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X-RANKS WIITH RESPECT TO A RATIONAL NORMAL CURVE IN POSITIVE CHARACTERISTIC

E. Ballico
Department of Mathematics
University of Trento
38 123 Povo (Trento) - Via Sommarive, 14, ITALY
e-mail: ballico@science.unitn.it

Abstract: Let $X \subset \mathbb{P}^n$ be a rational normal curve defined over an algebraically closed field $\mathbb{K}$. Here we extends to the case $p := \text{char}(\mathbb{K}) > n$ the description of the X-ranks proved by Comas and Seiguer. We discuss the case $p \le n$.

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Let $X \subset \mathbb{P}^n$ be an integral and non-degenerate curve defined over an algebrically closed field $\mathbb{K}$. Fix any $P \in \mathbb{P}^n$. The X-rank $r_X(P)$ of X is the minimal cardinality of a finite set $S \subset X$ such that $P \in \langle S \rangle$. Here we always assume $p := \text{char}(\mathbb{K}) > 0$ and that X is a rational normal curve. Firstly we extend to the case $p > n$ the following Proposition 1 proved in characteristic zero by G. Comas and M. Seiguer (see [4] or [8], Theorem 4.1) (see also [7], Theorem 1.43). We will follow the proof of [4], using at some point results of [3]. For any integral and non-degenerate curve $T \subset \mathbb{P}^n$ and any integer $k \ge 1$ let $S^k(T) \subseteq \mathbb{P}^n$ (not $S^{k-1}(T)$) denote the closure of the union of all $(k-1)$-dimensional linear subspaces spanned by k points of T. Thus $S^1(T) = T$. Set $S^0(T) = \emptyset$. We have $\dim(S^k(T)) = \min\{n, 2k-1\}$ (see [1], Remark 1.6). Hence $S^k(T) = \mathbb{P}^n$ if and only if $k \ge \lceil (n+1)/2 \rceil$.

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Proposition 1. *Let $X \subset \mathbb{P}^n$ be a rational normal curve. Assume $p > n$. Fix $P \in \mathbb{P}^n$. Let k be the only integer such that $1 \le k \le \lceil (r+1)/2 \rceil$ and $P \in S^k(X) \backslash S^{k-1}(X)$.*

(a) Assume $k < \lceil (n+1)/2 \rceil$. There is a unique zero-dimensional scheme $Z \subset X$ such that $P \in \langle Z \rangle$ and length$(Z) = k$. If Z is reduced, then $r_X(P) = k$. If Z is not reduced, then $r_X(P) = n + 2 - k$.

(b) If $k = \lceil (n+1)/2 \rceil$, then $k \le r_X(P) \le k + 1$.

Proof. The integer k exists, because $S^k(X) = \mathbb{P}^n$ if $k = \lceil (n+1)/2 \rceil$. The existence of Z is proved in [3], Proposition 2.8. Now we check the uniqueness of Z if $k < \lceil (n+1)/2 \rceil$. The cohomology of line bundles on $\mathbb{P}^1$ and the linear normality of X gives that any subscheme $J \subset X$ such that length$(J) \le n+1$ is linearly independent, i.e. $\dim(\langle J \rangle) = \text{length}(J) - 1$. Assume $k < \lceil (n+1)/2 \rceil$ and that $P \in \langle Z \rangle \cap \langle Z' \rangle$ with length$(Z') \le k$ and $Z \ne Z'$. Since $P \notin S^{k-1}(X)$, we have length$(Z') = k$. Set $w := \text{length}(Z \cap Z')$ and $W := Z \cup Z'$. Since X is a smooth curve, length$(W) = \deg(Z) + \deg(Z') - \deg(Z \cap Z') = 2k - w$. We just proved that $Z \cap Z'$ and W are linearly independen. Since $P \in \langle Z \rangle \cap \langle Z' \rangle$ and $Z \ne Z'$, the linear space $\langle W \rangle$ has dimension at most $2k - 2 - w$, contradicting the linear independence of W. Hence Z is unique. The proof of [4], Lemma 7, works because the catalecticant description of the secant varieties of X works if $p > n$ (see [7], §1.3). The form of Bertini's Theorem quoted in the proof of [4], Lemma 10, may be substitued by the following observation. Let $f : \mathbb{P}^1 \to \mathbb{P}^1$ be any morphism such $\deg(f) \le n$. Since $p > n \ge \deg(f)$, the morphism f is separable. Hence a general fiber of f is reduced. In the proof of [4], Lemma 7, quote [6], Lemma 5.2, as a characteristic free reference for the quoted [5], Lemma 9.8 p. 103). The proof of part (b) is similar. □

We observed in [2] that Proposition 1 is not true if $p = n = 2$ (the strange point of the conic X has X-rank equal to 3), but it holds if $n = 3$ for arbitrary p. Here we consider the case $n = 4$ (see Proposition 3) and the case of points on the tangent developable TX of X (see Proposition 2).

Proposition 2. *Fix $Q \in X$ and $P \in T_Q X \backslash \{Q\}$. Then $r_X(P) = n$.*

Proof. Since X is not strange, $r_X(P) \le n$ (see [2], Theorem 1 or Theorem 2). The inequality $r_X(P) \ge n$ is proved as in the first part of the proof of Proposition 1. □

Proposition 3. *Assume $n = 4$. Then Proposition 1 is true, even if $p = 2, 3$.*

Proof. Fix $P \in \mathbb{P}^4$. Let $\ell_P : \mathbb{P}^4 \backslash \{P\} \to \mathbb{P}^3$ denote the linear projection from

P. If $P \in X$, then $r_X(P) = 1$. If $P \in S^2(X)\backslash TX$, then obviously $r_X(P) = 2$. If $P \in TX\backslash X$, then apply Proposition 2. Nos assume $P \notin S^2(X)$. We need to prove $r_X(P) = 3$. Since X is not strange, $r_X(P) \leq 4$ (see [2], Theorem 1 or Theorem 2). Since $P \notin S^2(X)$, $C := \ell_P(X)$ is a smooth and rational curve of degree 4. Hence it has a one-dimensional family Γ of lines D such that $\sharp(D \cap C) = 3$. Fix any $D \in \Gamma$ and let $M \subset \mathbb{P}^4$ be the only plane such that $P \in M$ and $\ell_P(M\backslash\{P\}) = D$. Since $\ell_P|X : X \to C$ is an isomorphism, $\sharp(M \cap X) = 3$. Since any length 3 subscheme of X is linearly independent, $M = \langle M \cap X\rangle$. Since $P \in M$, we get $r_X(P) \leq 3$. □

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