if $K = \{0\} \cup \{a_n = \frac{1}{\log n}\}_{n \in \mathbb{N}}$ then $\dim_F(K, \mathbb{R}) = 1$. If E is a Hilbert space, $\{e_n\}$ is an orthonormal basis in E, and $K = \{0\} \cup \{\frac{e_n}{\log n}\}_{n \geq 2}$, then $\dim_F(K, E) = \infty$.

Fractals

(cf. K. Falconer [48] and [49]) A *self similar set* in $\mathbb{R}^n$ is a set which is invariant under a collection of similarities. It is the union of a number of smaller similar copies of itself (cf. [48], p. 117 and [49], p. 35). The following examples are self similar sets whose fractal and Hausdorff dimensions are equal.

In $\mathbb{R}$ the dimensions of the Cantor set ([95] and [48], p. 43) are $\frac{\log 2}{\log 3}$.

In $\mathbb{R}^2$, the Sierpinski triangle, Figure 2, also called the Sierpinski gasket, has dimensions $\frac{\log 3}{\log 2}$ ([48], p.120), whereas the Sierpinski carpet (Figure 3, p. 91) has dimensions $\frac{\log 8}{\log 3}$ ([49], p. 36). Figure 2 (resp. 3) represents pre-fractals $E^4 = F^k(A)$ (cf. Section 1.1), where A is the equilateral triangle (resp. the square(. The Sierpinski triangle and carpet respectively are the unions of closed blue sets. The Koch curve, whose prefractal E^3 is represented in Figure 4, p. 91, has dimensions $\frac{\log 4}{\log 3}$ ([49] p. 36).

1.7. Dirac Operator

History

In 1928, P.A.M. Dirac was searching for a first order, linear, Lorentz-invariant, differential operator D such that

$$D^2 = \frac{\partial^2}{\partial t^2} - \Delta, \quad \Delta = \sum_{k=1}^{3} \frac{\partial^2}{\partial x_k} \tag{13}$$

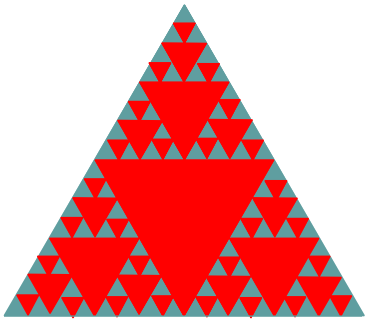

Figure 2: The Sierpinski gasket

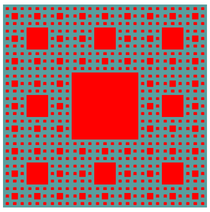

Figure 3: The Sierpinski carpet

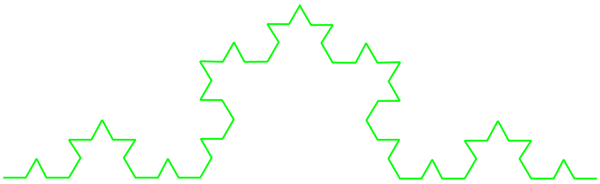

Figure 4: The von Koch curve

acting on vector-valued functions $\psi = (\psi_1, \psi_2, \psi_3, \psi_4)$. Setting $t = x_o$ and writing

$$D = \sum_{k=0}^{3} \gamma_k \frac{\partial}{\partial x_k},$$

where the γ_k are 4×4 complex matrices, (13) implies the commutation relations

$$\sum_{j,k=0}^{3} (\gamma_j \gamma_k + \gamma_k \gamma_j) = 2 \pm \delta_{jk} I_4, \tag{14}$$

where δ_{jk} is the Kronecker symbol and I_4 the 4×4 unit matrix. In particular, $\gamma_o^2 = 2I_4, \gamma_k^2 = -2I_4$ for $k = 1, 2, 3$. Let

$$\sigma_o = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

be the 2×2 Pauli matrices (cf. [45]), which satisfy the relations

$$\sigma_j \sigma_k + \sigma_k \sigma_j = 2\delta_{jk} I_2.$$

Then, the 4×4 matrices, called Dirac matrices

$$\gamma_o = \begin{pmatrix} \sigma_o & 0 \\ 0 & -\sigma_o \end{pmatrix}, \quad \gamma_k = \begin{pmatrix} 0 & \sigma_k \\ \sigma_k & 0 \end{pmatrix}, \quad k = 1, 2, 3.$$

satisfy relations (14). The corresponding operator D leads to the **Dirac equation** which is a system of four linear homogeneous partial equations of the first order with constant coefficients:

$$\gamma_k \frac{\partial \psi}{\partial x_k} - \mu \psi = 0, \quad k = 0, 1, 2, 3, \tag{15}$$

where $\mu = \frac{mc}{\hbar}$. The system (15) describes the motion of an electron in an electromagnetic field.

Remark 2. In (15), ψ is, in fact, a spinor-valued function. For a simple introduction to the theory of spinors see Elie Cartan [24].

Remark 3. For the relativistic theory of the electron, see P.A.M. Dirac [37].

Since the initial work of P.A.M. Dirac, many problems have been connected with Dirac-type operators. See, for instance, J.-P. Bourguignon et al (Ed-s) [19].

A One Dimensional Dirac-Type Operator

Consider the n dimensional periodic operator of the form:

$$Ly(x) = i \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \frac{dy}{dx} + \begin{bmatrix} 0 & P(x) \\ Q(x) & 0 \end{bmatrix} y, \ x \in \mathbb{R}, \ y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix},$$

where P and Q are periodic of period 1 and are in $L^2(0, 1)$.

In [38], P. Djakov, B. Mityagin suppose $Q(x) = \overline{P(x)}$. Then, L, considered with periodic or antiperiodic boundary conditions, is self-adjoint. If $\lambda_n^-, \lambda_n^+, n \in \mathbb{Z}$ are its eigenvalues close to $n\pi$ as $|n|$ is large, and $\gamma_n = \lambda_n^+ - \lambda_n^-$, the relationship between the decay rate of γ_n as $|n| \to \infty$, and the smoothness of P is investigated.

1.8. Distances

Distances in a Metric Space

Let (E, d) be a metric space and $X, Y \subseteq E$.

The Hausdorff Non-Symmetric- (or Semi-) Distance between sets X, Y in E is

$$\mathrm{dist}_E(X, Y) = \sup_{x \in X} \inf_{y \in Y} d(x, y)$$

The Hausdorff Symmetric Distance between sets X, Y in E is

$$\text{dist-sym}_E(X, Y) = \max(\mathrm{dist}_E(X, Y), \mathrm{dist}_E(Y, X)).$$

Quasi-Distance, Doubling Quasi Metric Space

On a set Y, a function $d : Y \times Y \to [0, \infty)$ is called a *quasi distance* if it is symmetric, strictly positive away from $\{x = y\}$ and such that for some constant $K \geq 1$

$$d(x, y) \leq K(d(x, z) + d(z, y))$$

for all $x, y \in Y$. The pair (Y, d) is called a *quasi metric space.* A positive measure μ on a σ-algebra of subsets of Y containing the d-balls is said to have the *doubling property* if there exists a positive constant C such that

$$0 < \mu(B(x, 2r)) \leq C\mu(B(x, r))$$

for all $x \in Y$ and $r > 0$. Then (Y, d, μ) is called a *doubling quasi metric space* (cf. G. Di Fazio, C.E. Gutierrez, E. Lanconelli [36]).

1.9. Douglis-Nirenberg Elliptic Systems

Notations: $\partial_i = \frac{\partial}{\partial x_i}, \partial = (\partial_1, ...\partial_n)$.

The Classical Case

(cf. A. Douglis, L. Nirenberg [42]) Let Ω be an open set in $\mathbb{R}^n$, and a differential system

$$U=(u_1,...,u_N)\to L(x,\partial)U=\sum_{j=1}^{N} l_{ij}(x,\partial)u_j(x)=F_i(x),\quad x\in\Omega, i=1,...,N \tag{16}$$

Definition 4. The system (16) is said to be elliptic if there exist *integers* s_i, t_j such that, for $i,j=1,...,N$, the degree of $l_{ij}(x,\xi)$, with respect to $\xi=(\xi_1,...,\xi_n)\in\mathbb{R}^n$, is s_i+t_j, ($l_{ij}=0$ if $s_i+t_j<0$), and $L'(x,\xi)=\det(l'_{ij}(x,\xi)\neq 0$, for $\xi\neq 0$ and $x\in\Omega$, where $l'(x,\xi)$ consists of the terms in $l_{ij}(x,\xi)$ which are just of degree s_i+t_j.

Example 9. (cf. S. Agmon, A. Douglis, L.Nirenberg [3], p. 40) In $\mathbb{R}^2$, the Laplace equation $\partial_1^2 u+\partial_2^2 u=0$ is reducible to the first order system, with constant coefficients

$$\begin{cases}\partial_1 u_1+\partial_2 u_2=0,\\ \partial_1 u-u_1=0,\\ \partial_2 u-u_2=0,\end{cases} \tag{17}$$

with respect to the new variables u, u_1, u_2. The usual characteristic determinant, for (17), is

$$\begin{vmatrix} 0 & \xi_1 & \xi_2\\ \xi_1 & 0 & 0\\ \xi_2 & 0 & 0\end{vmatrix}=0$$

and (17) is not elliptic in the usual sense. But, with the weights

$$\begin{aligned} t_o&=2,\quad t_1=t_2=1,\\ s_o&=0,\quad s_1=s_2=-1,\end{aligned}$$

we have

$$\det(l'_{ij}(\xi))=\begin{vmatrix} 0 & \xi_1 & \xi_2\\ \xi_1 & -1 & 0\\ \xi_2 & 0 & -1\end{vmatrix}=\xi_1^2+\xi_2^2\neq 0$$

and, therefore, the system is elliptic in the sense of Definition 4

This result is general, as it is pointed out by M.F. Atiyah, I.M. Singer [10]. in the sense of Definition 4, every elliptic system can be reduced to an elliptic first order system.

Example 10. (cf. [42]) Let $n=2$, $x_1=x$, $x_2=y$, $\xi_1=\xi$, $\xi_2=\eta$.

Consider the system

$$\begin{cases} u_{1x} - u_{2yy} = f_1, \\ u_{1yy} + au_{1y} + u_{2xxx} + u_{2xy} = f_2. \end{cases}$$

With the weights $s_1 = 0,\ s_2 = 1,\ t_1 = 1,\ t_2 = 2$, we have

$$\det(l'_{ij}) = \begin{vmatrix} \xi & -\eta^2 \\ \eta^2 & \xi^3 \end{vmatrix} = \xi^4 + \eta^4$$

and the system is elliptic in the above sense.

The Parameter-Elliptic Pseudo-Differential Case

Notation: for $\xi = (\xi_1, ..., \xi_n) \in \mathbb{R}^n, \langle\xi\rangle = (1 + |\xi|^2)^{\frac{1}{2}}$.

A Symbol Class (cf. H. Kumano-go [69])

Definition 5. Let $m \in \mathbb{R}, 0 < \rho \leq 1, 0 \leq \delta < 1$

$$S^m_{\rho,\delta} = \{a(x,\xi) \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n); |\partial^\alpha_\xi \partial^\beta_x a(x,\xi)| \leq C_{\alpha,\beta}\langle\xi\rangle^{m+\delta|\beta|-\rho|\alpha|}\},$$

where α, β are the usual multiindices. For $\rho = 1$ we set $S^m_\delta = S^m_{1,\delta}$, and for $\delta = 0,\ S^m = S^m_0$.

Pseudo-Differential Operators Let $a(x,\xi) \in S^m_{\rho,\delta}$. The assosiated pseudo-differential operator $a(x, D)$ is defined, for $u \in \mathcal{S}(\mathbb{R}^n)$, by

$$a(x,D)u = (\frac{1}{2\pi})^n \int_{\mathbb{R}^n} e^{ix.\xi} a(x,\xi)\hat{u}(\xi)d\xi,$$

where $\mathcal{S}$ denote L. Schwartz's space of rapidly decreasing functions (cf. [87]), and $\hat{u}$ is the Fourier transform of u:

$$\hat{u}(\xi) = \int_{\mathbb{R}^n} e^{-ix.\xi} u(x)dx.$$

In the context of Sobolev spaces, $a(x, D)$ is continuous $H^s_p(\mathbb{R}^n) \to H^{s-m}_p(\mathbb{R}^n), 1 < p < \infty, s \in \mathbb{R}$.

Parameter-Elliptic Douglis-Nirenberg Systems. (cf. R.Denk, J. Saal, J. Seiler [35]). For $0 < \omega < \pi$, consider the sector:

$$\Lambda(\omega) = \{re^{i\varphi}; r \geq 0, \omega \leq \varphi \leq 2\pi - \omega\}.$$

In [35], the authors give two equivalent definitions of Λ-elliptic system. One of them is as follows

Definition 6. A Λ-elliptic Douglis-Nirenberg system is a $N \times N$-matrix of pseudo-differential operators

$$A(x, D) = (a_{ij}(x, D))_{1 \le i,j \le N} \tag{18}$$

such that

i) There exist *real* numbers $l_1, ...l_N$, $m_1, ..., m_N$ with the property that

$$a_{ij}(x, \xi) \in S_\delta^{l_i + m_j}(\mathbb{R}^n \times \mathbb{R}^n), \quad i, j = 1, ..., N \tag{19}$$

and the numbers $r_i = l_i + m_i$ satisfy

$$r_1 > r_2 > ...r_N \ge O \tag{20}$$

ii) For $\kappa = 1, ..., N$ and

$$A[\kappa](x, D) = (a_{ij}(x, D)_{1 \le i,j \le \kappa}, \quad E_\kappa = \text{diag}(0, ..., 0, 1) \in \mathbb{C}^\kappa \times \mathbb{C}^\kappa,$$

we have

$$|\det(A[\kappa](x, \xi) - \lambda E_\kappa)| \ge C \langle \xi \rangle^{r_1 + ... + r_{\kappa - 1}} (\langle \xi \rangle^{r_\kappa} + |\lambda|), \quad x \in \mathbb{R}, |\xi| \ge R, \lambda \in \Lambda$$

with suitable constants $C > 0, R \ge 0$, and every $1 \le \kappa \le N$.

They introduce suitable perturbations K of a Λ-elliptic Douglis-Nirenberg system $A(x, D)$, with the property that the operator $\mathcal{A} = A + K$ is unbounded and closed in $\mathcal{H} = \bigoplus_{i=1}^{N} H_p^{s - l_i}(\mathbb{R}^n)$, for some $s \in \mathbb{R}, 1 < p < \infty$. They show the existence of $\alpha_o \ge 0$ such that, for each $\alpha \ge \alpha_o$, $\mathcal{A} + \alpha$ admits a bounded H_∞-calculus with respect to Λ in $\mathcal{H}$ (cf Section 2.3).

Example 11. (The Generalized Thermoelastic Plate System, cf. [35], p. 405) Let $\eta > 0, L = (-\Delta)^\eta$ on $\mathbb{R}^n$. The symbol of L is $|\xi|^{2\eta}$. The generalized thermoelastic plate equations are

$$\begin{aligned} v_{tt} + Lv - L^\beta w = 0, \\ w_t + L^\alpha w + L^\beta v_t = 0, \end{aligned} \tag{21}$$

together with suitable initial conditions. Here α, β are parameters with $0 \le \alpha, \beta \le 1$. In what follows it is supposed that α, β satisfy the conditions

$$\alpha > \beta, \quad 2\beta - \alpha > \frac{1}{2}. \tag{22}$$

Introducing the new variables $u_1 = w, u_2 = v_t, u_3 = L^{\frac{1}{2}} v$, system (21) becomes $u_t + \tilde{A}(D) u = 0$, with

$$\tilde{A}(\xi) = \begin{pmatrix} |\xi|^{2\alpha\eta} & |\xi|^{2\beta} & 0 \\ -|\xi|^{2\beta\eta} & 0 & |\xi|^\eta \\ 0 & -|\xi|^\eta & 0 \end{pmatrix}.$$

Let $\chi(\xi)$ denote a 0-excision function and set $A(\xi) = \chi(\xi) \tilde{A}(\xi]$. With the weights:

$$m_1 = 2\eta(\alpha - \beta), \quad m_2 = 0, \quad m_3 = 2\eta(\frac{1}{2} + \alpha - 2\beta);$$

$$l_1 = 2\beta\eta, \quad l_2 = 2\eta(2\beta - \alpha), \quad l_3 = \eta,$$

under conditions (22), we have

$$r_1 = 2\alpha\eta > r_2 = 2\eta(2\beta - \alpha) > r_3 = 2\eta(1 + \alpha - 2\beta) > 0.$$

Thanks to Definition 6, a direct computation shows that $A(x, D)$ is a Λ-elliptic Douglis-Nirenberg system, with $N = 3$, for any sector Λ which does not contain the positive half axis. Moreover, $\tilde{A}(D)$ is an admissible perturbation of A in $\mathcal{H}$. Therefore there exists $\lambda_o \geq 0$ such that $\tilde{A} + \lambda$ admits, for any $\lambda \geq \lambda_o$ a bounded H_∞-calculus (cf. Section 2.3) and, hence, maximal regularity (cf. Section 2.6).

1.10. Fokker-Planck Equation

Forward Kolmogorov Equation

Let $p(s, x, t, y)$ be a transition density of a Markov process, with a drift coefficient $\alpha(t, x)$ and a diffusion coefficient $\sigma^2(t, x)$. Under suitable assumptions on p, α, σ^2, p is a fundamental solution of the *forward Kolmogorov equation*:

$$\frac{\partial p}{\partial t} = -\frac{\partial}{\partial y}(\alpha(t, y)p) + \frac{1}{2}\frac{\partial^2}{\partial y^2}(\sigma^2(t, y)p)$$

known as the *Fokker-Planck equation* (cf. Encyclopaedia of Mathematics, Springer Online reference Works, [44])

The Kuramoto Model

In [71], Y. Kuramoto studies a large set of coupled oscillators $Q_1, ..., Q_N$ with natural frequencies w_i. He considers the simple possible case of purely sinusoidal coupling, for which the governing equations are

$$\frac{\partial \theta_i}{\partial t} = \omega_i + \frac{K}{N}\sum_{j=1}^{N} \sin(\theta_j - \theta_i), \quad i = 1, ..., N, \tag{23}$$

where $K \geq 0$ is the coupling strength. The example of a Lorentzian distribution of the native frequency is studied. Setting

$$re^{i\psi} = \frac{1}{N}\sum_{j=1}^{N} e^{i\theta_j} \tag{24}$$

called the complex parameter, where $r(t)$ measures the phase coherence and $\psi(t)$ the average phase, (23) becomes

$$\frac{\partial \theta_j}{\partial t} = \omega_j + Kr\sin(\psi - \theta_j), \quad j = 1, ..., N \tag{25}$$

(cf. Y. Kuramoto [72]).

From Kuramoto to Crawford

In [94], S.H. Strogatz tells the story of coupled oscillators, from Y. Kuramoto's work to J.D. Crawford's contributions.

The Infinite-N Version of (23). The frequencies ω are distributed according to the probability density $g(\omega)$. Let $\rho(\theta, t, \omega)$ denote the fraction of oscillators with natural frequency ω and phase in $[\theta, \theta + d\theta]$, at time t. Then, ρ is non-negative, 2π- periodic in θ, and satisfies the normalization condition $\int_0^{2\pi} \rho(\theta, t, \omega)d\theta = 1$, for all t, ω. The evolution of ρ is governed by the continuity condition

$$\frac{\partial \rho}{\partial t} = -\frac{\partial}{\partial \theta}(\rho v)$$

which expresses the conservation of oscillators with frequency ω. The infinite-N versions of (24) and (25) are respectively

$$re^{i\psi} = \int_0^{2\pi} \int_{-\infty}^{+\infty} e^{i\theta} \rho(\theta, t, \omega) g(\omega) d\omega d\theta$$

and

$$v(\theta, t, \omega) = \omega + K \int_0^{2\pi} \int_{-\infty}^{+\infty} \sin(\theta' - \theta) \rho(\theta', t, \omega') g(\omega') d\theta' d\omega'.$$

Finally, the governing equation for ρ becomes

$$\frac{\partial \rho}{\partial t} = -\frac{\partial}{\partial \theta}(\rho[\omega + K \int_0^{2\pi} \int_{-\infty}^{+\infty} \sin(\theta' - \theta) \rho(\theta', t, \omega') g(\omega') d\theta' d\omega']). \tag{26}$$

Introduction of Noise Processes. In [85], H.Sakaguchi considers the Fokker-Planck equation

$$\frac{\partial \rho}{\partial t} = D\frac{\partial^2 \rho}{\partial \theta^2} - \frac{\partial}{\partial \theta}(\rho v), \tag{27}$$

where $D \geq 0$ is a noise strength. Equation (27) reduces to (26) for $D = 0$.

Crawford's Work. J.D. Crawford worked on Kuramoto-Sakaguchi's model (27) (cf. [30]). Later on, he considered a more general model where $\sin(\theta' - \theta)$ is replaced by $f(\theta' - \theta)$.

The Acebron-Spigler Model

In [1], J.A. Acebron, R. Spigler, are intersted in self-synchronization among the coupled oscillators, both in phase and in frequency. They point out that the Kuramoto-Sakaguchi model with noise terms does not allow for simultaneous synchronization in both phase and frequency. They propose a new model in which frequencies of the oscillators vary in time along with their phase. The governing equations, when $N \gg 1$ and noise effects are included, are a set of Langevin equations, which lead, when $N \to \infty$, to a Fokker-Planck type equation of the form (with normalized parameters)

$$\frac{\partial \rho}{\partial t} = \frac{\partial^2 \rho}{\partial \omega^2} - \omega \frac{\partial \rho}{\partial \theta} + \frac{\partial}{\partial \omega}[(\omega - \Omega - K\mathcal{K}(t,\theta))\rho], \tag{28}$$

where

$$\mathcal{K}(t,\theta) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \int_{0}^{2\pi} g(\Omega) sin(\varphi - \theta)\rho(\varphi, \omega, t, \Omega) d\varphi d\omega d\Omega$$

associated with the conditions $\rho|_{\theta=0} = \rho|_{\theta=2\pi}, \rho|_{t=0} = \rho_o(\theta,\omega,\Omega), \rho(\theta,\omega,t,\Omega) > 0$ and $\int_0^{2\pi}\int_{-\infty}^{+\infty} \rho(\theta,\omega,t,\Omega) d\omega d\theta = 1$ for all t, Ω. Here $\theta(t)$ is the phase and $\omega(t) = \frac{d\theta}{dt}(t)$ the frequency, at time t, and $g(\Omega)$ represents a natural frequency distribution.

The Fokker-Planck Type Ultraparabolic Integro-Differential Equation

In [4] and [5], D.R. Akhmetov, M.M. Lavrentiev JR. and R. Spigler study existence and uniqueness of solutions to (28), in $\mathcal{Q}_T = \{(\theta,\omega,t,\Omega) \in [0,2\pi] \times \mathbb{R} \times [0,T] \times [-G,+G]\}$, when $g \in L^1(\mathbb{R})$ is bounded and compactly supported in $[-G,+G]$ and the initial condition ρ_o has an exponential decay in ω at infinity, namely

$$\sup_{\theta\in\mathbb{R},\Omega\in[-G,+G]} |D^{l_1,l_2,l_3}_{\theta,\omega,\Omega} \rho_o(\theta,\omega,\Omega)| \le Ce^{-M\omega^2}$$

for all $\omega \in \mathbb{R}, l_1 + l_2 + l_3 \le l_o$. Equation (28) may be interpreted as an ultraparabolic (integro-differential) equation with $t_1 = \theta, t_2 = t$ or as a parabolic equation in (ω, θ), degenerate with respect to θ (cf. Section 2.15). In [4] and [5] existence and uniqueness of a solution ρ to problem (28), either in anisotropic Sobolev and Hölder spaces or in suitable classes of decaying functions, are considered. Optimal decay estimates for ρ were obtained in D.R. Akhmetov, R. Spigler [7]. The proofs are based on suitable parabolic regularization of (28).

Conclusion. In this entry, I was mostly interested in the successive forms of the Fokker-Planck type equation, from the Kuramoto model to the recently studied ultraparabolic equations. The analysis of these equations and their innumerable applications will be found in the references.

2. From Gierer-Meinhardt System to Wavefront Sets

2.1. Gierer-Meinhardt System

The Gierer-Meinhardt Model

The Gierer-Meinhardt model is a system of two coupled nonlinear parabolic equations:

$$A_t = d\Delta A - A + \frac{A^2}{H}, \quad H_t = D\Delta H - H + A^2, \ x \in \Omega, t > 0,$$

associated with the boundary conditions

$$\frac{\partial A}{\partial n} = \frac{\partial H}{\partial n} = 0, \ x \in \partial\Omega, t \geq 0,$$

where Ω is a bounded Lipschitzian domain, in $\mathbb{R}$ or $\mathbb{R}^2$, $A = A(x,t)$ and $H = H(x,t)$ are the concentrations of an activator substance and an inhibator substance, respectively, in cell biology and physiology (cf. Y. You [102]).

Example 12. In [102], the author considers the ground state solutions of the Gierer-Meinhard-type system in $\mathbb{R}$, i.e. the solutions of the system of ordinary differential equations:

$$u" - u + \frac{u^2}{v} = 0, \quad v" - \sigma^2 v + u^2 = 0, \quad x \in \mathbb{R} \tag{29}$$

associated with the conditions $u, v > 0$ and $u, v \to 0$ as $|x| \to \infty$, where $\sigma^2 = \frac{d}{D} \ll 1$.

The author is interested in an approximation of the number $k(\sigma)$ of spikes of the solutions to (29) one can expect to occur, as $\sigma \to 0$. He shows that, for σ small enough, there exists a solution (u, v) of (29) such that u has $k =$const $\sigma^{-\beta}$ spikes where β can be arbitrarily close to $\frac{1}{2}$.

Example 13. In [99], J. Wei and M. Winter consider the Gierer-Meinhardt system, in $\Omega = (-1, +1)$, with $t \geq 0$

$$\begin{cases} a_t = \epsilon^2 a_{xx} - a + \dfrac{a^2}{h}, \\ \tau h_t = (D(x) h_x)_x - h + a^2, \end{cases}$$

associated with the Neumann boundary conditions

$$a_x(-1) = a_x(+1) = 0, \quad h_x(-1) = h_x(+1) = 0.$$

Here the diffusion coefficient $D(x)$ is the step function $D(x) = D_1$ on $-1 < x < x_b, D(x) = D_2$ on $x_b < x < 1$, $D_1 > 0, D_2 > 0$, $D_1 \neq D_2$, ϵ, τ are constants with $0 < \epsilon \ll 1, \tau \geq 0$. They study, in adapted Sobolev-type spaces, existence and location of stationary spike solutions of the above problem, far away from x_b, and next, at a distance of order ϵ from x_b. The proofs involve the Fisher function and a nonlocal eigenvalue problem (cf. Section 2.9).

2.2. Gray-Scott Systems

The Gray–Scott Model

This reaction-diffusion system models an irreversible reaction involving two reactants in a gel reactor, where the reactor is maintained in contact with a reservoir of one of the two chemicals in the reaction. In dimensionless units it can be written as

$$\begin{cases} V_t = b\Delta V - (F + k)V + UV^2, \text{ in } \Omega, \\ U_t = a\Delta U + F(1 - U) - UV^2, \text{ in } \Omega, \\ \dfrac{\partial U}{\partial \nu} = \dfrac{\partial V}{\partial \nu} = 0, \text{ on } \partial\Omega, \end{cases}$$

where the unknowns $U(x, t)$ and $V(x, t)$ represent the concentrations of the two biochemicals at a point $x \in \Omega \subset \mathbb{R}^{\mathbb{N}}, N \leq 3$ at the time t; Ω is a bounded smooth domain, a, b are the diffusion coefficients of U and V respectively, F denotes the rate at which U is fed from the reservoir into the reactor, and k is a reaction-time constant. Various Gray-Scott problems have been intensively studied in the recent years, with different values of the parameters N, a, b, F, k and different choices for Ω (cf. T. Kolokolnikov, J. Wei [68])

Example 14. In [39] and [40], A. Doelman, R.A. Gardner, T.J. Kaper consider the following system:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} - uv^2 + \delta^2 a(1-u),$$
$$\frac{\partial v}{\partial t} = \delta^2 \frac{\partial^2 v}{\partial x^2} + uv^2 - \delta^\beta bv,$$

where $x \in \mathbb{R}$, $a, b = \mathcal{O}(1)$ with respect to δ and $0 \leq \beta < 1$.

Example 15. In [80], D.S. Morgan and T.J. Kaper study, in R^2, stationary ring solutions, for $r \in [0, r_{max}]$, of the system

$$\frac{\partial^2 U}{\partial r^2} + \frac{1}{r}\frac{\partial U}{\partial r} = UV^2 - A + AU,$$
$$D(\frac{\partial^2 V}{\partial r^2} + \frac{1}{r}\frac{\partial V}{\partial r}) = -UV^2 + BV,$$

with boundary conditions $U_r(0) = U_{r_{max}}(0) = V_r(0) = V_{r_{max}}(0) = 0$ where A, B, D, are constants with $D \ll 1$ (see also Section 2.9).

Example 16. In [68], the following Gray-Scott systems and the associated stationary system in a bounded smooth domain $\Omega \subset \mathbb{R}^N$ are considered

$$\begin{aligned}&\frac{\partial v}{\partial t} = \epsilon^2 \Delta v - v + Auv^2,\\ &\tau\frac{\partial u}{\partial t} = \Delta u - (6\epsilon)^{-1} uv^2 + (1-u),\\ &u > 0,\ v > 0,\end{aligned} \tag{30}$$

associated with the boundary condition $\frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0$ on $\partial\Omega$, mostly when $N = 2, 3$ and Ω is a ball B_R of center 0 and radius R; the cases $\Omega = \mathbb{R}^N$, or $\mathbb{R}^N \setminus B_R$, or $B_{R_2} \setminus B_{R_1}$ are also discussed.

Example 17. In [25], W. Chen and M.J. Ward consider the singularly perturbed model

$$v_t = \epsilon^2 v_{xx} - v + \mathcal{A}uv^2,$$
$$\tau u_t = Du_{xx} + (1-u) - \epsilon^{-1}uv^2,$$

on $|x < 1|$, with boudary conditions $u_x = v_x = 0$ at $x = \pm 1$. Here ϵ is a positive small parameter, which goes to zero, $\tau > 0$ and $D = O(1)$. Spike solutions in the intermediate regim $O(1) \ll \mathcal{A} \ll O(\epsilon^{-\frac{1}{2}})$ are investigated.

2.3. H_∞-Functional Calculus

This calculus concerns sectorial operators in a Banach space X. Let A be an operator in X, with *dense* domain, whose spectrum and resolvent set are denoted by $\sigma(A)$ and $\rho(A)$ respectively. The space of bounded operators in X, with its isual norm, will be denoted by $\mathcal{L}(X)$.

Definitions

Cf A. Mcintosh [76].

Sectorial Operators. For $0 < \omega < \pi$, let S_ω and S^o_ω be the sectors

$$S_\omega = \{z \in \mathbb{C}; z = 0 \text{ or } |\arg z| \leq \omega\},$$

$$S^o_\omega = \{z \in \mathbb{C}; z \neq 0, \text{ and } |\arg z| < \omega\},$$

i.e. S^o_ω is the interior of S_ω. Set

$$\Lambda(\omega) : \{\mathbb{C} \setminus S^0_\omega\} = \{re^{i\varphi}; r \geq 0, \omega \leq \varphi \leq 2\pi - \omega\}. \tag{31}$$

Definition 7. The operator A is called sectorial of angle ω if it is *closed* and satifies the conditions:

i) $\sigma(A) \subset S_\omega$;

ii) $\|(A - \lambda I)^{-1}\| \leq C_\mu |\lambda|^{-1}$, for all $\lambda \in \mathbb{C}$, with $|\arg \lambda| \geq \mu$, and all $\mu > \omega$.

Spaces of Analytic Functions

Definition 8. The following spaces will be needed:

$$H_\infty(S^o_\omega) = \{f : S^o_\omega \to \mathbb{C}, \text{ f is analytique and } \|f\|_\infty < \infty\},$$

where $\|f\|_\infty = \sup\{|f(z)|, z \in S^o_\omega\}$

$$H(\omega) = \{f \in H_\infty(S^o_\omega) : \exists s > 0, c > 0 \text{ such that } |f(z)| \leq \frac{c|z|^s}{1 + |z|^{2s}} \text{ for all } z \in S^o_\omega\}$$

H_∞-Calculus for a Sectorial Operator

(cf. [76] and C. Kunstmann, L. Weis [70]). Let A be a sectorial operator of angle ω on X. The aim of a bounded H_∞- functional calculus for A, with respect to ω, is to define, for $0 \leq \omega < \mu < \pi$, a map

$$\Phi_A : f \in H_\infty(S^o_\mu) \to f(A) \in \mathcal{L}(X) \tag{32}$$

such that:

i)

$$(\alpha f + g)(A) = \alpha f(A) + g(A), \quad \forall f, g \in H_\infty(S^o_\mu), \alpha \in \mathbb{C};$$

ii)

$$(fg)(A) = f(A)g(A) \in \mathcal{L}(X), \quad \forall f, g \in H_\infty(^o_\mu);$$

iii)

$$\|f(A)\|_{\mathcal{L}(X)} \leq C \, \|f\|_\infty, \quad \forall f \in H_\infty(S^o_\mu);$$

iv)

$$\forall \lambda_1 \notin S_\mu \text{ and } f(\lambda) = (\lambda_1 - \lambda), \text{ then } f(A) = (\lambda_1 - A)^{-1}.$$

For that, Φ_A is defined on a suitable subspace of $H_\infty(S^o_\mu)$ and, next, is extended to the whole space.

Example 18. In [76], X is a Hilbert space and Φ_A is defined on $H(\mu), \omega < \mu$, in the following way:

$$f(A) = \frac{1}{2\pi i} \int_{\gamma_\theta} (A - \lambda I)^{-1} f(\lambda) d\lambda, \ \omega < \theta < \mu, \tag{33}$$

where γ_θ is the contour

$$g(t) = \begin{cases} -te^{-i\theta}, & -\infty < t \leq 0, \\ te^{i\theta}, \ 0 \leq t < \infty. \end{cases} \tag{34}$$

Example 19. In R. Denk, J. Saal, J. Seiler [35], the authors consider a sector S_ω, $0 < \omega < \pi$, and, in a Banach space X, a sectorial operator A of angle ω such that:

i) $\Lambda(\omega) \setminus 0$ is contained in the resolvent set of A, $\|\lambda(\lambda - A)^{-1}\|_{\mathcal{L}(X)}$ is uniformly bounded in $0 \neq \lambda \in \Lambda(\omega)$.

iii) A is one-to-one with dense range.

Thanks to condition i), for $f \in H(\omega)$, the integral (33) is defined on the contour γ_ω. Therefore, Φ_A is well defined on $H(\omega)$.

Example 20. See parameter-elliptic Douglis-Nirenberg system p. 95, and Example 11

Remark 4. For complements on the assumptions on A, namely: densely defined, closed, injective, with dense range, see, for instance, [70].

2.4. Kirchoff-Type Equations

The Kirchhoff Model

In [66], G. Kirchhoff studies the integro-diffrential equation:

$$u_{tt} - \left(\epsilon^2 + \frac{1}{2\ell}\int_0^{\ell} u_s^2 ds\right) \cdot u_{ss} = 0, \tag{35}$$

which appears in transversal vibrations of an elastic string of length ℓ. This Kirchhof model is briefly described in S. Spagnolo [91]. Since then, Kirchhoff-type equations have been intensively studied in different situations.

The Cauchy Problem for a Kirchhoff-Type Equation

Let Ω be an open subset of $\mathbb{R}^n$. We note that equation (35) is a particular case of

$$u_{tt} - \varphi(\int_{\Omega} |\nabla u|^2 dx)\Delta u = 0, \tag{36}$$

where φ is a regular function satisfying $\varphi(s) \geq \lambda_o > 0, \forall s > 0$ (in (35) $\varphi(s) = \epsilon^2 + \frac{1}{2\ell}s$). More generaly, in [91] a Kirchhoff-type equation is defined as:

$$u_{tt} - \sum_{i,j,=1}^{n} \varphi_{ij}\left(||D^{\beta_1}u||_{L^2(\Omega)}, ..., ||D^{\beta_p}u||_{L^2(\Omega)}\right) u_{x_i x_j} = 0, \tag{37}$$

where the $\varphi_{ij}(s_1, ..., s_p)$ are regular functions satisfying

$$\sum_{i,j=1}^{n} \varphi_{ij}(s_1, ..., s_p)\xi_i\xi_j \geq \lambda_o|\xi|^2, \quad \lambda_o \geq 0,$$

and $\beta_j \in \mathbb{N}^n$. Being inspired by S. Bernstein's paper [16], the author investigates, in suitable functional spaces, local in t and global existence, uniqueness of solutions u to equation (37) associated with initial conditions $u(0,x) = u_o(x), u_t(0,x) = u_1(x), x \in \Omega$, when Ω is an interval of $\mathbb{R}$ and $u(t,\cdot)$ is Ω-periodic, or $\Omega = \mathbb{R}^n$.

Pseudo-Differential Operators of Kirchhoff-Type

In [32], P. D'Ancona and S. Spagnolo show the existence of a unique solution, in a suitable functional space, to the pseudo-differential equation

$$u_{tt} + F\left(||a_1(-iD_x)u||^2_{L^2}, ..., ||a_p(-iD_x)u||^2_{L^2}; -iD_x\right) u = 0 \tag{38}$$

associated with initial conditions of the form

$$u(0,x) = \epsilon u_o(x), \quad u_t(0,x) = \epsilon u_1(x),$$

where $a_j(\xi)$ are smooth functions on $\mathbb{R}^n \backslash 0$ satisfying growth conditions, $F(s_1, ..., s_p; \xi_1, ..., \xi_n)$ is a C^2 real function defined on $\mathbb{R}^p \times \mathbb{R}^n$, with $|s| \leq 1, |\xi| \neq 0$, and satisfying for some real numbers $\nu > 0, m \geq 1$:

$$F(s; \lambda\xi) = \lambda^{2m} F(s, \xi), \quad \forall \lambda \geq 0,$$

$$\nu^2 |\xi|^{2m} \leq F(s,\xi) \leq \nu^{-2}|\xi|^{2m}, \quad |\nabla_\xi F(s,\xi)| \leq \nu^{-1}|\xi|^{2m-1},$$

and ϵ small enough. A particular case of (38) is the equation:

$$u_{tt} + (-1)^m \sum_{|\alpha|=2m} f_\alpha \left(||D_x^{\beta_1} u||^2_{L^2}, ..., ||D_x^{\beta_p} u||^2_{L^2}\right) D_x^\alpha u = 0, \tag{39}$$

where $f_\alpha(s_1, ..., s_p)$ are C^2 real functions on $\mathbb{R}^p$, satisfying the strict hyperbolicity condition

$$\sum_{|\alpha|=2m} f_\alpha(s_1, ..., s_p)\xi^\alpha \geq \nu^2 |\xi|^{2m}, \quad \nu > 0.$$

The simplest example of equation (39) is equation (36), where $\Omega = \mathbb{R}^n$.

Abstract Setting of Kirchhoff-Type Equation

Let H be a Hilbert space, A a selfadjoint non-negative operator on H, with dense domain $D(A)$. Then there exists a unique self-adjoint positive operator $A^{\frac{1}{2}}$ such that $(A^{\frac{1}{2}})^2 = A$ (for the definition and properties of $A^\alpha, 0 < \alpha < 1$, see, for instance, D. Huet [60], T. Kato [64] or M. Schechter [86]). If $H = L^2(\Omega), A = -\Delta, D(A) = H^2(\Omega) \cap H^1_o(\Omega)$, the equation

$$u_{tt} + \varphi(|A^{\frac{1}{2}} u|^2) Au = O \tag{40}$$

becomes equation (36) ($|\ |$ denotes the norm in H). The abstract Cauchy problem

$$u''(t) + \delta u'(t) + m(|A^{\frac{1}{2}} u(t)|^2) Au(t) = 0,\ t \geq 0, \quad u(0) = u_o, \quad u'(0) = u_1, \tag{41}$$

is investigated in M. Ghisi, M. Gobino [53], where $\delta > 0$ and $m : [0, +\infty[\to [0, +\infty[$ is a locally Lipschitz continuous function. When $(u_o, u_1) \in D(A) \times$

$D(A^{\frac{1}{2}})$ and under the nondegeneracy condition $m(|A^{\frac{1}{2}}u_o|^2) > 0$, existence, uniqueness, in suitable functional spaces, of u solution to (41), and the asymptotic behavior of $(u(t), u'(t), u''(t))$, in $D(A) \times D(A^{\frac{1}{2}}) \times H$, as $t \to \infty$, are studied.

Singular Perturbations in Kirchhoff Type Equations

In M. Ghisi, M. Gobino [54], time-decay estimates, as $t \to \infty$, are obtained for the singularly perturbed Cauchy problem

$$\epsilon u''_\epsilon + u'_\epsilon + m(|A^{\frac{1}{2}}u_\epsilon|^2)Au_\epsilon = 0, \quad u_\epsilon(0) = u_o, \quad u'_\epsilon(0) = u_1, \tag{42}$$

and for the first order limit problem

$$u' + m(|A^{\frac{1}{2}}u|^2)Au = O, \quad u(0) = u_o, \tag{43}$$

where $\epsilon > 0$, $(u_o, u_1) \in D(A) \times D(A^{\frac{1}{2}})$ and m is of class C^1. Decay estimates are stated for equation (43) under the nondegeneracy condition $|A^{\frac{1}{2}}u_o|^2 m(|A^{\frac{1}{2}}u_o|^2) > 0$, and, for equation (42), under a weaker condition $m(|A^{\frac{1}{2}}u_o|^2) > 0$. In this last case, most of the constants which appear in decay estimates are independent of ϵ. A clear comparison with similar previous estimates, obtained by several authors, is presented.

2.5. Martingale Difference Sequences

(cf. D.L. Burkholder [21], p. 273) Let B be a Banach space.

Definition 9. A sequence $\{f_i\}$ is a B-valued martingale in $L^1((0,1), B)$ if its *difference sequence* $\{d_i\}$ defined by $d_i = f_i - f_{i-1}$ satisfies

$$\int_0^1 \varphi(d_1(x), ..., d_n(x))d_{n+1}(x)dx = 0 \quad (\text{in } B) \tag{44}$$

for all bounded continuous function $\varphi : B \times ... \times B \to \mathbb{R}$, and all $n \geq 1$.

Example 21. Let F be an arbitrary function $[0,1] \to B$ and, for any positive integer n, a partition P_n of $[0,1)$ into a finite number of left-closed and right-open intervals, such that each interval in P_n is the union of intervals in P_{n+1}. Then, if $[a,b) \in P_i$, $\{f_i\}$ defined by

$$f_i(t) = \frac{F(b) - F(a)}{b - a}, \quad t \in [a,b),$$

is a B-valued martingale in $L^1((0,1)B)$.

Application. The martingale difference sequences are essential in the theory of UMD spaces and, therefore, in maximal regularity.

2.6. Maximal L_p-Regularity

(cf. P.C. Kunstmann, L. Weis [70])

Definitions

i) Let Ω be a smooth open set in $\mathbb{R}^n$, A be an elliptic operator on $X = L^q(\Omega)$, $1 < q < \infty$, and consider the Cauchy problem

$$y' = Ay(t) + f(t), \quad y(0) = 0, \tag{45}$$

where $f \in L_q([0,T], X)$. Then A *has maximal L_p-regularity* if the solution $y(t)$ to (45) satisfies the a priori estimate

$$\int_0^T \|y'(t)\|^p_{L_q(\Omega)} dt + \int_0^T \|Ay(t)\|^p_{L_q(\Omega)} dt \le C \int_0^T \|f(t)\|^p_{L_q(\Omega)} dt.$$

ii) More generaly, consider the abstract Cauchy problem

$$y' = Ay + f, \quad y(0) = x_o, \tag{46}$$

where A is the generator of a bounded analytic semigroup on a Banach space X. A is said to have *maximal L_p-regularity* for $1 < p < \infty$ on $[0, T), 0 < T \le \infty$, if for $x_o = 0$ and all $f \in L_p([0,T), X)$, the solution y of the Cauchy problem (46) is differentiable a.e., takes its values in the domain $D(A)$ of A a.e., and y' and Ay belong to $L_p([0,T), X)$, i.e. y' and Ay cannot be in a better space than f. Then, there exists a constant C, which depends on p, such that

$$\|y'\|_{L^p([0,T),X)} + \|Ay\|_{L^p([0,T),X)} \le C\|f\|_{L^p([0,T),X)}.$$

Examples

i) On a **Hilbert space**, every generator of a bounded analytic semi-group has maximal L_p-regularity.

ii) This is not true in a Banach space. But, in a Banach space, if an operator A has L_p-maximal regularity, then it is the generator of an analytic semi-group.

iii) All analytic generators with a bounded H^∞-functional calculus on a UMD-space (cf Section 2.16) has maximal L^p-regularity.

Many further applications and examples are presented in [70].

2.7. The Moser-Trudinger Inequality

The Moser-Trudinger inequality states that:

$$C(\Omega) = \sup_{u \in H_o^1(\Omega); ||\nabla u||_2 = 1} \int_\Omega e^{4\pi u^2} dx < \infty,$$

where Ω is a smooth bounded domain in $\mathbb{R}^2$ (see N.S. Trudinger [97] and J. Moser [82]).

A Sharp Form of the Moser-Trudinger Inequality

Let $\lambda_1(\Omega) > 0$ be the first eigenvalue of the Laplacian with Dirichlet boundary conditions in Ω, and

$$C_\alpha(\Omega) = \sup_{u \in H_o^1(\Omega), ||\nabla u||_2 = 1} \int_\Omega e^{4\pi u^2(1+\alpha||u||_2^2)} dx \tag{47}$$

In [2], Adimurthi and O. Druet proved that

$$C_\alpha(\Omega) < +\infty \ \text{ if } \ 0 \leq \alpha < \lambda_1(\Omega)$$

and

$$C_\alpha(\Omega) = +\infty \text{ if } \alpha \geq \lambda_1(\Omega).$$

In [101], Y. Yang, shows the existence of an extremal function for which the supremum, in (47), is attained in the case $0 \leq \alpha < \lambda_1(\Omega)$.

2.8. Moving Planes and Sliding Methods

The Classical Method

The classical method of moving planes is described in J. Serrin [89], and used to prove the following result: if Ω is a bounded open connected domain in $\mathbb{R}^n$, whose boundary is of class C^2, and if there exists $u \in C^(\bar{\Omega})$ satisfying $\Delta u = -1$ in Ω, $u = 0$ and $\frac{\partial u}{\partial n} =$ constant on $\partial\Omega$, then Ω is a ball.

Example 22. In [15], H. Berestycki and L. Nirenberg used the moving plane and sliding methods in proving monotonicity or symmetry in the x_1 direction for solutions to nonlinear elliptic equations $F(x, u, Du, D^2u) = 0$ in a bounded domain $\Omega \subset \mathbb{R}^n$, which is convex in the x_1 direction. "Both methods compare values of the solution of the equation at two different points. In the moving plane method, one point is the reflection of the other in a hyperplane

$x_1 = \lambda$ and then, the plane is moved up to a critical position. In the sliding method, the second point is obtained from the first by sliding the domain in the x_1 direction and the domain is slid up to a critical position". Many forms of the maximum principle are crucial in this paper.

Technique of the Domain Reflection

A different form of the classical moving plane analysis is used in E.M. Harrell, P. Kröger, K. Kurata [59]. Let Ω be a bounded open domain in $\mathbb{R}^n$, and P a hyperplane of dimension $n-1$ which intersects Ω. For any connected set S which does not intersect P, S^P denotes its reflection through P. If there exists a connected component Ω_s of $\Omega \setminus P$ such that Ω_s^P is a proper subset of the other connected component Ω_b of $\Omega \setminus P$, *P is called a hyperplane of interior reflection for* Ω. Moreover Ω_s (resp. Ω_b) is called the small (resp. big) side of Ω, and Ω *is said to have the interior reflection property with respect to P.*

Example 23. Using the technique of the domain reflection, the authors study how to place, inside Ω, an obstacle or a well in order to minimize or maximize the principal eigenvalue of the Laplacian operator in Ω with zero Dirichlet boundary condition on the boundary $\partial\Omega$. The obstacle B may be hard (i.e. zero Dirichlet conditions are additionally imposed on B) or replaced by a central non-negative (resp. non-positive) function supported by B, i.e. the investigated operator has the form

$$-\Delta + \alpha\chi_B(x),$$

where χ_B is the indicator function of B, and $\alpha > 0$ (resp. < 0). Several examples with very clear figures are presented. Similar problems are studied for the stationnary Schrödinger operator in Ω.

Example 24. A similar problem is investigated, with the same technique of domain reflection, in P. Exner, A. Mantile [47]. Here B is replaced by a single point interaction $x_o \in \Omega$. The domain Ω is an open set in $\mathbb{R}^d$, $d = 2, 3$, bounded and connected, with a piecewise C^1 boundary. Let $\alpha \in \mathbb{R}, \lambda \in \mathbb{C} \setminus \mathbb{R}$. The authors consider a family of operarors H_α, whose domain is

$$D(H_\alpha) = \left\{\psi \in L^2(\Omega),\ \psi = \Phi^\lambda + q\mathcal{G}_o^\lambda(.,x_o),\ \phi^\lambda \in H^2 \cap H_o^1(\Omega)\right\}$$

and defined by

$$H_\alpha\psi = -\Delta\phi^\lambda - \lambda q\mathcal{G}_o^\lambda(\cdot, x_o),$$

associated with a suitable condition for $\Phi^\lambda(x_o)$, where $\mathcal{G}_o^\lambda$ denotes the Green function of $-\Delta + \lambda$ in Ω, corresponding with homogeneous Dirichlet boundary conditions on $\partial\Omega$. How to place the point interaction center x_o to minimize the principal eigenvalue of H_α is discussed.

2.9. Nonlocal Eigenvalue Problem

1) This type of problem arises in the study of the stability of solutions of *Gray-Scott problems* under small perturbations of the form $e^{\lambda t}(U(x,t), V(x,t))$. It appears in the following form:

$$y" + f(t)y = Cg(t)\int_{-\infty}^{+\infty} h(t)y(t)dt, \tag{48}$$

where f, g, h, are known functions. It is called nonlocal because of the presence, in the right member of (48), of the nonlocal term $\int_{-\infty}^{+\infty} h(t)y(t)dt$.

Example 25. In A. Doelman, R.A. Gardner, T.J. Kaper [39],

$$f(t) = (\frac{12}{\cosh^2 t} - P),\ g(t) = \frac{1}{\cosh^4 t},\ h(t) = \frac{1}{\cosh^2 t}.$$

Example 26. In D.S. Morgan, T.J. Kaper [80],

$$f(t) = (12\sinh^2 t - P),\quad g(t) = sech^4 t,\quad h(t) = sech^2 t$$

(see Section 2.2)

2) In their investigation of spikes for a **Gierer-Meinhardt system** (cf. Section 2.1), J. Wei and M. Winter [99] p.305 consider the nonlocal eigenvalue problem:

$$\Delta\phi - \phi + 2w\phi - \gamma\frac{\int_{\mathbb{R}} w\phi dy}{\int_{\mathbb{R}} w^2 dy}w^2 = \lambda\phi, \quad \phi \in H^1(\mathbb{R}),$$

where $w(y) = \frac{3}{2}sech^2(\frac{y}{2})$. .

2.10. Scattering Metric

The following definitions are given in R.B. Melrose [78].

Euclidean Metric Space

Let $S^n_+ = \{t = (t_1, ..., t_{n+1}) \in \mathbb{R}^{n+1},\ t_{n+1} \geq 0\}$ and SP be the stereographic projection

$$z \in \mathbb{R}^n \to (\frac{z}{\sqrt{1+|z|^2}}, \frac{1}{\sqrt{1+|z|^2}}) \in S^n_+.$$

The image of $\mathbb{R}^n$ under this map is the interior of S^n_+. The function $x = \frac{1}{|z|} = \frac{t_{n+1}}{\sqrt{1-t^2_{n+1}}}$ is a defining function for the boundary of S^n_+, which is defined on $t_{n+1} < 1$ and can be smoothly extented across $z = 0$. If we identify the collar neighborhood $\{0 \leq t_{n+1} < 1\}$ of ∂S^n_+ with $(0, +\infty) \times S^{n-1}$ by the mapping

$$t \to (\frac{t'}{\sqrt{1-t^2_{n+1}}}, \frac{t_{n+1}}{\sqrt{1-t^2_{n+1}}}) = (\omega, x),$$

the Euclidian metric becomes

$$\frac{dx^2}{x^4} + \frac{d\omega^2}{x^2},$$

where $d\omega^2$ is the standard round metric on S^{n-1}. Thus, if $\mathbb{R}^n$ is replaced by its compactification, the Euclidian metric becomes a scattering metric.

The Compact Manifold Case

More generaly, let $\mathcal{M}$ be a compact manifold with boundary, and x be a boundary defining function for $\mathcal{M}$, i.e. x is a smooth positive function on $\mathcal{M}$ such that $x = 0$ exactly on $\partial\mathcal{M}$ and $dx \neq 0$ on $\partial\mathcal{M}$. A scattering metric g on $\mathcal{M}$ is a Riemannian metric on the interior of $\mathcal{M}$, which has, near the boundary, an expression

$$g = \frac{dx^2}{x^4} + \frac{h}{x^2},$$

where h is a smooth symmetric 2-tensor on $\mathcal{M}$, whose restriction to $\partial\mathcal{M}$ is a metric

Applications

See wavefront sets, Section 2.17, Application.

2.11. Sobolev-Type Spaces $M^{1,p}$ on Metric-Measure Spaces

Definition of a Doubling Measure

Let $X = (X, d, \mu)$ be a metric space equipped with a Borel regular outer measure μ. Moreover μ is assumed to be doubling, i.e. there exists a constant $C_\mu > 0$ such that $0 < \mu(2B) \leq C_\mu\ \mu(B) < \infty$ for all ball $B \subset X$. Then, there exists a constant C_o such that

$$\frac{\mu(B)}{\mu(B_o)} \geq C_o(\frac{r}{r_o})^s, \tag{49}$$

whenever $B_o = B(x_o, r_o), B = B(x, r), x \in B_o, 0 < r \leq r_o, s = \log_2(C_\mu)$ The smallest constant s for which (49) holds is called the doubling dimension of μ.

Space $M^{1,p}(X)$, Definition, Properties

The space $M^{1,p}(X), 0 \leq p < \infty$ is the space of $u \in L^p(X)$ such that there exists a generalized gradient $g \in L^p(X), g \geq 0$ with

$$|u(x) - u(y)| \leq d(x,y)(g(x) + g(y)),$$

μ-almost every where in X. Equipped with the norm

$$||u||_{1,p} = (||u||^p_{L^p} + \inf_g ||g||^p_{L_p})^{\frac{1}{p}},$$

$M^{1,p}(X)$is a Banach space if $p \geq 1$. These spaces were defined by P. Hajlasz [57]. A good introduction to these spaces and other Sobolev type spaces on metric measure spaces, and applications such as geometric analysis, quasi conformal mappings, non-linear subelliptic equations, differential geometry, analysis of graphs, are presented in P. Hajlasz [58]. Pointwise behavior such as Lebesgue points, Hölder continuity, for $M^{1,p}$ functions have been studied by P.Hajlasz and others. In particular, the case $p = 1$ was studied recently by J. Kinunen, H. Tuominen [65]. We recall that $M^{1,p}(\mathbb{R}^n)$ coincide with the usual Sobolev space $W^{1,p}(\mathbb{R}^n)$ if $p > 1$. This is not true for $p = 1$. And the Hardy-Littlewood maximal operator $\mathcal{M}$ defined by

$$\mathcal{M}h(x) = \sup_{r>0} \frac{1}{\mu(B(x,r))} \int_{B(x,r)} |h| d\mu$$

is bounded in $L^p(X)$ for $p > 1$, but is not bounded in $L^1(X)$.

2.12. Steklov Eigenproblems

Steklov and Membrane Eigenvalues

(see J.R. Kuttler, V.G. Sigillito [73]). Steklov eigenvalues are interpreted physically as the natural frequencies of a vibrating membrane with its mass concentrared around the edge. Three Steklov eigenvalue problems:

$$\Delta u = 0 \text{ on } D, \quad \frac{\partial u}{\partial n} = pu \qquad \text{on } \partial D,$$

$$\Delta^2 u = 0 \ \text{ on } D, \quad \frac{\partial u}{\partial n} = \frac{\partial \Delta u}{\partial n} + \xi u = 0 \qquad \text{on } \partial D,$$
$$\Delta^2 u = 0 \ \text{ on } D, \quad u = \Delta u - q\frac{\partial u}{\partial n} = O \qquad \text{on } \partial D,$$

and two membrane eigenvalue problems:

$$\Delta u + \lambda u = 0 \ \text{ on } D, \quad u = 0 \quad \text{on } \partial D \qquad \text{(fixed membrane)},$$
$$\Delta u + \mu u = 0 \ \text{ on } D, \quad \frac{\partial u}{\partial n} = 0 \text{ on } \quad \partial D \qquad \text{(free membrane)},$$

are considered on a bounded, smooth domain $D \subset \mathbb{R}^2$. The authors obtain sharp relationships connecting the first non-zero eigenvalues p_2, ξ_2, q_1 of Steklov problems and membrane sigenvalues.

The Harmonic Steklov Eigenproblem

A function $s \in H^1(\Omega)$ is said to be a harmonic Steklov eigenfunction on Ω corresponding to the Steklov eigenvalue δ, if s satisfies:

$$\int_\Omega \nabla s \cdot \nabla v \, dx = \delta |\partial\Omega|^{-1} \int_{\partial\Omega} sv \, d\sigma, \quad \forall v \in H^1(\Omega).$$

Here Ω is a smooth bounded domain in $\mathbb{R}^n$, cf. G. Auchmuty [11]. This problem arises, in particular, as a model for the sloshing of a perfect fluid in a tank.

Applications

In [12], G. Auchmuty uses harmonic Steklov eigenfunction expansion to characterize the trace spaces $H^s(\partial\Omega)$, under quite general conditions on Ω and its boundary, see Section 2.14

2.13. Strichartz-Type Inequality

Strichartz's Inequality

In [92], R. Strichartz considers the solution $u(x,t)$ of

$$iu_t + \Delta u = g(x,t), \quad (x,t) \in \mathbb{R}^n \times \mathbb{R}, \quad u(x,0) = f(x),$$

where $f \in L^2(\mathbb{R}^n), g \in L^p(\mathbb{R}^{n+1})$ with $p = \frac{2(n+2)}{n+4}$. He shows that $u \in L^q(\mathbb{R}^{n+1})$ for $q = \frac{2(n+2)}{n}$ and satisfies the inequality

$$\|u\|_q \leq C(\|f\|_2 + \|g\|_p).$$

Strichartz-Type Inequalities

Since then, *Strichartz-type inequalities* have been proved by several authors in different situations. In [84], L. Robbiano, C. Zuily study the initial value problem:

$$\imath\frac{\partial u}{\partial t} - Pu = 0 \;\; u(.,0) = u_o \in L^2(\mathbb{R}^n),$$

where

$$P = \sum_{j,k=1}^{n} D_j(g^{jk}(x)D_k) + \sum_{j=1}^{n}(D_j b_j(x) + b_j(x)D_j) + V(x), \quad D_j = \frac{1}{\imath}\frac{\partial}{\partial x_j}.$$

The assumptions are: the coefficients $g^{jk} = g^{kj}, b_j$ and V are real valued, b_j and $g^{jk} - \delta_{jk}$ belong to the space $\mathcal{B}_{\sigma o}$ of symbols which decay like $\langle x\rangle^{-1-\sigma_o}$, δ_{jk} is the Kronecker symbol, $\langle x\rangle = (1+|x|^2)^{\frac{1}{2}})$, $V \in L^\infty(\mathbb{R}^n)$, there exists $\nu > 0$ such that, for every $(x,\xi) \in \mathbb{R}^n \times \mathbb{R}^n$, the principal symbol $p(x,\xi)$ of P satisfies $p(x,\xi) \geq \nu|\xi|^2$; moreover, the bicharacteristic flow associated to $p(x,\xi)$ is not trapped backward not forward. Let $T > 0$ and $(q > 2, r)$ be a couple of real numbers verifying $\frac{2}{q} = \frac{n}{2} - \frac{n}{r}$, then the existence of a constant $C > 0$ such that

$$\|e^{-itP}u_o\|_{(L^q[-T+T],L^r(\mathbb{R}^n))} \leq C\|u_o\|_{L^2(\mathbb{R}^n)}$$

for all $u_o \in L^2(\mathbb{R}^n)$, is proved.

2.14. Trace Space $H^s(\partial\Omega)$: An Intrinsic Steklov Eigenfunction Expansion Definition

The following result is proved in [12] (cf. Section 2.12). Let Ω be a bounded, connected, open subset of $\mathbb{R}^n$, whose boundary $\partial\Omega$ satisfies quite general conditions; in particular, $\partial\Omega$ may be the union of a finite number of disjoint closed Lipschitz surfaces, each surface having finite area. The usual real sobolev space $H^1(\Omega)$ is equipped with the inner product $(u,v)_\partial = \int_{\partial\Omega} \nabla u{\cdot}\nabla v \; dx + |\partial\Omega|^{-1}\int_{\partial\Omega} uv \; d\sigma$. Denote by Γ the trace map $H^1(\Omega) \to L^2(\Omega)$. Let $0 = \delta_o < \delta_1 \leq ... \leq \delta_j \leq ...$ (resp. $s_j, j \geq 0$) be the harmonic Steklov eigenvalues (resp. the correponding $\partial-$orthonormal eigenfunctions), and $\hat{s}_j = \sqrt{1+\delta_j}\Gamma s_j$. The following definition is given, for $s \geq 0$:

$$H^s(\partial\Omega) = \{g \in L^2(\partial\Omega); \sum_{j=0}^{\infty}(1+\delta_j)^{2s}|g_j|^2 < \infty\},$$

where $g_j = |\partial\Omega|^{-1}\int_{\partial\Omega} g\hat{s}_j \; d\sigma$, equipped with the natural inner product. These spaces satisfy the same properties as the usual trace spaces (see J.-L. Lions, E.

Magenes [74]). For $s < 0$, $H^s(\partial\Omega)$ is defined by duality.

2.15. Ultraparabolic Equations

Definition 10. A linear ultraparabolic equation is an equation of the form

$$\sum_{1}^{m} k_i(x,t)v_{t_i} = \sum_{i,j=1}^{n} a_{ij}(x,t)v_{x_i,x_j} + \sum_{i=1}^{n} b_i(x,t)v_{x_i} + c(x,t)v + f(x,t),$$

where $x = (x_1, ...x_n) \in \mathbb{R}^n$, $t = (t_1, ...t_m) \in \mathbb{R}^m, m \geq 2$ and

$$\sum_{i,j=1}^{n} a_{ij}(x,t)\xi_i\xi_j \geq a_o \sum_{i=1}^{n} \xi_i^2,$$

with $a_o > 0$, for every $\xi \in \mathbb{R}^n$ and for all values (x,t) in some domain $\mathcal{Q}_T$.

Singularly Perturbed Regularized Parabolic Problem

The following problem

$$Lv = v_t + k(x,y,t)v_y - a(x,y,t)v_{xx} - b(x,y,t)v_x - c(x,y,t)v = f(x,y,t) \quad (50)$$

has been, recently, studied by D.R. Akhmetov, M.M. Lavrentiev and R. Spigler [6] with the condition of periodicity $v|_{y=0} = v|_{y=1}$ and the initial condition $v(x,y,0) = \varphi(x,y)$, on different domains. Equation (50) is an ultraparabolique equation with $y = t_1, t = t_2$. The coefficients in L and f (resp. φ) are defined on $\mathbb{R} \times \mathbb{R} \times [0,T]$ (resp. $\mathbb{R}^2$), are smooth enough and y periodic with period 1. In D.R. Akhmetov, R.Spigler [7], $Q_T = \mathbb{R} \times [0,1] \times [0,T]$. The singularly perturbed regularized parabolic problem

$$Lu^\epsilon - \epsilon u^\epsilon_{yy} = f(x,y,t) \text{ in } Q_T,$$

$$(u^\epsilon, u^\epsilon_y)|_{y=0} = (u^\epsilon, u^\epsilon_y)|_{y=1}, \quad u^\epsilon(x,y,0) = \varphi(x,y),$$

is introduced and it is proved that it does not have any boundary layer, as $\epsilon \to 0$. Then ϵ-uniform estimates of u^ϵ are obtained in suitable functional spaces, according to the regularity and decay assumptions on f, φ, and the coefficients in L. The limit of u^ϵ as $\epsilon \to 0$, leads to a solution v of problem (50), which belongs to the anisotropic Sobolev space $W^{3,2,1}(Q_T)$, and the anisotropic Hölder space $C^{\lambda,\lambda,\frac{1}{12}}(Q_T \cap \{x \in [-K,K]\})$ for $\lambda \in (0,1)$ and $K > 0$. Moreover, estimates of the form $|v(x,y,t)| \leq C_p e^{-p|x|}, \forall p > 0$ or $|v(x,y,t)| \leq \frac{C_m}{1+|x|^m}, m = 0,1...$ hold in Q_T. Similar results were obtained previously, for problem (50), by D.R. Akhmetov, M.M. Lavrentiev, R. Spigler [6], where $Q_T = [0,1]^2 \times [0,T]$ with additional boundary conditions on v. Other examples and applications to Markov processess, atomic physics, transport theory, chemistry, hydrodynamics, are mentioned in loc.cit. and references there.

A Fokker-Planck Type Equation

(see Section 1.10). In [7], the authors consider the non linear equation

$$\frac{\partial \rho}{\partial t} = \frac{\partial^2 \rho}{\partial \omega^2} + \frac{\partial}{\partial \omega}[(\omega - \Omega - \mathcal{K}(\theta, t))\rho] - \omega \frac{\partial \rho}{\partial \theta},$$

in the unbounded domain $Q_T = \{(\theta, \omega, t, \Omega) \in [0, 2\pi] \times \mathbb{R} \times [O, T] \times [-G, G]\}$, associated with the boundary and initial data $\rho|_{\theta=0} = \rho|_{\theta=2\pi}, \rho|_{t=0} = \rho_o(\theta, \omega, \Omega)$. Here

$$\mathcal{K}(\theta, t) = K \int_{-G}^{+G} \int_{-\infty}^{+\infty} \int_0^{2\pi} g(\Omega) \sin(\varphi - \theta) \rho(\varphi, \omega, t, \Omega) d\varphi d\omega d\Omega.$$

The authors prove that the solution obtained in [4] satisfies the estimate

$$|\rho(\theta, \omega, t, \Omega) \leq Ce^{-M\omega^2}.$$

2.16. UMD Spaces

(cf. D.L. Burkholder [21]) Let B be a Banach space, $1 < p < \infty$, $X = L^p((0, 1), B)$.

Definition 11. $B \in UMD$ (i.e. B has the unconditionality property for martingale differences) if there exists a positive real number c such that, if $\{d_i\}$ is a martingale difference sequence in X (i.e $d_i \in X$ and satisfies the condition (44)) and $\{\epsilon_i\}$ is a sequence of numbers in $\{-1, +1\}$, then

$$|| \sum_{k=1}^{n} \epsilon_k d_k ||_X \leq c || \sum_{k=1}^{n} d_i ||_X, \quad \forall n \geq 1. \tag{51}$$

ξ-Convex Banach Spaces

Definition 12. Let $\xi : B \times B \to \mathbb{R}$ be a symmetric, biconvex, function. A Banach space B is ξ-convex if $\xi(0, 0) > 0$ and

$$\xi(x, y) \leq ||x + y|| \text{ for } ||x||_B \leq 1 \leq ||y||_B.$$

Property. $B \in UMD$ if and only if B is ξ-convex.

The Vector-Valued Hilbert Transform

In the scalar case, it is well known that, if $f \in L^p, 1 \leq p < \infty$, its Hilbert transform $Hf(x)$ defined by

$$Hf(x) = p.v.\frac{1}{\pi}\int_{\infty}^{+\infty}\frac{f(x-t)}{t}dt = \lim_{\epsilon\to 0}\int_{|t|>\epsilon}\frac{f(x-t)}{t}dt \qquad (52)$$

exists, is in L^p and satisfies the Riesz inequality

$$||Hf||_p \leq C||f||_p, 1 < p < \infty. \qquad (53)$$

Here $p.v.$ means the Cauchy principal value.

Now, if B is a Banach space, $X = L^p(\mathbb{R}, B), 1 \leq p \leq \infty$, and $f \in X$, then, the formulas (52) and (53) (with the norms in X) are valid if and only if $B \in UMD$ (cf. [21] for the suffisance of the condition and J. Bourgain [18] for its necessity).

Example 27. In D.J. Aldous [8], it is proved that, if $X = L^p(\mathbb{R}, B)$ has an unconditional basis, then $B \in UMD$. But B may have an conditional basis while X fails to have one, for any p.

Example 28. In P.C. Kunstmann, L. Weis [70], it is mentioned that Sobolev spaces, Hardy spaces, and other well known spaces of analysis are UMD if they are reflexive.

Application. UMD spaces are closely related with Maximal L_p-regularity (cf. Section 2.6).

2.17. Wavefront Sets

The Wavefront Set of a Distribution

(see F. Treves [96]). Let Ω be an open set in $\mathbb{R}^n$. A distribution $u \in \mathcal{D}'(\Omega)$ (cf. L. Schwartz [87]) is said to be C^∞ in a neighborhood of $(x_o, \xi_o) \in \Omega \times (\mathbb{R}^n \setminus \{0\})$, if there exits a function $g \in \mathcal{D}(\Omega)$, $g \equiv 1$ in a neigborhood U of x_o and an open cone Γ_o in $\mathbb{R}^n$ containing ξ_o such that:

$$\forall M \geq 0, \quad \exists C_M \geq 0 \text{ such that } |\widehat{(gu)}(\xi)| \leq C_M(1 + |\xi|)^{-M}, \quad \forall \xi \in \Gamma_o.$$

Here $\widehat{(gu)}(\xi)$ is the Fourier transform of $gu \in \mathcal{E}'(\Omega)$, and a cone is a subset of $\mathbb{R}^n$, stable under the dilatations $\xi \to \rho\xi,\ \rho > 0$.

Wavefront Set and the Weyl Quantization

For the change of quantization see A. Martinez [77]. For $a(x,\xi) \in C^{\infty}(\mathbb{R}^d \times \mathbb{R}^d)$, its Weyl quantization, for $u \in \mathcal{S}(\mathbb{R}^d)$, is denoted by

$$a^w(x, D_x)u(x) = (2\pi)^{-d} \int\int e^{i(x-y)\cdot\xi} a(\frac{x+y}{2}, \xi)u(y)dyd\xi.$$

Let $u \in \mathcal{S}'(\mathbb{R}^d)$, then, $(x,\xi) \notin WF(u), \xi \neq 0$ if and only if there exists $\varphi \in \mathcal{D}(\mathbb{R}^{2d})$ such that $\varphi(x,\xi) \neq 0$ and

$$||\varphi^w(x, hD_x)u|| \quad \leq C_N h^N, \quad h \in (0,1],$$

for any $N \in \mathbb{Z}_+$, where $||\cdot||$ denotes the L^2 norm.

Homogeneous Wavefront Set

The notion of homogeneous wavefront set was introduced by S. Nakamura [83]. Let $u \in \mathcal{S}'(\mathbb{R}^d)$, then $(x,\xi) \in \mathbb{R}^{2d} \setminus 0$ is not in the homogeneous wavefront set of u, denoted by HWF(u), if and only if there exists $\varphi \in \mathcal{D}(\mathbb{R}^{2d})$ such that $\varphi(x,\xi) \neq 0$ and

$$||\varphi^w(hx, hD_x)u|| \quad \leq C_N h^N, \quad h \in (0,1],$$

for any $N \in \mathbb{Z}_+$.

Application

In [62], K. Ito studies the propagation of singularities, in terms of wavefront sets, for the solution $u_t = e^{-itH}u_o$ to the time-dependent Schrödinger equation

$$i\frac{du}{dt} = Hu, \quad H = \frac{1}{2}\Delta + V, \quad u(0) = u_o \in L^2(\mathbb{R}^n),$$

where Δ is the Laplace-Beltrami operator with respect to a scattering metric g given on S^n_+ (cf. Section 2.12), and V is a smooth subquadratic potential. Suppose $(z_o, \zeta_o) \in T^*(\mathbb{R}^n)$ is backward nontrapping and let $\omega_- = -\lim_{t\to-\infty} \frac{z(t;z_o,\zeta_o)}{|z(t;z_o,\zeta_o)|}$, where $z(t; z_o, \zeta_o)$ is solution of the Hamilton equations. It is proved that, if there exists $t_o > 0$ such that $(-t_o\omega_-, \omega_-) \notin HWF(u_o)$, then $(z_o, \zeta_o) \notin WF(u_{t_o})$.

I mention the notion of **quadratic scattering wavefrontset** introduced in R.B. Melrose [78] and also used in [62].

Acknowledgments

I wish to thank our librarians Raymonde Michel and Nathalie Pierache who always provide me with indispensable books and articles and Didier Gemmerlé for his constant help in the practice of LaTeX, and in the realization of the figures.

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