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CAUSAL RIESZ POTENTIAL AND CAUSAL POISSON OPERATORS

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Abstract: Relationship between Riesz potentials and the Poisson kernel in the radial case has been studied by Samko (cf. [9]) and Rudin (cf. [8]) among others authors. The last author give a theorem that establish that the Riesz potential can be expressed as an unidimensional integral whose integrand is a Poisson's integral. In this paper analogue relations between ultrahyperbolic causal Riesz potential and causal Poisson's integral is obtained.

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1. Preliminaries

Let $x = (x_1, x_2, .., x_n)$ be a point of the Euclidean space $\mathbb{R}^n$. We write

$$P = P(x) = x_1^2 + ... + x_p^2 - x_{p+1}^2 - ... - x_{p+q}^2; \tag{1.1}$$

a non degenerate quadratic form, where $p + q = n$.

The generalized function $(P \pm i0)^\lambda$ was introduced by Gelfand and Shilov (cf. [1]) as the following limit

$$(P \pm i0)^\lambda = \lim_{\varepsilon \to 0} \left(P \pm i\varepsilon|x^2|\right)^\lambda, \tag{1.2}$$

where λ is a complex number, $\varepsilon > 0$ and $|x|^2 = x_1^2 + ... + x_n^2$.

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Let $\{h_\alpha(P \pm i0, n)\}$ $\alpha \in \mathbb{C}$ be the family of the causal (anticausal) distributions introduced by Trione (cf. [10]) defined by

$$H_\alpha(P \pm i0, n) = \frac{e^{i\frac{\pi}{2}\alpha} e^{i\frac{\pi}{2}q} \Gamma\left(\frac{n-\alpha}{2}\right)}{2^\alpha \pi^{\frac{n}{2}} \Gamma(\frac{\alpha}{2})} (P \pm i0)^{\frac{\alpha-n}{2}}, \tag{1.3}$$

where Γ is the Euler Gamma function.

The Fourier transform of $H_\alpha(P \pm i0, n)$ is given by

$$\mathcal{F}[H_\alpha(P \pm io, n)](\xi) = \frac{1}{(2\pi)^{\frac{n}{2}}} \int e^{i\langle x, \xi\rangle} H_\alpha(P \pm io, n) dx = \frac{(Q \mp io)^{-\frac{\alpha}{2}}}{(2\pi)^{\frac{n}{2}}}, \tag{1.4}$$

where Q is the quadratic form

$$Q = \xi_1^2 + ... + \xi_p^2 - \xi_{p+1}^2 - ... - \xi_{p+q}^2$$

and $p + q = n$.

By $S(\mathbb{R}^n) = S$ we denote the Schwartz space infinitely differentiable rapidly decreasing funtions in $\mathbb{R}^n$.

Definition 1. Let φ be a function which belongs to S. The causal (anticausal) Riesz potential of order $\alpha > 0$ is defined by the convolution

$$R^\alpha \varphi = H_\alpha(P \pm i0, n) * \varphi \tag{1.5}$$

(cf. [1]).

This definition in analogue to the one due to Samko (cf. [9, p. 556]).

The integral in (1.5) results convergent in the case $\alpha > n-2$; and it admits an analytical continuation with respect to α for $\alpha \leq n-2$.

If the number q, the negative term of the form P, is equal to zero, P reduces to $|x|^2$ and then (1.5) reduces to the integral operator

$$(I^\alpha \varphi) = \frac{1}{C(n, \alpha, q = 0)} \int_{\mathbb{R}^n} \frac{\varphi(y)}{\|x - y|^n} dy, \tag{1.6}$$

$$C(n, \alpha, q = 0) = \pi^{\frac{n}{2}} 2^\alpha \Gamma\left(\frac{\alpha}{2}\right) \Gamma\left(\frac{n - \alpha}{2}\right),$$

called the elliptic Riesz potential of order α (cf. [7], also [8]).

As it is known the distributional functions $R^\alpha \varphi$ are causal (anticausal) analogues to the ultrahyperbolic potential introduced by Nozaki (cf. [6]) defined by the convolution

$$\mathcal{U}^\alpha f = \Phi_\alpha * f, \tag{1.7}$$

where Φ_α is given by

$$\Phi = \frac{r_+^{\alpha-n}}{C_n(\alpha)}, \tag{1.8}$$

$r_+^{\alpha-n} = P^{\frac{\alpha-n}{2}}$, with $x_1 > 0$, P given by (1.1), and the constant $C_n(\alpha)$ is

$$C_n = \frac{\pi^{\frac{n-1}{2}}\Gamma\left(\frac{2-\alpha-n}{2}\right)\Gamma\left(\frac{1-\alpha}{2}\right)\Gamma\left(\alpha\right)}{\Gamma\left(\frac{2+\alpha-p}{2}\right)\Gamma\left(\frac{p-\alpha}{2}\right)}. \tag{1.9}$$

It can be observed that if $p = 1$ is considered, from (1.8) and (1.9) we obtain

$$M_\alpha(u) = \begin{cases} \frac{u^{\frac{\alpha-n}{2}}}{N_n(\alpha)} & \text{if } x \in \Gamma_+, \\ 0 & \text{if } x \notin \Gamma_+, \end{cases} \tag{1.10}$$

where $u = x_1^2 - x_2^2 - ... - x_n^2$, denotes the cone $\Gamma_+ = \{x \in \mathbb{R}^n : x_1 > 0, u > 0\}$ and the constant $N_n(\alpha) = \pi^{\frac{n-2}{2}} 2^{\alpha-1}\Gamma\left(\frac{\alpha-n+2}{2}\right)\Gamma\left(\frac{\alpha}{2}\right)$.

$M_\alpha(u)$ is the hyperbolic Riesz kernel (cf. [7], also [11]).

By putting $n = 1$ in (1.10) and taking into account the Legendre duplication formula for the Gamma function

$$\Gamma(z) = 2^{2z-1}\pi^{-\frac{1}{2}}\Gamma(z)\Gamma\left(z + \frac{1}{2}\right),$$

we obtain

$$I_\alpha(x) = \begin{cases} \frac{x^{\alpha-1}}{\Gamma(\alpha)} & \text{if } \ x > 0, \\ 0 & \text{if } \ x \leq 0, \end{cases} \tag{1.11}$$

or equivalently $I_\alpha = \frac{x_+^{\alpha-1}}{\Gamma(\alpha)}$, where $x_+^{\alpha-1}$ is the generalized function

$$x_+^\lambda = \begin{cases} x^\lambda & \text{if } \ x > 0, \\ 0 & \text{if } \ x \leq 0 \end{cases} \tag{1.12}$$

(cf. [3]) for $\lambda = \alpha - 1$. $I_{\alpha(x)}$ is precisely the singular Riemann-Liouville kernel (cf. [7]).

Formally analogues to the $(P \pm i0)^\lambda$ generalized functions are the $(t^2 + P \pm i0)^\lambda$ distributions defined by the limit

$$(t^2 + P \pm i0)^\lambda = \lim_{\varepsilon \to 0}\left(t^2 + P \pm i\varepsilon|x|^2\right)^\lambda, \tag{1.13}$$

where t is a positive real number, $\varepsilon > 0$, $|x|^2 = x_1^2 + ... + x_n^2$ and $P = P(x)$ as in (1.1).

It can be observed that $(t^2 + P \pm i0)^\lambda$ are entire distributional functions of λ (cf. [10]). They will be used to introduce a kernel that may be considered as generalization of the Poisson kernel.

To invert causal Riesz potentials we have used the causal hypersingular integrals technique defining the causal Riesz derivative of order α, (cf. [1]) as

$$D^{\alpha}\varphi = \frac{1}{d_{n,l}(\alpha)} T_l^{\alpha}\varphi,$$

where $T_l^{\alpha}\varphi$ is the causal hypersingular integral given by the following

$$T_l^{\alpha}\varphi(x) = \int_{\mathbb{R}^n} \frac{\sum_{k=0}^{l}\binom{l}{k}(-1)^k \varphi(x-kt)}{(P+i0)^{\frac{n+\alpha}{2}}} dt.$$

The normalizing coefficient $\frac{1}{d_{n,l}(\alpha)}$ is chosen that the Fourier transform is given by

$$\mathcal{F}[D^{\alpha}\varphi] = (Q-i0)^{-\frac{\alpha}{2}}\mathcal{F}[\varphi].$$

Then, if $\varphi = R^{\alpha}f$, $f \in S$, we have that

$$D^{\alpha}\varphi = f.$$

In this paper firstly we will introduce a kernel that generalize the Poisson's one and then, by using the so called causal Poisson integrals we invert causal Riesz potential by means of one dimensional integral.

We follow the ideas by Rudin (cf. [8]) expossed to invert elliptic Riesz potentials.

2. The Causal Poisson Kernel and the Causal Poisson Integral

In this section we introduce a generalization of the radial Poisson kernel by means of distributions connected whit the quadratic form P.

Definition 2. Let (x,t) be point of $\mathbb{R}^n \times \mathbb{R}_+$,$x \in \mathbb{R}^n$, $t \in \mathbb{R}_+$. For $t \in \mathbb{R}_+$ consider the following generalized function

$$\mathcal{P}(P+i0,t) = \frac{e^{i\frac{\pi}{2}q}\Gamma\left(\frac{n+1}{2}\right)}{\pi^{\frac{n+1}{2}}} \frac{t}{(t^2+P+i0)^{\frac{n+1}{2}}}, \tag{2.1}$$

where $(t^2+P+i0)^{\frac{n+1}{2}}$ is given by (1.13) for $\lambda = \frac{n+1}{2}$.

We can observe that when $q=0$, (2.1) reduces at the Poisson kernel

$$\mathcal{P}(x,t) = \frac{\Gamma\left(\frac{n-1}{2}\right)}{\pi^{\frac{n+1}{2}}} \frac{t}{\left(t^2+\|x\|^2\right)^{\frac{n+1}{2}}} \tag{2.2}$$

(cf [9, p. 457], also [8, p. 217]).

Taking into account the Fourier transform of the $(t^2+P+i0)^{\lambda}$ distribution, the partial Fourier transform in x of (2.1) is

$$\mathcal{F}_x\left[\mathcal{P}(P+i0,t)\right](\xi) = \frac{e^{i\frac{\pi}{2}}2^{-\frac{n+1}{2}+1}(2\pi)^{\frac{n}{2}}t^{-\frac{1}{2}}}{\Gamma\left(\frac{n+1}{2}\right)}(Q-i0)^{\frac{1}{4}}\frac{\pi}{2}K_{-\frac{1}{2}}\left(t(Q-i0)^{\frac{1}{2}}\right). \quad (2.3)$$

From the well-known relationship (cf. [4] and also [5, p. 112])

$$K_{\pm\frac{1}{2}(z)} = \left(\frac{\pi}{2z}\right)^{\frac{1}{2}} e^{-z}. \quad (2.4)$$

From (2.3) and (2.4) it results

$$\mathcal{F}_x\left[\mathcal{P}(P+i0,t)\right](\xi) = \frac{e^{i\frac{\pi}{2}q}\pi^{\frac{n+1}{2}}}{2^{\frac{1}{2}}\Gamma\left(\frac{n+1}{2}\right)}e^{-t(Q-i0)^{\frac{1}{2}}}\frac{\Gamma\left(\frac{n+1}{2}\right)}{\pi^{\frac{n+1}{2}}} \quad (2.5)$$

$$= e^{i\frac{\pi}{2}q}e^{-t(Q-i0)^{\frac{1}{2}}}. \quad (2.6)$$

If in (2.5) $q=0$, we have

$$\mathcal{F}_x\left[\mathcal{P}(x,t)\right](\xi) = e^{-t|x|}, \quad (2.7)$$

that is the Fourier transform of the Poisson elliptic kernel (cf. [9], p. 498).

2.1. Causal Poisson Integral

Definition 3. Let φ be function which belongs to S – the Schwartzian space of functions.

The causal Poisson integral is defined by the following convolution

$$(P_t\varphi)(x) = \int_{\mathbb{R}^n} \mathcal{P}(y,t)\varphi(x-y)dy, \quad \text{for } t>0. \quad (2.8)$$

Some elementary properties of these integrals are pointed in the following.

Lemma 1. *1. For any $t>0$ the partial Fourier transform is*

$$\mathcal{F}_x\left[P_t(\varphi)\right](\xi) = e^{-t(Q-i0)^{\frac{1}{2}}}\mathcal{F}_x(\varphi)(\xi).$$

2. For $\varphi \in S$,

$$\lim_{t\to 0}(P\varphi)(x,t) = \varphi(x).$$

For the proof of this lemma we refer to [2].

Now we will present another expression of the $(P \pm i0)^{\lambda}$ distribution which will be used in further discussions.

Lemma 2. *For α a real number, $0<\alpha<n$, we have the following integral*

representation of the $(Q-i0)^{\frac{\alpha-n}{2}}$ *generalized function*

$$(Q-i0)^{\frac{\alpha-n}{2}} = \frac{2}{B\left(\frac{\alpha+1}{2}, \frac{n-\alpha}{2}\right)} \int_0^\infty \frac{t^\alpha}{(t^2+P+i0)^{\frac{n+1}{2}}} dt. \tag{2.9}$$

Proof. Consider the following well known generalized function

$$G(Q-i0) = (Q-i0)^{\frac{\alpha-n}{2}} \frac{1}{2} B\left(\frac{\alpha+1}{2}, \frac{n-\alpha}{2}\right) \tag{2.10}$$

where $B(x,y)$ is the Beta function $B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$.

According to a Trione's Theorem that allows us computing the Fourier transform of a generalized function depending on $(Q \pm i0)$ distribution as if it is a radial distribution (cf. [10]), i.e. making the formal substitution $(Q \pm i0) \to |y|^2$, we have

$$\mathcal{F}\left[G(Q-i0)\right] = \mathcal{F}\left[G\left(|y|^2\right)\right]_{|\xi|^2 \to P+i0}. \tag{2.11}$$

The symbol on the right-hand member of (2.11) has the following meaning, we evaluate the Fourier transform as it was a radial function and then replace $|\xi|^2$ by $(P+i0)$. Then

$$\mathcal{F}\left[G\left(y\right)\right](\xi) = \mathcal{F}\left[|y|^{\alpha-n} \frac{1}{2}\right] B\left(\frac{\alpha+1}{2}, \frac{n-\alpha}{2}\right)(\xi). \tag{2.12}$$

Taking into account formula (3.251.2) from [4] and putting $\alpha = \mu - 1$, and $\frac{n+1}{2} = 1 - \mu$, we have

$$G\left(|y|\right) = |y|^{\alpha-n} \int_0^\infty \frac{x^\alpha}{(1+x^2)^{\frac{n+1}{2}}} dx = \int_0^\infty \frac{(|y|x)^\alpha}{(|y|^2)^{\frac{n+1}{2}} (1+x^2)^{\frac{n+1}{2}}} |y| dx. \tag{2.13}$$

Making the substitution $|y|x = t$, we have

$$G(|y|) = \int_0^\infty \frac{t^\alpha}{(t^2+|y|^2)^{\frac{n+1}{2}}} |y| dt. \tag{2.14}$$

Then from (2.12) and (2.14), we can rewrite

$$(Q-i0)^{\frac{\alpha-n}{2}} \frac{1}{2} B\left(\frac{\alpha+1}{2}, \frac{n-\alpha}{2}\right) = \int_0^\infty \frac{t^2}{(t^2+Q+i0)^{\frac{n+1}{2}}} dt, \tag{2.15}$$

that is the thesis of the lemma.

(2.15) and other analogue expressions will be used in the next discussions and we will deal with in the same mode.

Notice that the left-hand side of (1.13) when α is replaced by $-\alpha$, for $\alpha > 0$, exists in the distributional sense, thus in this case the right-hand side must be understood in the same mode. $\square$

3. Representation of Causal Riesz Potentials by Means of Causal Poisson Integrals

In the section we given an integral representation of the causal Riesz potential by means of a causal Poisson integral.

Theorem 1. *Let α be a complex number that $Re(\alpha) < n$ and let φ be a function belonging to S. Then the causal Riesz potential of order α, $R^\alpha\varphi$ admits a representation given by the following integral:*

$$(R^\alpha\varphi)(x) = \frac{e^{i\frac{\pi}{2}q}}{\Gamma(\alpha)}\int_0^\infty t^{\alpha-1}(\mathcal{P}_t\varphi)(t)dt. \tag{3.1}$$

Proof. By considering the left-hand member of (3.1) we have:

$$\begin{aligned} I &= \frac{e^{i\frac{\pi}{2}q}}{\Gamma(\alpha)}\int_0^\infty t^{\alpha-1}(\mathcal{P}_t\varphi)(t)dt \\ &= \frac{e^{i\frac{\pi}{2}q}}{\Gamma(\alpha)}\int_0^\infty t^{\alpha-1}\left(\int_{\mathbb{R}^n}\frac{C_n\varphi(x-y)}{(t^2+P+i0)^{\frac{n+1}{2}}}tdy\right)dt \\ &= \frac{C_n e^{i\frac{\pi}{2}q}}{\Gamma(\alpha)}\int_{\mathbb{R}^n}\varphi(x-y)dy\int_0^\infty\frac{t^\alpha}{(t^2+P+i0)^{\frac{n+1}{2}}}dt. \end{aligned} \tag{3.2}$$

Taking into account (2.15), it results

$$I = \frac{C_n\Gamma\left(\frac{\alpha+1}{2}\right)\Gamma\left(\frac{n-\alpha}{2}\right)}{2\Gamma(\alpha)\Gamma\left(\frac{n+1}{2}\right)}\int_{\mathbb{R}^n}\varphi(x-y)(P+i0)^{\frac{\alpha-n}{2}}dy, \tag{3.3}$$

where

$$C_n = \frac{\Gamma\left(\frac{n+1}{2}\right)}{\pi^{\frac{n+1}{2}}}. \tag{3.4}$$

We conclude that

$$I = \frac{e^{i\frac{\pi}{2}q}\Gamma\left(\frac{n+1}{2}\right)\Gamma\left(\frac{n-\alpha}{2}\right)2^\alpha\pi^{\frac{n}{2}}\Gamma\left(\frac{\alpha}{2}\right)}{2\Gamma(\alpha)\pi^{\frac{n-1}{2}}e^{i\frac{\pi}{2}q}e^{i\pi\alpha}\Gamma\left(\frac{n-\alpha}{2}\right)}R^\alpha\varphi = \frac{2^{\alpha-1}\Gamma\left(\frac{\alpha+1}{2}\right)\Gamma\left(\frac{\alpha}{2}\right)}{2\Gamma(\alpha)\pi^{\frac{1}{2}}}R^\alpha\varphi. \tag{3.5}$$

Applying formula (1.2.3), p. 3 from [5], we have

$$\Gamma\left(\frac{\alpha+1}{2}\right) = \frac{\sqrt{\pi}\Gamma(\alpha)}{2^{\alpha-1}\Gamma\left(\frac{\alpha}{2}\right)}. \tag{3.6}$$

Then, from (3.5) and (3.6), it results

$$I = R^\alpha\varphi, \tag{3.7}$$

which is the thesis of Theorem 1. ☐

Theorem 1 allows us to invert causal Riesz potential by using unidimensional integral that involves causal Poisson integral. In fact, we have the following.

Theorem 2. *Let α by a positive real number, $\alpha \neq 0, 1, 2, \ldots$, and let φ be a function belonging to S, an let $D^{\alpha}\varphi(x)$ the causal Riesz derivative of order α.*

Then $D^{\alpha}\varphi(x)$ admits a representation given by the following integral

$$D^{\alpha}\varphi(x) = \frac{e^{i\frac{\pi}{2}q}}{\Gamma(-\alpha)} \int_0^{\infty} \frac{1}{t^{\alpha+1}} \left(\sum_{k=0}^{l} \binom{l}{k} (-1)^k (P_{kt}\varphi)(x) \right) dt. \qquad (3.8)$$

Proof. By the definition we have that the causal Riesz derivative is given by the following n dimensional integral (cf. [1])

$$\begin{aligned} D^{\alpha}\varphi(x) &= \frac{(T_l^{\alpha}\varphi)(x)}{d_{n,l}(\alpha)} \\ &= \frac{1}{d_{n,l}} \int_{\mathbb{R}^n} \sum_{k=0}^{l} \binom{l}{k} (-1)^k (P + i0)^{-\frac{n+\alpha}{2}} \varphi(x - kt) dt, \qquad (3.9) \end{aligned}$$

where

$$d_{n,l}(\alpha) = \frac{\pi^{\frac{n}{2}+1} e^{i\frac{\pi}{2}q} \mathcal{A}_l(\alpha)}{2^{\alpha}\Gamma\left(1 + \frac{\alpha}{2}\right)\Gamma\left(\frac{n-\alpha}{2}\right) \sin \frac{\pi}{2}\alpha} \qquad (3.10)$$

and $\mathcal{A}_l(\alpha)$ is the function of the parameter α given by the following expression

$$\mathcal{A}_l(\alpha) = \sum_{k=0}^{l} \binom{l}{k} (-1)^k k^{\alpha}, \quad a > 0 \quad \text{(cf. [3], also [1])}.$$

Let us show that $(T_l^{\alpha}\varphi)(x)$ may be written as an integral which integrand depends on the ultrahyperbolic Poisson's kernel. In fact

$$(T_l^{\alpha}\varphi)(x) = \frac{1}{d_{n,l}} \sum_{k=0}^{l} \binom{l}{k} (-1)^k \int_{\mathbb{R}^n} (P + i0)^{-\frac{n+\alpha}{2}} \varphi(x - kt) dt. \qquad (3.11)$$

Making the substitution $y = kt$, it result

$$\begin{aligned} \frac{(T_l^{\alpha}\varphi)(x)}{d_{n,l}(\alpha)} &= \frac{1}{d_{n,l}} \sum_{k=0}^{l} \binom{l}{k} (-1)^k k^{\alpha} \int_{\mathbb{R}^n} (P + i0)^{-\frac{n+\alpha}{2}} \varphi(x - kt) dt \\ &= \frac{1}{d_{n,l}} \sum_{k=0}^{l} \binom{l}{k} (-1)^k k^{\alpha} \frac{2^{-\alpha}\pi^{\frac{n}{2}}\Gamma\left(-\frac{\alpha}{2}\right)}{e^{-i\frac{\pi}{2}q}\Gamma\left(\frac{n+\alpha}{2}\right)} R^{-\alpha}\varphi(x) \\ &= \frac{1}{d_{n,l}(\alpha)} \mathcal{A}_t(\alpha) \frac{2^{-\alpha}\pi^{\frac{n}{2}}\Gamma\left(-\frac{\alpha}{2}\right)}{e^{-i\frac{\pi}{2}q}\Gamma\left(\frac{n+\alpha}{2}\right)} R^{-\alpha}\varphi(x). \qquad (3.12) \end{aligned}$$

From (3.10), (1.3) and (3.12) we have

$$\begin{aligned}\frac{(T_l^\alpha\varphi)(x)}{d_{n,l}(\alpha)} &= \frac{2^\alpha\Gamma\left(1+\frac{\alpha}{2}\right)\Gamma\left(\frac{n-\alpha}{2}\right)\sin\frac{\pi}{2}\alpha}{\pi^{\frac{n}{2}}\pi e^{i\frac{\pi}{2}q}\mathcal{A}_t(\alpha)}\frac{\mathcal{A}_t(\alpha)e^{i\frac{\pi}{2}\alpha}\pi^{\frac{n}{2}}\Gamma\left(-\frac{\alpha}{2}\right)}{e^{i\frac{\pi}{2}q}\Gamma\left(\frac{n+\alpha}{2}\right)}R^{-\alpha}\varphi(x)\\ &= R^{-\alpha}\varphi(x). \qquad (3.13)\end{aligned}$$

Let us consider the integral in the right-hand side of (3.7). Formally we have

$$I(\varphi) = \frac{1}{C(\alpha,l)}\int_0^\infty\left(\sum_{k=0}^{l}\binom{l}{k}(-1)^k(P_{kt}(x))\right)\frac{dt}{t^{\alpha+1}}, \qquad (3.14)$$

where

$$C(\alpha,l) = \Gamma(-\alpha)\sum_{k=0}^{l}\binom{l}{k}(-1)^k k^\alpha. \qquad (3.15)$$

Applying the definition of the causal Poisson kernel, we have

$$I(\varphi) = \frac{C_n}{C(\alpha,l)}\int_0^\infty\sum_{k=0}^{l}\binom{l}{k}(-1)^k\left(\int_{\mathbb{R}^n}\frac{kt}{(P+i0+(kt)^2)^{\frac{n+1}{2}}}\varphi(x-y)dy\right)\frac{dt}{t^{\alpha+1}}.$$

Changing the order of integration

$$I(\varphi) = \frac{C_n}{C(\alpha,l)}\int_{\mathbb{R}^n}\sum_{k=0}^{l}\binom{l}{k}(-1)^k k^\alpha\left(\int_0^\infty\frac{(z)^{-\alpha}}{(P+i0+(z)^2)^{\frac{n+1}{2}}}dt\right)\varphi(x-y)dy.$$

Making the change $kt=z$, we obtain

$$I(\varphi) = \frac{C_n}{C(\alpha,l)}\int_{\mathbb{R}^n}\sum_{k=0}^{l}\binom{l}{k}(-1)^k k^{\alpha+1}$$
$$\times\left(\int_0^\infty\frac{(kt)^{-\alpha}}{(P+i0+(kt)^2)^{\frac{n+1}{2}}}dt\right)\varphi(x-y)dy.$$

Applying Lemma 2, we have

$$I(\varphi) = \frac{C_n}{C(\alpha,l)}\sum_{k=0}^{l}\binom{l}{k}(-1)^k k^\alpha\frac{1}{2}B\left(\frac{-\alpha+1}{2},\frac{n+\alpha}{2}\right)$$
$$\times\int_{\mathbb{R}^n}(P+i0)^{-(\frac{\alpha+n}{2})}\varphi(x-y)dy.$$

From (3.4) and the definition of the Beta function we have

$$I(\varphi) = \frac{\Gamma(\frac{n+1}{2})}{\pi^{\frac{n+1}{2}}}\frac{1}{2}\frac{\Gamma(-\frac{\alpha}{2}+\frac{1}{2})\Gamma(\frac{n+\alpha}{2})}{\Gamma(\frac{n+1}{2})}\frac{1}{\Gamma(-\alpha)}\int_{\mathbb{R}^n}(P+i0)^{-(\frac{\alpha+n}{2})}\varphi(x-y)dy.$$

From formula (1.2.3) of [5] we have that

$$2^{2z-1}\Gamma(z)\Gamma(z+\frac{1}{2}) = \sqrt{\pi}\,\Gamma(2z),$$

for $z = -\frac{\alpha}{2}$, we obtain

$$I(\varphi) = \frac{\Gamma(-\alpha)\Gamma(\frac{n+\alpha}{2})}{\Gamma(-\alpha)2^{-\alpha}\Gamma(-\frac{\alpha}{2})\pi^{\frac{n}{2}}} \int_{\mathbb{R}^n} (P+i0)^{-(\frac{\alpha+n}{2})}\varphi(x-y)dy, \tag{3.16}$$

$$I(\varphi) = H_{-\alpha}(P+i0) * \varphi = R^{-\alpha}\varphi. \tag{3.17}$$

Then, from (3.8), (3.12), (3.14) and (3.17) we obtain an integral representation of the causal Riesz derivative in terms of causal Poisson integral given by

$$D^{\alpha}\varphi = \frac{1}{\Gamma(-\alpha)} \int_0^{\infty} \left(\sum_{k=0}^{l} \binom{l}{k} (-1)^k (P_{kt}\varphi)(x) \right) \frac{dt}{t^{\alpha+1}},$$

wich is the thesis of Theorem 2. □

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