International Journal of Pure and Applied Mathematics
Volume 62 No. 3 2010, 287-289

AN EXTREMUM PROPERTY OF WHITTAKER FUNCTIONS

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Abstract: We establish an extremum property for Whittaker functions for GL$(3,\mathbb{R})$, which is not unexpected for a powerful kernel. The proof extends in a natural way for Whittaker functions on GL$(n,\mathbb{R})$.

AMS Subject Classification: 11F03, 11F11, 11F12, 11F72
Key Words: eigenvalues, Laplacian

1. Introduction

Let

$$v_1 = 1/3 + it_1, \qquad \pi_1 = 2v_1 + v_2,$$
$$v_2 = 1/3 + it_2, \qquad \pi_2 = v_1 + 2v_2.$$

Following Bump [1], we consider

$$W_{n_1,n_2}^{\pi_1,\pi_2}(x_1,x_2,y_1,y_2) = e^{2\pi i(n_1x_1+n_2x_2)}W(|n_1|y_1,|n_2|y_2),$$

where, since Whittaker functions are real analytic,

$$W(y_1,y_2) = y_1^{\pi_1}y_2^{\pi_2}\sum_{m_1=0}^{\infty}\sum_{m_2=0}^{\infty} a_{m_1m_2}y_1^{m_1}y_2^{m_2}$$

and $W(y_1,y_2)$ satisfies simultaneously

Received: June 10, 2010

$$\left\{y_2^2\frac{\partial^2}{\partial y_1^2}+y_2^2\frac{\partial^2}{\partial y_2^2}-y_1y_2\frac{\partial^2}{\partial y_1\partial y_2}-4\pi^2(y_1^2+y_2^2)\right\}W(y_1,y_2)=\lambda W(y_1,y_2); \quad (1)$$

$$\left\{-y_1^2y_2\frac{\partial^3}{\partial y_1^2\partial y_2}+y_1y_2^2\frac{\partial^3}{\partial y_1\partial y_2^2}+4\pi^2y_1^2y_2\frac{\partial}{\partial y_2}-4\pi^2y_1y_2^2\frac{\partial}{\partial y_1}\right.$$
$$\left.+y_1^2\frac{\partial^2}{\partial y_1^2}-y_2^2\frac{\partial^2}{\partial y_2^2}-4\pi^2y_1^2+4\pi^2y_2^2\right\}W(y_1,y_2)=\mu W(y_1,y_2), \qquad (2)$$

where

$$\lambda=(\pi_1^2+\pi_2^2)-(\pi_1+\pi_2)-\pi_1\pi_2,$$
$$\mu=(\pi_1^2-\pi_2^2)-(\pi_1-\pi_2)+(\pi_1\pi_2^2-\pi_1^2\pi_2).$$

Let $W=\{w_0,w_1,w_2,w_3,w_4,w_5\}$ be the Weyl group for $\mathrm{GL}(3,\mathbb{R})$.

As explained by Bump in [1], the elements of W act on π_1 and π_2 as follows:

$$w_0(\pi_1,\pi_2)=(\pi_1,\pi_2),$$
$$w_1(\pi_1,\pi_2)=((2-\pi_2),(2-\pi_1)),$$
$$w_2(\pi_1,\pi_2)=(\pi_1,(\pi_1-\pi_2)+1)),$$
$$w_3(\pi_1,\pi_2)=((\pi_2-\pi_1)+1,\pi_2),$$
$$w_4(\pi_1,\pi_2)=((2-\pi_2),(\pi_1-\pi_2)+1),$$
$$w_5(\pi,\pi_2)=((\pi_2-\pi_1)+1,(2-\pi_1)).$$

Since these represent all possible permutations of the factors of the eigenvalues, there are exactly six linearly independent $W(y_1,y_2)$. Throughout, we disregard the obvious, trivial pathelogical cases.

2. Result

We make the following curious observation.

Let $\epsilon>0$ be arbitrarily small. Let $A>0$, $B>0$ be arbitrarily large. We have

$$I=\int_0^A\int_0^B|W(y_1,y_2)|^2\frac{dy_1dy_2}{y_1^{3-\epsilon}\,y_2^{3-\epsilon}}$$
$$=\int_0^A\int_0^B|y_1^{\pi_1}|^2|y_2^{\pi_2}|^2\left|\sum_{m_1=0}^{\infty}\sum_{m_2=0}^{\infty}a_{m_1m_2}y_1^{m_1}y_2^{m_2}\right|^2\frac{dy_1dy_2}{y_1^{3-\epsilon}\,y_2^{3-\epsilon}}$$

$$= \sum_{m_1=0}^{\infty} \sum_{m_2=0}^{\infty} b_{m_1 m_2} \int_0^A \int_0^B y_1^{-3+2Re\pi_1+m_1+\epsilon} y_2^{-3+2Re\pi_2+m_2+\epsilon} dy_1 dy_2$$
$$< \infty,$$

since it is trivial to see that $3 - 2Re\pi_1 - \epsilon < 1$ and $3 - 2Re\pi_2 - \epsilon < 1$.

The same proof will establish the same result for each of the five other $W(y_1, y_2)$ that arise from the operation of the Weyl group on (π_1, π_2).

Remark. The fact that Whittaker functions are *not* square integrable plays a fundamental role in the theory of $\mathrm{GL}(n, R)$.

Acknowledgments

I thank Dorian Goldfeld for several conversations concerning Whittaker functions.

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