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FURTHER REMARKS ON PACHPATTE'S INEQUALITY AND ITS ANALOGUES

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Abstract: In this paper, by using the real function techniques and the theory of inequality, we obtain a general class of inequalities similar to the Pachpatte's integral inequality. As applications, we consider several particular results.

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1. Introduction

The well known Hardy-Hilbert's inequality is (see [3]):

Theorem 1.1. *If $p > 1, p' = p/(p-1)$ and $\sum_{n=1}^{\infty} a_n^p \leq A, \sum_{n=1}^{\infty} b_n^{p'} \leq B$, then*

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{m+n} < \frac{\pi}{\sin(\pi/p)}(A)^{1/p}(B)^{1/p'}, \tag{1.1}$$

unless the sequence $\{a_m\}$ or $\{b_n\}$ is null.

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The integral form can be stated as follows.

Theorem 1.2. *If $p > 1, p' = p/(p-1)$ and $\int_0^\infty f^p(x)\mathrm{d}x \le F, \int_0^\infty g^{p'}(x)\mathrm{d}x \le G$, then*

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x+y}\mathrm{d}x\mathrm{d}y < \frac{\pi}{\sin(\pi/p)} F^{1/p} G^{1/p'}, \tag{1.2}$$

unless $f \equiv 0$ or $g \equiv 0$.

The following two theorems was studied by Pachpatte (see [8]).

Theorem 1.3. *Let $p \ge 1, q \ge 1$ and $f(\sigma) \ge 0, g(\tau) \ge 0$ for $\sigma \in (0,x)$, $\tau \in (0,y)$, where x, y are positive numbers and define $F(s) = \int_0^s f(\sigma)\mathrm{d}\sigma$, and $G(t) = \int_0^t g(\tau)\mathrm{d}\tau$, for $s \in (0,x), t \in (0,y)$. Then*

$$\int_0^x \int_0^y \frac{F^p(s)G^q(t)}{s+t}\mathrm{d}s\mathrm{d}t \le D(p,q,x,y) \left\{ \int_0^x (x-s)(F^{p-1}(s)f(s))^2 \mathrm{d}s \right\}^{1/2}$$
$$\times \left\{ \int_0^y (y-t)(F^{q-1}(t)f(t))^2 \mathrm{d}t \right\}^{1/2}, \tag{1.3}$$

unless $f \equiv 0$ or $g \equiv 0$, where $D(p,q,x,y) = \frac{1}{2}pq\sqrt{xy}$.

Theorem 1.4. *Let f, g, F, G be as in the above theorem, let $p(\sigma)$ and $q(\tau)$ be two positive functions defined for $\sigma \in (0,x)$, $\tau \in (0,y)$ and define $P(s) = \int_0^s p(\sigma)\mathrm{d}\sigma$ and $Q(t) = \int_0^t q(\tau)\mathrm{d}\tau$ for $s \in (0,x), t \in (0,y)$, where x, y are positive real numbers. Let ϕ and ψ be two real-valued, nonnegative, convex, and sub-multiplicative functions defined on $R_+ = [0,\infty)$. Then*

$$\int_0^x \int_0^y \frac{\phi(F(s))\psi(G(t))}{s+t}\mathrm{d}s\mathrm{d}t \le L(x,y) \left\{ \int_0^x (x-s) \left[p(s)\phi\left(\frac{f(s)}{p(s)}\right)\right]^2 \mathrm{d}s \right\}^{1/2}$$
$$\times \left\{ \int_0^y (y-t) \left[q(t)\psi\left(\frac{g(t)}{q(t)}\right)\right]^2 \mathrm{d}t \right\}^{1/2}, \tag{1.4}$$

where

$$L(x,y) = \frac{1}{2} \left(\int_0^x \left[\frac{\phi(P(s))}{P(s)}\right]^2 \mathrm{d}s \right)^{1/2} \left(\int_0^y \left[\frac{\psi(Q(t))}{Q(t)}\right]^2 \mathrm{d}t \right)^{1/2}.$$

The above two inequalities were studied extensively and numerous variants, generalizations, and extensions appeared in the literature [1]-[12].

The main purpose of the present article is to establish comprehensive inequalities according to the Pachpatte's integral inequalities by using the method of analysis and the theory of inequality. As applications, we consider some particular cases.

2. Main Results

Lemma 2.1. *Suppose that* $p > 0, \frac{1}{p}+\frac{1}{q} = 1, F, G \geq 0$, *and* $F \in L^p(E), G \in L^q(E)$, *one has Hölder's inequality as follows, see [6]:*

(1) If $p > 1$, *then*

$$\int_E F(t)G(t)\mathrm{d}t \leq \left(\int_E F^p(t)\mathrm{d}t\right)^{1/p} \left(\int_E G^q(t)\mathrm{d}t\right)^{1/q}. \tag{2.1}$$

(2) If $0 < p < 1$, *then*

$$\int_E F(t)G(t)\mathrm{d}t \geq \left(\int_E F^p(t)\mathrm{d}t\right)^{1/p} \left(\int_E G^q(t)\mathrm{d}t\right)^{1/q}. \tag{2.2}$$

Here the equalities hold if and only if there exist real numbers $A, B(A^2 + B^2 \neq 0)$, *such that* $AF^p(t) = BG^q(t)$ *a.e. in* E.

If $K(x, y)$ is a real measurable function and satisfies $K(ux, uy) = u^{-\lambda}K(x, y)$ $(\lambda > 0, u > 0)$ for $(x, y) \in (0, \infty)\times(0, \infty)$, then $K(x, y)$ is called a homogeneous function of $-\lambda$-degree.

Lemma 2.2. *Assume that* $r > 1, \frac{1}{r} + \frac{1}{w} = 1, \lambda > 0$, *setting* $K(x, y)(\geq 0)$ *is a homogeneous kernel function of* $-\lambda$*-degree, and define the weight functions* $\varpi_1(\lambda, w, x), \varpi_2(\lambda, r, y)$ *as*

$$\varpi_1(\lambda, w, x) := \int_0^\infty K(x, y)\frac{x^{\frac{\lambda}{r}}}{y^{1-\frac{\lambda}{w}}}\mathrm{d}y, x \in (0, \infty), \tag{2.3}$$

$$\varpi_2(\lambda, r, y) := \int_0^\infty K(x, y)\frac{x^{\frac{\lambda}{w}}}{y^{1-\frac{\lambda}{r}}}\mathrm{d}x, y \in (0, \infty), \tag{2.4}$$

$$C_\lambda(r) := \int_0^\infty K(u, 1)u^{\frac{\lambda}{r}-1}\mathrm{d}u, \tag{2.5}$$

then one has

$$\varpi_1(\lambda, w, x) = \varpi_2(\lambda, r, y) = C_\lambda(r). \tag{2.6}$$

Proof. In view of the $-\lambda$ homogeneity of the kernel $K(x, y)$, for $x > 0$, setting $u = x/y$, we obtain

$$\varpi_1(\lambda, w, x) = \int_0^\infty K\left(\frac{x}{y}\cdot y, y\right)\frac{(\frac{x}{y}\cdot y)^{\frac{\lambda}{r}}}{y^{1-\frac{\lambda}{w}}}\cdot\frac{-y^2}{x}\mathrm{d}\frac{x}{y} = \int_0^\infty K(u,1)x^{\frac{\lambda}{r}-1}\mathrm{d}u = C_\lambda(r).$$

Similarly, for $y > 0$, letting $u = x/y$, we get

$$\varpi_2(\lambda, r, y) = \int_0^\infty K\left(\frac{x}{y}\cdot y, y\right)\frac{y^{\frac{\lambda}{w}}}{(\frac{x}{y}\cdot y)^{1-\frac{\lambda}{r}}}\cdot y\mathrm{d}\frac{x}{y} = \int_0^\infty K(u,1)x^{\frac{\lambda}{r}-1}\mathrm{d}u = C_\lambda(r).$$

Hence equation (2.6) is valid. The lemma is proved. □

Theorem 2.3. *Let $m, n \geq 1, p_i > 1, \frac{1}{p_i} + \frac{1}{q_i} = 1$ for $i = 0, 1, 2, 3, 4$, noting $p_0 = p, q_0 = q; p_3 = k, q_3 = l; p_4 = r, q_4 = w$, $f(\sigma) \geq 0, g(\tau) \geq 0$ for $\sigma, \tau \in \mathbb{R}$, $F(s) = \int_0^s f(\sigma)\mathrm{d}\sigma$ and $G(t) = \int_0^t g(\tau)\mathrm{d}\tau$ for $s, t \in \mathbb{R}$. Suppose that $K(s,t) \geq 0$ is a measurable homogeneous kernel function of $-\lambda$-degree and the weight function $C_\lambda(r) = \int_0^\infty K(u,1)u^{\frac{\lambda}{r}-1}\mathrm{d}u$ is a positive number depending only on the parameters λ, r, then*

$$\int_0^\infty\int_0^\infty \frac{K(s,t)F^m(s)G^n(t)}{ls^{k/p_1} + kt^{l/p_2}}\mathrm{d}s\mathrm{d}t \leq E(m,n,k,l,\lambda)\left\{\int_0^\infty s^{p(1-\frac{\lambda}{r})-1}F_f^p(s)\mathrm{d}s\right\}^{1/p}$$
$$\times\left\{\int_0^\infty t^{q(1-\frac{\lambda}{w})-1}G_g^q(t)\mathrm{d}t\right\}^{1/q}, \tag{2.7}$$

unless $f \equiv 0$ or $g \equiv 0$, where the constant factor $E(m,n,k,r,\lambda) = \frac{mn}{kl}C_\lambda(r)$, $F_f(s) = \{\int_0^s (F^{m-1}(\sigma)f(\sigma))^{q_1}\mathrm{d}\sigma\}^{1/q_1}$ and $G_g(t) = \{\int_0^t (G^{n-1}(\tau)g(\tau))^{q_2}\mathrm{d}\tau\}^{1/q_2}$.

Proof. From the hypotheses, we can easily observe that

$$F^m(s) = m\int_0^s F^{m-1}(\sigma)f(\sigma)\mathrm{d}\sigma, \quad s \in (0,\infty), \tag{2.8}$$

$$G^n(t) = n\int_0^t G^{n-1}(\tau)g(\tau)\mathrm{d}\tau, \quad t \in (0,\infty), \tag{2.9}$$

From (2.8) and (2.9), applying Hölder's inequality, equality (2.6) and Young's inequality: $ab \leq \frac{a^p}{p} + \frac{b^q}{q}$, where $a \geq 0,\ b \geq 0,\ p > 1, \frac{1}{p} + \frac{1}{q} = 1$. We obtain

$$F^m(s)G^n(t) = mn\left(\int_0^s F^{m-1}(\sigma)f(\sigma)\mathrm{d}\sigma\right)\left(\int_0^t G^{n-1}(\tau)g(\tau)\mathrm{d}\tau\right)$$
$$\leq mns^{1/p_1}t^{1/p_2}\left\{\int_0^s (F^{m-1}(\sigma)f(\sigma))^{q_1}\mathrm{d}\sigma\right\}^{1/q_1}\left\{\int_0^t (G^{n-1}(\tau)g(\tau))^{q_2}\mathrm{d}\tau\right\}^{1/q_2} \leq$$
$$mn\left(\frac{s^{k/p_1}}{k} + \frac{t^{l/p_2}}{l}\right)\left\{\int_0^s (F^{m-1}(\sigma)f(\sigma))^{q_1}\mathrm{d}\sigma\right\}^{1/q_1}\left\{\int_0^t (G^{n-1}(\tau)g(\tau))^{q_2}\mathrm{d}\tau\right\}^{1/q_2}.$$

Note that

$$F_f(s)=\left\{\int_0^s (F^{m-1}(\sigma)f(\sigma))^{q_1}\mathrm{d}\sigma\right\}^{1/q_1}, G_g(t)=\left\{\int_0^t (G^{n-1}(\tau)g(\tau))^{q_2}\mathrm{d}\tau\right\}^{1/q_2}.$$

Hence

$$\int_0^\infty\int_0^\infty \frac{K(s,t)F^m(s)G^n(t)}{ls^{k/p_1}+kt^{l/p_2}}\mathrm{d}s\mathrm{d}t \le \frac{mn}{kl}\int_0^\infty\int_0^\infty K(s,t)F_f(s)G_g(t)\mathrm{d}s\mathrm{d}t$$
$$=\frac{mn}{kl}\int_0^\infty\int_0^\infty K(s,t)\left[F_f(s)\frac{s^{(1-\frac{\lambda}{r})/q}}{t^{(1-\frac{\lambda}{w})/p}}\right]\left[G_g(t)\frac{t^{(1-\frac{\lambda}{w})/p}}{s^{(1-\frac{\lambda}{r})/q}}\right]\mathrm{d}s\mathrm{d}t$$
$$\le\frac{mn}{kl}\left\{\int_0^\infty\int_0^\infty K(s,t)F_f^p(s)\frac{s^{(p-1)(1-\frac{\lambda}{r})}}{t^{(1-\frac{\lambda}{w})}}\mathrm{d}s\mathrm{d}t\right\}^{1/p}$$
$$\times\left\{\int_0^\infty\int_0^\infty K(s,t)G_g^q(t)\frac{t^{(q-1)(1-\frac{\lambda}{w})}}{s^{(1-\frac{\lambda}{r})}}\mathrm{d}s\mathrm{d}t\right\}^{1/q}=$$
$$\frac{mn}{kl}\left\{\int_0^\infty \varpi_1(\lambda,w,s)s^{p(1-\frac{\lambda}{r})-1}F_f^p(s)\mathrm{d}s\right\}^{1/p}\left\{\int_0^\infty \varpi_2(\lambda,r,t)t^{q(1-\frac{\lambda}{w})-1}G_g^q(t)\mathrm{d}t\right\}^{1/q}$$
$$=\frac{mn}{kl}C_\lambda(r)\left\{\int_0^\infty s^{p(1-\frac{\lambda}{r})-1}F_f^p(s)\mathrm{d}s\right\}^{1/p}\left\{\int_0^\infty t^{q(1-\frac{\lambda}{w})-1}G_g^q(t)\mathrm{d}t\right\}^{1/q}$$
$$=E(m,n,k,r,\lambda)\left\{\int_0^\infty s^{p(1-\frac{\lambda}{r})-1}F_f^p(s)\mathrm{d}s\right\}^{1/p}\left\{\int_0^\infty t^{q(1-\frac{\lambda}{w})-1}G_g^q(t)\mathrm{d}t\right\}^{1/q}.$$

This completes the proof. □

Theorem 2.4. *Let $p_i,q_i,f,g,F,G,K(s,t)$ be as in the above theorem, let $p(\sigma),q(\tau)$ be two positive functions defined for $\sigma,\tau\in\mathbb{R}$, assume that $P(s)=\int_0^s p(\sigma)\mathrm{d}\sigma$ and $Q(t)=\int_0^t q(\tau)\mathrm{d}\tau$ for $s,t\in\mathbb{R}$. Let ϕ,ψ be real-valued, nonnegative, convex, and sub-multiplicative function defined in $R_+=[0,\infty)$, then*

$$\int_0^\infty\int_0^\infty \frac{K(s,t)\phi(F(s))\psi(G(t))}{ls^{k/p_1}+kt^{l/p_2}}\mathrm{d}s\mathrm{d}t$$
$$\le E(k,r,\lambda)\left\{\int_0^\infty s^{p(1-\frac{\lambda}{r})-1}\phi_1^p(s)\mathrm{d}s\right\}^{1/p}\left\{\int_0^\infty t^{q(1-\frac{\lambda}{w})-1}\psi_1^q(t)\mathrm{d}t\right\}^{1/q}, \quad (2.10)$$

where $E(k,r,\lambda)=\frac{1}{kl}C_\lambda(r)$, $\phi_1(s)=\frac{\phi(P(s))}{P(s)}\left\{\int_0^s\left[p(\sigma)\phi\left(\frac{f(\sigma)}{p(\sigma)}\right)\right]^{q_1}\mathrm{d}\sigma\right\}^{1/q_1}$ and $\psi_1(t)=\frac{\psi(Q(t))}{Q(t)}\left\{\int_0^t\left[q(\tau)\psi\left(\frac{g(\tau)}{q(\tau)}\right)\right]^{q_2}\mathrm{d}\tau\right\}^{1/q_2}$.

Proof. Applying Jensen's inequality and the Hölder's inequality, in view of the hypotheses, it is clear to observe that

$$\phi(F(s)) = \phi\left(\frac{P(s)\int_0^s p(\sigma)\frac{f(\sigma)}{p(\sigma)}\mathrm{d}\sigma}{\int_0^s p(\sigma)\mathrm{d}\sigma}\right) \leq \frac{\phi(P(s))}{P(s)}\int_0^s \left[p(\sigma)\phi\left(\frac{f(\sigma)}{p(\sigma)}\right)\right]\mathrm{d}\sigma$$
$$\leq \frac{\phi(P(s))}{P(s)} s^{\frac{1}{p_1}} \left\{\int_0^s \left[p(\sigma)\phi\left(\frac{f(\sigma)}{p(\sigma)}\right)\right]^{q_1}\mathrm{d}\sigma\right\}^{1/q_1}, \quad (2.11)$$

and similarly,

$$\psi(G(t)) \leq \frac{\phi(Q(t))}{Q(t)} t^{\frac{1}{p_2}} \left\{\int_0^t \left[q(\tau)\psi\left(\frac{g(\tau)}{q(\tau)}\right)\right]^{q_2}\mathrm{d}\tau\right\}^{1/q_2}. \quad (2.12)$$

Using Young's inequality, we obtain that

$$\phi(F(s))\psi(G(t)) \leq \left(\frac{s^{k/p_1}}{k} + \frac{t^{l/p_2}}{l}\right)\frac{\phi(P(s))}{P(s)}\left\{\int_0^s \left[p(\sigma)\phi\left(\frac{f(\sigma)}{p(\sigma)}\right)\right]^{q_1}\mathrm{d}\sigma\right\}^{1/q_1}$$
$$\times\frac{\psi(Q(t))}{Q(t)}\left\{\int_0^t \left[q(\tau)\psi\left(\frac{g(\tau)}{q(\tau)}\right)\right]^{q_2}\mathrm{d}\tau\right\}^{1/q_2}. \quad (2.13)$$

Note that

$$\phi_1(s) = \frac{\phi(P(s))}{P(s)}\left\{\int_0^s \left[p(\sigma)\phi\left(\frac{f(\sigma)}{p(\sigma)}\right)\right]^{q_1}\mathrm{d}\sigma\right\}^{1/q_1},$$
$$\psi_1(t) = \frac{\psi(Q(t))}{Q(t)}\left\{\int_0^t \left[q(\tau)\psi\left(\frac{g(\tau)}{q(\tau)}\right)\right]^{q_2}\mathrm{d}\tau\right\}^{1/q_2}.$$

Dividing both sides of (2.13) by $ls^{k/p_1} + kt^{l/p_2}$, multiplying by $K(s,t)$ both sides, integrating both sides of the above inequality over s from 0 to ∞ and over t from 0 to ∞, applying Hölder's inequality, we get

$$\int_0^\infty\int_0^\infty \frac{K(s,t)\phi(F(s))\psi(G(t))}{ls^{k/p_1}+kt^{l/p_2}}\mathrm{d}s\mathrm{d}t \leq \frac{1}{kl}\int_0^\infty\int_0^\infty K(s,t)\phi_1(s)\psi_1(t)\mathrm{d}s\mathrm{d}t$$
$$= \frac{mn}{kl}\int_0^\infty\int_0^\infty K(s,t)\left[\phi_1(s)\frac{s^{(1-\frac{\lambda}{r})/q}}{t^{(1-\frac{\lambda}{w})/p}}\right]\left[\psi_1(t)\frac{t^{(1-\frac{\lambda}{w})/p}}{s^{(1-\frac{\lambda}{r})/q}}\right]\mathrm{d}s\mathrm{d}t$$
$$\leq \frac{1}{kl}\left\{\int_0^\infty \varpi_1(\lambda,w,s)s^{p(1-\frac{\lambda}{r})-1}\phi_1^p(s)\mathrm{d}s\right\}^{1/p}\left\{\int_0^\infty \varpi_2(\lambda,r,t)t^{q(1-\frac{\lambda}{w})-1}\psi_1^q(t)\mathrm{d}t\right\}^{1/q}$$
$$= \frac{1}{kl}C_\lambda(r)\left\{\int_0^\infty s^{p(1-\frac{\lambda}{r})-1}\phi_1^p(s)\mathrm{d}s\right\}^{1/p}\left\{\int_0^\infty t^{q(1-\frac{\lambda}{w})-1}\psi_1^q(t)\mathrm{d}t\right\}^{1/q}$$
$$= E(k,r,\lambda)\left\{\int_0^\infty s^{p(1-\frac{\lambda}{r})-1}\phi_1^p(s)\mathrm{d}s\right\}^{1/p}\left\{\int_0^\infty t^{q(1-\frac{\lambda}{w})-1}\psi_1^q(t)\mathrm{d}t\right\}^{1/q}.$$

This completes the theorem. □

Theorem 2.5. *Let $p_i, q_i, f, g, K(x,y)$ be as in Theorem 2.3. Assume that $F(s) = \frac{1}{s}\int_0^s p(\sigma)\mathrm{d}\sigma$ and $G(t) = \frac{1}{t}\int_0^t q(\tau)\mathrm{d}\tau$ for $s, t \in \mathbb{R}$. Let ϕ, ψ be real-valued, nonnegative, convex, and sub-multiplicative function defined in $R_+ = [0,\infty)$, then*

$$\int_0^\infty \int_0^\infty \frac{st}{ls^{k/p_1} + kt^{l/p_2}} \cdot K(s,t)\phi(F(s))\psi(G(t))\mathrm{d}s\mathrm{d}t$$
$$\leq E(k,r,\lambda)\left\{\int_0^\infty s^{p(1-\frac{\lambda}{r})-1}\phi_f^p(s)\mathrm{d}s\right\}^{1/p}\left\{\int_0^\infty t^{q(1-\frac{\lambda}{w})-1}\psi_g^q(t)\mathrm{d}t\right\}^{1/q}, \quad (2.14)$$

where the constant factor $E(k,r,\lambda) = \frac{1}{kl}C_\lambda(r)$, $\phi_f(s) = \left\{\int_0^s [(\phi(f(\sigma))]^{q_1}\,\mathrm{d}\sigma\right\}^{1/q_1}$ and $\psi_g(t) = \left\{\int_0^t [\psi(g(\tau))]^{q_2}\,\mathrm{d}\tau\right\}^{1/q_2}$. In particular, for $l = k = p_1 = p_2 = 2$, (2.14) reduces to

$$\int_0^\infty \int_0^\infty \frac{st}{s+t} \cdot K(s,t)\phi(F(s))\psi(G(t))\mathrm{d}s\mathrm{d}t$$
$$\leq \frac{1}{2}C_\lambda(r)\left\{\int_0^\infty s^{p(1-\frac{\lambda}{r})-1}\phi_f^p(s)\mathrm{d}s\right\}^{1/p}\left\{\int_0^\infty t^{q(1-\frac{\lambda}{w})-1}\psi_g^q(t)\mathrm{d}t\right\}^{1/q}. \quad (2.15)$$

Proof. From the hypotheses and by applying Jensen's inequality and the Hölder's inequality, we have

$$\phi(F(s)) = \phi\left(\frac{1}{s}\int_0^s f(\sigma)\mathrm{d}\sigma\right) \leq \frac{1}{s}\int_0^s [\phi(f(\sigma))]\mathrm{d}\sigma$$
$$\leq \frac{1}{s}\cdot s^{\frac{1}{p_1}}\left\{\int_0^s [\phi(f(\sigma))]^{q_1}\mathrm{d}\sigma\right\}^{1/q_1}, \quad (2.16)$$

and similarly,

$$\psi(G(t)) \leq \frac{1}{t}\cdot t^{\frac{1}{p_2}}\left\{\int_0^t [\psi(g(\tau))]^{q_2}\mathrm{d}\tau\right\}^{1/q_2}. \quad (2.17)$$

Using Young's inequality, we obtain that

$$\phi(F(s))\psi(G(t)) \leq \frac{1}{st}\left(\frac{s^{k/p_1}}{k} + \frac{t^{l/p_2}}{l}\right)\left\{\int_0^s [\phi(f(\sigma))]^{q_1}\mathrm{d}\sigma\right\}^{1/q_1}$$
$$\times\left\{\int_0^t [\psi(g(\tau))]^{q_2}\mathrm{d}\tau\right\}^{1/q_2}. \quad (2.18)$$

Note that

$$\phi_f(s) = \left\{\int_0^s [\phi(f(\sigma))]^{q_1}\mathrm{d}\sigma\right\}^{1/q_1}, \psi_g(t) = \left\{\int_0^t [\psi(g(\tau))]^{q_2}\mathrm{d}\tau\right\}^{1/q_2}.$$

Dividing both sides of (2.18) by $ls^{k/p_1} + kt^{l/p_2}$, multiplying by $K(s,t)$ both sides, integrating both sides of the above inequality over s from 0 to ∞ and over t from 0 to ∞, and applying Hölder's inequality, we get

$$\int_0^\infty \int_0^\infty \frac{st}{ls^{k/p_1} + kt^{l/p_2}} \cdot K(s,t)\phi(F(t))\psi(G(t))\mathrm{d}s\mathrm{d}t$$

$$\leq \frac{1}{kl}\int_0^\infty \int_0^\infty K(s,t)\left\{\int_0^s [\phi(f(\sigma))]^{q_1}\mathrm{d}\sigma\right\}^{1/q_1}\left\{\int_0^t [\psi(g(\tau))]^{q_2}\mathrm{d}\tau\right\}^{1/q_2}\mathrm{d}s\mathrm{d}t$$

$$\leq \frac{1}{kl}\int_0^\infty \int_0^\infty K(s,t)\phi_f(s)\psi_g(t)\mathrm{d}s\mathrm{d}t$$

$$\leq \frac{1}{kl}\left\{\int_0^\infty \varpi_1(\lambda,w,s)s^{p(1-\frac{\lambda}{r})-1}\phi_f^p(s)\mathrm{d}s\right\}^{1/p}\left\{\int_0^\infty \varpi_2(\lambda,r,t)t^{q(1-\frac{\lambda}{w})-1}\psi_g^q(t)\mathrm{d}t\right\}^{1/q}$$

$$= \frac{1}{kl}C_\lambda(r)\left\{\int_0^\infty s^{p(1-\frac{\lambda}{r})-1}\phi_f^p(s)\mathrm{d}s\right\}^{1/p}\left\{\int_0^\infty t^{q(1-\frac{\lambda}{w})-1}\psi_g^q(t)\mathrm{d}t\right\}^{1/q}$$

$$= E(k,r,\lambda)\left\{\int_0^\infty s^{p(1-\frac{\lambda}{r})-1}\phi_f^p(s)\mathrm{d}s\right\}^{1/p}\left\{\int_0^\infty t^{q(1-\frac{\lambda}{w})-1}\psi_g^q(t)\mathrm{d}t\right\}^{1/q}.$$

This completes the theorem. □

Theorem 2.6. *Let $p_i, q_i, f, g, p, q, P, Q$ be as in Theorem 2.3. Assume that $F(s) = \frac{1}{P(s)}\int_0^s p(\sigma)f(\sigma)\mathrm{d}\sigma$ and $G(t) = \frac{1}{Q(t)}\int_0^t q(\tau)g(\tau)\mathrm{d}\tau$ for $s,t \in \mathbb{R}$. Let ϕ, ψ be real-valued, nonnegative, convex, and sub-multiplicative function defined in $R_+ = [0,\infty)$, then*

$$\int_0^\infty \int_0^\infty \frac{K(s,t)P(s)Q(t)\phi(F(s))\psi(G(t))}{ls^{k/p_1} + kt^{l/p_2}}\mathrm{d}s\mathrm{d}t$$

$$\leq E(k,r,\lambda)\left\{\int_0^\infty s^{p(1-\frac{\lambda}{r})-1}\phi_2^p(s)\mathrm{d}s\right\}^{1/p}\left\{\int_0^\infty t^{q(1-\frac{\lambda}{w})-1}\psi_2^q(t)\mathrm{d}t\right\}^{1/q}, \quad (2.19)$$

where $E(k,r,\lambda) = \frac{1}{kl}C_\lambda(r)$, $\phi_2(s) = \left\{\int_0^s [(p(\sigma)\phi(f(\sigma))]^{q_1}\,\mathrm{d}\sigma\right\}^{1/q_1}$ and $\psi_2(t) = \left\{\int_0^t [q(\tau)\psi(g(\tau))]^{q_2}\,\mathrm{d}\tau\right\}^{1/q_2}$.

Proof. From the hypotheses and by applying Jensen's inequality and the Hölder's inequality, we have

$$\phi(F(s)) = \phi\left(\frac{1}{P(s)}\int_0^s p(\sigma)f(\sigma)\mathrm{d}\sigma\right) \leq \frac{1}{P(s)}\int_0^s [p(\sigma)\phi(f(\sigma))]\mathrm{d}\sigma$$

$$\leq \frac{1}{P(s)} \cdot s^{\frac{1}{p_1}}\left\{\int_0^s [\phi(p(\sigma)f(\sigma))]^{q_1}\mathrm{d}\sigma\right\}^{1/q_1}, \quad (2.20)$$

and similarly,

$$\psi(G(t)) \le \frac{1}{Q(t)} \cdot t^{\frac{1}{p_2}} \left\{ \int_0^t [q(\tau)\psi(g(\tau))]^{q_2} \mathrm{d}\tau \right\}^{1/q_2}. \tag{2.21}$$

Using Young's inequality, we obtain that

$$\phi(F(s))\psi(G(t)) \le \frac{1}{P(s)Q(t)} \left(\frac{s^{k/p_1}}{k} + \frac{t^{l/p_2}}{l} \right) \left\{ \int_0^s [p(\sigma)\phi(f(\sigma))]^{q_1} \mathrm{d}\sigma \right\}^{1/q_1} \times \left\{ \int_0^t [q(\tau)\psi(g(\tau))]^{q_2} \mathrm{d}\tau \right\}^{1/q_2}. \tag{2.22}$$

Note that

$$\phi_2(s) = \left\{ \int_0^s [p(\sigma)\phi(f(\sigma))]^{q_1} \mathrm{d}\sigma \right\}^{1/q_1}, \psi_2(t) = \left\{ \int_0^t [q(\tau)\psi(g(\tau))]^{q_2} \mathrm{d}\tau \right\}^{1/q_2}.$$

The rest of the proof can be completed by following the same steps as in the proof of Theorem 2.3, thus we omit the details. □

3. Some Particular Cases

Corollary 3.1. *Set* $m = n = 1, k = l = p_1 = p_2 = 2, \frac{1}{p} + \frac{1}{q} = 1, \alpha > 0, K(s,t) = \frac{1}{(s^\alpha + t^\alpha)^{\lambda/\alpha}}$, $f(\sigma) \ge 0, g(\tau) \ge 0$ *for* $\sigma, \tau \in \mathbb{R}$, $F(s) = \int_0^s f(\sigma)\mathrm{d}\sigma$ *and* $G(t) = \int_0^t g(\tau)\mathrm{d}\tau$ *for* $s, t \in \mathbb{R}$, *then*

$$\int_0^\infty \int_0^\infty \frac{F(s)G(t)}{(s+t)(s^\alpha + t^\alpha)^{\lambda/\alpha}} \mathrm{d}s\mathrm{d}t \le \frac{1}{2\alpha} B\left(\frac{\lambda}{r\alpha}, \frac{\lambda}{w\alpha} \right) \left\{ \int_0^\infty s^{p(1-\frac{\lambda}{r})-1} F_f^p(s)\mathrm{d}s \right\}^{1/p} \times \left\{ \int_0^\infty t^{q(1-\frac{\lambda}{w})-1} G_g^q(t)\mathrm{d}t \right\}^{1/q}, \tag{3.1}$$

unless $f \equiv 0$ *or* $g \equiv 0$, *where* $F_f(s) = \{\int_0^s (F^{m-1}(\sigma)f(\sigma))^2 \mathrm{d}\sigma\}^{1/2}$ *and* $G_g(t) = \{\int_0^t (G^{n-1}(\tau)g(\tau))^2 \mathrm{d}\tau\}^{1/2}$. *In particular:*

(i) for $\alpha = 1$, *(3.1) reduces to*

$$\int_0^\infty \int_0^\infty \frac{F(s)G(t)}{(s+t)^{1+\lambda}} \mathrm{d}s\mathrm{d}t \le \frac{1}{2} B\left(\frac{\lambda}{r}, \frac{\lambda}{w} \right) \left\{ \int_0^\infty s^{p(1-\frac{\lambda}{r})-1} F_f^p(s)\mathrm{d}s \right\}^{1/p} \times \left\{ \int_0^\infty t^{q(1-\frac{\lambda}{w})-1} G_g^q(t)\mathrm{d}t \right\}^{1/q}, \tag{3.2}$$

(ii) for $\alpha = \lambda$, we get

$$\int_0^\infty \int_0^\infty \frac{F(s)G(t)}{(s+t)(s^\lambda + t^\lambda)} \mathrm{d}s\mathrm{d}t \le \frac{\pi}{2\lambda \sin(\pi/r)} \left\{ \int_0^\infty s^{p(1-\frac{\lambda}{r})-1} F_f^p(s)\mathrm{d}s \right\}^{1/p}$$
$$\times \left\{ \int_0^\infty t^{q(1-\frac{\lambda}{w})-1} G_g^q(t)\mathrm{d}t \right\}^{1/q}. \tag{3.3}$$

Proof. It is obvious that $K(s,t) = \frac{1}{(s^\alpha + t^\alpha)^{\lambda/\alpha}}$ is a nonnegative measurable homogeneous kernel function of $-\lambda$-degree in $(0,\infty) \times (0,\infty)$. In view of Theorem 2.3, we just need to show that $C_\lambda(r) := \int_0^\infty K(u,1)u^{\frac{\lambda}{r}-1}\mathrm{d}u$ is a positive number depending only on the parameters λ, r. Setting $v = u^\alpha$, we obtain

$$C_\lambda(r) = \int_0^\infty K(u,1)u^{\frac{\lambda}{r}-1}\mathrm{d}u = \int_0^\infty \frac{1}{(u^\alpha+1)^{\lambda/\alpha}} u^{\frac{\lambda}{r}-1}\mathrm{d}u$$
$$= \frac{1}{\alpha}\int_0^\infty \frac{1}{(v+1)^{\lambda/\alpha}} v^{\frac{\lambda}{r\alpha}-1} dv = \frac{1}{\alpha} B\left(\frac{\lambda}{r\alpha}, \frac{\lambda}{w\alpha}\right).$$

In view of $k = l = 2$, we get (3.1).

In addition, (3.2) and (3.3) can be obtain directly by setting $\alpha = 1$ and $\alpha = \lambda$ separately. This completes the proof. □

Corollary 3.2. *Set $m = n = 1, k = l = p_1 = p_2 = 2, \frac{1}{p} + \frac{1}{q} = 1, \alpha > -1, K(s,t) = \frac{|\ln(s/t)|^\alpha}{(\max\{s,t\})^\lambda}$, $f(\sigma) \ge 0, g(\tau) \ge 0$ for $\sigma, \tau \in \mathbb{R}$, $F(s) = \int_0^s f(\sigma)\mathrm{d}\sigma$ and $G(t) = \int_0^t g(\tau)\mathrm{d}\tau$ for $s, t \in \mathbb{R}$, then*

$$\int_0^\infty \int_0^\infty \frac{|\ln(s/t)|^\alpha F(s)G(t)}{(s+t)(\max\{s,t\})^\lambda} \mathrm{d}s\mathrm{d}t \le \frac{\Gamma(\alpha+1)}{2\lambda^\alpha}(r^{\alpha+1} + s^{\alpha+1})$$
$$\times \left\{ \int_0^\infty s^{p(1-\frac{\lambda}{r})-1} F_f^p(s)\mathrm{d}s \right\}^{1/p} \left\{ \int_0^\infty t^{q(1-\frac{\lambda}{w})-1} G_g^q(t)\mathrm{d}t \right\}^{1/q}, \tag{3.4}$$

unless $f \equiv 0$ or $g \equiv 0$, where $\Gamma(a) := \int_0^\infty \mathbf{e}^{-t} t^{a-1}\mathrm{d}t$ is the Gamma function, $F_f(s) = \{\int_0^s (F^{m-1}(\sigma)f(\sigma))^2 \mathrm{d}\sigma\}^{1/2}$ and $G_g(t) = \{\int_0^t (G^{n-1}(\tau)g(\tau))^2 \mathrm{d}\tau\}^{1/2}$.

Proof. It is clear that $K(s,t) = \frac{|\ln(s/t)|^\alpha}{(\max\{s,t\})^\lambda}$ is a nonnegative measurable homogeneous kernel function of $-\lambda$-degree in $(0,\infty) \times (0,\infty)$. Similar to the above corollary, we obtain

$$C_\lambda(r) = \int_0^\infty K(u,1)u^{\frac{\lambda}{r}-1}\mathrm{d}u = \int_0^\infty \frac{|\ln u|^\alpha}{(\max\{u,1\})^\lambda} u^{\frac{\lambda}{r}-1}\mathrm{d}u$$
$$= \int_0^1 (-\ln u)^{(\alpha+1)-1} u^{\frac{\lambda}{r}-1}\mathrm{d}u + \int_1^\infty (\ln u)^{(\beta+1)-1} u^{-\frac{\lambda}{w}-1}\mathrm{d}u$$

$$= \frac{\Gamma(\alpha+1)}{\lambda^{\alpha}}(r^{\alpha+1}+s^{\alpha+1}).$$

In view of $k=l=2$, we get (3.4) directly. This completes the proof. □

Corollary 3.3. *Set* $m=n=1, k=l=p_1=p_2=2, \frac{1}{p}+\frac{1}{q}=1, \lambda>0, K(s,t)=\frac{1}{(s+t)^{\lambda-\alpha}(\max\{s,t\})^{\alpha}}$, $f(\sigma)\geq 0, g(\tau)\geq 0$ *for* $\sigma,\tau\in\mathbb{R}$, $F(s)=\int_0^s f(\sigma)\mathrm{d}\sigma$ *and* $G(t)=\int_0^t g(\tau)\mathrm{d}\tau$ *for* $s,t\in\mathbb{R}$, *then*

$$\int_0^{\infty}\int_0^{\infty}\frac{F(s)G(t)}{(s+t)^{1+\lambda-\alpha}(\max\{s,t\})^{\alpha}}\mathrm{d}s\mathrm{d}t \leq \frac{1}{2}C_{\lambda}(r)\left\{\int_0^{\infty}s^{p(1-\frac{\lambda}{r})-1}F_f^p(s)\mathrm{d}s\right\}^{1/p}$$
$$\times\left\{\int_0^{\infty}t^{q(1-\frac{\lambda}{w})-1}G_g^q(t)\mathrm{d}t\right\}^{1/q}, \tag{3.5}$$

unless $f\equiv 0$ *or* $g\equiv 0$, *where* $C_{\lambda}(r)=\sum_{i=0}^{\infty}\begin{pmatrix}\alpha-\lambda\\ i\end{pmatrix}\frac{(\lambda+2i)rw}{(\lambda+ri)(\lambda+wi)}$, $F_f(s)=\{\int_0^s(F^{m-1}(\sigma)f(\sigma))^2\mathrm{d}\sigma\}^{1/2}$ *and* $G_g(t)=\{\int_0^t(G^{n-1}(\tau)g(\tau))^2\mathrm{d}\tau\}^2$. *In particular, for* $\alpha=\lambda=1$, *(3.5) reduces to*

$$\int_0^{\infty}\int_0^{\infty}\frac{F(s)G(t)}{(s+t)\max\{s,t\}}\mathrm{d}s\mathrm{d}t \leq \frac{1}{2}C_1(r)\left\{\int_0^{\infty}s^{p(1-\frac{\lambda}{r})-1}F_f^p(s)\mathrm{d}s\right\}^{1/p}$$
$$\times\left\{\int_0^{\infty}t^{q(1-\frac{\lambda}{w})-1}G_g^q(t)\mathrm{d}t\right\}^{1/q}, \tag{3.6}$$

Proof. It is obvious that $K(s,t)=\frac{1}{(s+t)^{\lambda-\alpha}(\max\{s,t\})^{\alpha}}$ is a nonnegative measurable homogeneous kernel function of $-\lambda$-degree in $(0,\infty)\times(0,\infty)$. Similarly,

$$C_{\lambda}(r)=\int_0^{\infty}K(u,1)u^{\frac{\lambda}{r}-1}\mathrm{d}u=\int_0^1 K(u,1)u^{\frac{\lambda}{r}-1}\mathrm{d}u+\int_0^1 K(1,u)u^{\frac{\lambda}{w}-1}\mathrm{d}u$$
$$=\int_0^1(u+1)^{\alpha-\lambda}(u^{\frac{\lambda}{r}-1}+u^{\frac{\lambda}{w}-1})\mathrm{d}u=\int_0^1\sum_{i=0}^{\infty}\begin{pmatrix}\alpha-\lambda\\ i\end{pmatrix}u^i(u^{\frac{\lambda}{r}-1}+u^{\frac{\lambda}{w}-1})\mathrm{d}u$$
$$=\sum_{i=0}^{\infty}\begin{pmatrix}\alpha-\lambda\\ i\end{pmatrix}\int_0^1(u^{\frac{\lambda}{r}+i-1}+u^{\frac{\lambda}{w}+i-1})\mathrm{d}u=\sum_{i=0}^{\infty}\begin{pmatrix}\alpha-\lambda\\ i\end{pmatrix}\frac{(\lambda+2i)rw}{(\lambda+ri)(\lambda+wi)}.$$

In view of $k=l=2$, we get (3.5). This completes the proof. □

Corollary 3.4. *Set* $m=n=1, l=k=p_1=p_2=2, \frac{1}{p}+\frac{1}{q}=1, 0<\alpha\leq\lambda<\alpha\min\{r,w\}, K(s,t)=\frac{\ln(s/t)}{(s^{\alpha}-t^{\alpha})(\min\{s,t\})^{\lambda-\alpha}}$, $f(\sigma)\geq 0, g(\tau)\geq 0$ *for* $\sigma,\tau\in\mathbb{R}$, $F(s)=\int_0^s f(\sigma)\mathrm{d}\sigma$ *and* $G(t)=\int_0^t g(\tau)\mathrm{d}\tau$ *for* $s,t\in\mathbb{R}$ *then*

$$\int_0^\infty\int_0^\infty \frac{\ln(s/t)F(s)G(t)}{(s+t)(s^\alpha-t^\alpha)(\min\{s,t\})^{\lambda-\alpha}}\mathrm{d}s\mathrm{d}t$$
$$\le \frac{1}{2}C_{\lambda,\alpha}(r)\left\{\int_0^\infty s^{p(1-\frac{\lambda}{r})-1}F_f^p(s)\mathrm{d}s\right\}^{1/p}$$
$$\times\left\{\int_0^\infty t^{q(1-\frac{\lambda}{w})-1}G_g^q(t)\mathrm{d}t\right\}^{1/q}, \quad (3.7)$$

unless $f\equiv 0$ *or* $g\equiv 0$*, where* $C_{\lambda,\alpha}(r)=\sum_{i=0}^\infty\left\{\frac{1}{[(i+1)\alpha-\lambda/w]^2}+\frac{1}{[(i+1)\alpha-\lambda/r]^2}\right\}$, $F_f(s)=\{\int_0^s(F^{m-1}(\sigma)f(\sigma))^2\mathrm{d}\sigma\}^{1/2}$ *and* $G_g(t)=\{\int_0^t(G^{n-1}(\tau)g(\tau))^2\mathrm{d}\tau\}^{1/2}$.

Proof. It is obviously that $K(s,t)=\frac{\ln(s/t)}{(s^\alpha-t^\alpha)(\min\{s,t\})^{\lambda-\alpha}}$ is a nonnegative measurable homogeneous kernel function of $-\lambda$-degree in $(0,\infty)\times(0,\infty)$. We obtain

$$\begin{aligned}
C_{\lambda,\alpha}(r) &= \int_0^\infty K(u,1)u^{\frac{\lambda}{r}-1}\mathrm{d}u=\int_0^\infty \frac{u^{\frac{\lambda}{r}-1}\ln u}{(u^\alpha-1)(\min\{u,1\})^{\lambda-\alpha}}\mathrm{d}u\\
&= \int_0^1\frac{-\ln u}{1-u^\alpha}u^{\alpha-\frac{\lambda}{w}-1}\mathrm{d}u+\int_1^\infty\frac{\ln u}{1-u^{-\alpha}}u^{-\alpha+\frac{\lambda}{r}-1}\mathrm{d}u\\
&= \int_0^1(-\ln u)\sum_{i=0}^\infty u^{(i+1)\alpha-\frac{\lambda}{w}-1}\mathrm{d}u+\int_1^\infty \ln u\sum_{i=0}^\infty u^{-(i+1)\alpha+\frac{\lambda}{r}-1}\mathrm{d}u\\
&= \sum_{i=0}^\infty\frac{1}{(i+1)\alpha-\lambda/w}\int_0^1(-\ln u)\mathrm{d}u^{(i+1)\alpha-\frac{\lambda}{w}}\\
&\quad +\sum_{i=0}^\infty\frac{1}{-(i+1)\alpha+\lambda/r}\int_1^\infty(\ln u)\mathrm{d}u^{-(i+1)\alpha+\frac{\lambda}{r}}\\
&= \sum_{i=0}^\infty\left\{\frac{1}{[(i+1)\alpha-\lambda/w]^2}+\frac{1}{[(i+1)\alpha-\lambda/r]^2}\right\}.
\end{aligned}$$

Thus (3.7) is valid. This completes the proof. □

Corollary 3.5. *Set* $m=n=1, k=l=p_1=p_2=2, \frac{1}{p}+\frac{1}{q}=1, 0<\lambda<1+\alpha, K(s,t)=\frac{1}{|s-t|^{\lambda-\alpha}(s+t)^\alpha}$, $f(\sigma)\ge 0, g(\tau)\ge 0$ *for* $\sigma,\tau\in\mathbb{R}$, $F(s)=\int_0^s f(\sigma)\mathrm{d}\sigma$ *and* $G(t)=\int_0^t g(\tau)\mathrm{d}\tau$ *for* $s,t\in\mathbb{R}$*, then*

$$\int_0^\infty\int_0^\infty\frac{F(s)G(t)}{|s-t|^{\lambda-\alpha}(s+t)^{1+\alpha}}\mathrm{d}s\mathrm{d}t\le\frac{1}{2}C_{\alpha,\lambda}(r)\left\{\int_0^\infty s^{p(1-\frac{\lambda}{r})-1}F_f^p(s)\mathrm{d}s\right\}^{1/p}$$
$$\times\left\{\int_0^\infty t^{q(1-\frac{\lambda}{w})-1}G_g^q(t)\mathrm{d}t\right\}^{1/q}, \quad (3.8)$$

unless $f \equiv 0$ *or* $g \equiv 0$, *where*

$$C_{\alpha,\lambda}(r) = \sum_{i=0}^{\infty} \binom{-\alpha}{i} \left[B\left(1+\alpha-\lambda, \frac{\lambda}{r}+i\right) + B\left(1+\alpha-\lambda, \frac{\lambda}{w}+i\right) \right],$$

$F_f(s) = \{\int_0^s (F^{m-1}(\sigma)f(\sigma))^2 \mathrm{d}\sigma\}^{1/2}$ *and* $G_g(t) = \{\int_0^t (G^{n-1}(\tau)g(\tau))^2 \mathrm{d}\tau\}^{1/2}$.

Proof. Obviously, $K(s,t) = \frac{1}{|s-t|^{\lambda-\alpha}(s+t)^{\alpha}}$ is a nonnegative measurable homogeneous kernel function of $-\lambda$-degree in $(0,\infty)\times(0,\infty)$. Similarly, by a simple calculation, we obtain

$$\begin{aligned} C_{\lambda,\alpha}(r) &= \int_0^{\infty} K(u,1)u^{\frac{\lambda}{r}-1}\mathrm{d}u = \int_0^1 K(u,1)u^{\frac{\lambda}{r}-1}\mathrm{d}u + \int_0^1 K(1,u)u^{\frac{\lambda}{w}-1}\mathrm{d}u \\ &= \int_0^1 (1+u)^{-\alpha}(1-u)^{\alpha-\lambda}u^{\frac{\lambda}{r}-1}\mathrm{d}u + \int_0^1 (1+u)^{-\alpha}(1-u)^{\alpha-\lambda}u^{\frac{\lambda}{w}-1}\mathrm{d}u \\ &= \int_0^1 \sum_{i=0}^{\infty} \binom{-\alpha}{i} u^i (1-u)^{\alpha-\lambda} u^{\frac{\lambda}{r}-1}\mathrm{d}u \\ &\quad + \int_0^1 \sum_{i=0}^{\infty} \binom{-\alpha}{i} u^i (1-u)^{\alpha-\lambda} u^{\frac{\lambda}{w}-1}\mathrm{d}u \\ &= \sum_{i=0}^{\infty} \binom{-\alpha}{i} \left[\int_0^1 (1-u)^{\alpha-\lambda} u^{\frac{\lambda}{r}+i-1}\mathrm{d}u + \int_0^1 (1-u)^{\alpha-\lambda} u^{\frac{\lambda}{w}+i-1}\mathrm{d}u \right] \\ &= \sum_{i=0}^{\infty} \binom{-\alpha}{i} \left[B\left(1+\alpha-\lambda, \frac{\lambda}{r}+i\right) + B\left(1+\alpha-\lambda, \frac{\lambda}{w}+i\right) \right]. \end{aligned}$$

Hence (3.8) is correct. This completes the proof. □

Remark 3.6. All the above corollaries are particular cases of Theorem 2.3, and some special cases of other theorems can be deduced according to the above corollaries and hence we omit the details.

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