International Journal of Pure and Applied Mathematics
Volume 63 No. 1 2010, 85-106

POSITIVE SOLUTIONS TO MULTI-POINT BVPS WITH p-LAPLACIAN ON TIME SCALES

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Abstract: Some new and more general sufficient conditions for the existence of at least twin or arbitrary even positive solutions to a multi-point BVPs for p-Laplacian dynamic equation on time scales are presented. The methods used in this paper are different from that of the paper by Su and Li (Y.H. Su, W.T. Li, Triple positive solutions of m-point BVPs for p-Laplacian dynamic equations on time scales, *Nonlinear Anal.*, **68** (2008), 1442-1452). Our results generalize and improve some known results. In particular, the results are even new for the special cases of differential and difference equations, as well as in the general time scale setting. As an application, two examples are given to illustrate the results.

AMS Subject Classification: 34B15, 39A10
Key Words: time scales, boundary value problem, positive solutions, p-Laplacian, fixed point theorem

1. Introduction

For convenience, we make the blanket assumption that $0, T$ are points in $\mathbb{T}$, for an interval $(0,T)_{\mathbb{T}}$ and we always mean $(0,T) \cap \mathbb{T}$. Other type of intervals are defined similarly.

Received: July 5, 2010

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Throughout this paper, we denote the p-Laplacian operator by $\varphi_p(u)$, i.e., $\varphi_p(u) = |u|^{p-2}u$, $p > 1, (\varphi_p)^{-1} = \varphi_q$, $1/p + 1/q = 1$. In addition, B_0 and B_1 satisfy

$$A'x \le B_i(x) \le Bx,\ x \in \mathbb{R}^+,\ i = 0, 1, \tag{1.1}$$

where A' and B are positive real numbers.

Recently, there is much attention be paid to the positive solutions of two-point, three-point BVPs for second order dynamic equation. For two-point BVPs, see [8, 11, 23, 24] and the references therein. As far as for three-point BVPs, see [9, 10, 12, 16, 17, 19].

For the multi-point BVPs

$$(\varphi_p(u^\Delta(t)))^\nabla + h(t)f(u(t)) = 0,\ t \in [0,T]_{\mathbb{T}}, \tag{1.2}$$

$$u(0) - B_0\left(\sum_{i=1}^{m-2} a_i u^\Delta(\xi_i)\right) = 0,\ u^\Delta(T) = 0, \tag{1.3}$$

or

$$u^\Delta(0) = 0,\ u(T) + B_1\left(\sum_{i=1}^{m-2} b_i u^\Delta(\xi_i')\right) = 0. \tag{1.4}$$

Su and Li [20] obtained the existence criteria for at least *three* positive solutions to problem (1.2) and (1.3) or (1.4).

On the one hand, little work [18, 20, 21, 22, 25, 26] has been done to the positive to multi-point BVPs with one-dimensional p-Laplacian dynamic equation on time scales. In [18, 21, 22, 26], the authors only obtained that there exists at least one positive solution by using the upper and lower solution method. However, they did not give the location and character of positive solution. On the other hand, in [20, 25], the authors failed to further provide comprehensible results of positive solution in spite of giving part character and location of positive solutions. In addition, the authors [20, 25] only obtain the existence of triple solution, however, they did not obtain the existence of arbitrary positive solutions. Hence, it is quite necessary to consider the existence of positive solutions for p-Laplacian multi-point BVPs on time scales in all respects.

In this paper, we all-sidedly consider the dynamic equation

$$(\varphi_p(u^\Delta(t)))^\nabla + h(t)f(t, u(t)) = 0,\ t \in [0,T]_{\mathbb{T}}, \tag{1.5}$$

subject to multi-point boundary conditions

$$u(0) - B_0\left(\sum_{i=1}^{m-2} a_i u^\Delta(\xi_i)\right) = 0,\ u^\Delta(T) = 0, \tag{1.6}$$

or

$$u^\Delta(0) = 0,\ u(T) + B_1\left(\sum_{i=1}^{m-2} b_i u^\Delta(\xi_i')\right) = 0, \tag{1.7}$$

where $\xi_i, \xi_i' \in [0,T]_{\mathbb{T}}$, and satisfy $0 \le \xi_1 < \xi_2 < \ldots < \xi_{m-2} < \rho(T)$, $\sigma(0) < \xi_1' < \xi_2' < \ldots < \xi_{m-2}' \le T$, $a_i, b_i \in [0,\infty)$ $(i = 1,2,\ldots,m-2)$. Some new and more general results are obtained for the existence of at least *twin* or arbitrary *even* positive solutions for the above problems by using fixed point theorem due to Avery and Henderson [3]. In particular, if $f(t,u) = f(u)$, our results include and extent the results of He [9, 11]; the main results of He [10] and Liu et al [16] are our special cases for $\mathbb{T} = \mathbb{Z}$. Our results are even new for the special cases of difference equations and differential equations as well as in the general time scale setting. As an application, two examples are given to illustrate the results.

We note that by a solution $u(t)$ of problems (1.5) and (1.6) or (1.7), we mean $u(t) : \mathbb{T} \to \mathbb{R}$ which is delta differential, $u^{\Delta}(t)$ and $(\varphi_p(u^{\Delta}))^{\nabla}(t)$ are both continuous on $\mathbb{T}^{\kappa} \cap \mathbb{T}_{\kappa}$, and $u(t)$ satisfies problems (1.5) and (1.6) or (1.7). If $(u^{\Delta})^{\nabla}(t) \le 0$, then we say $u(t)$ is concave on $[0,T]_{\mathbb{T}}$.

Now, we present some basic definitions which can be found in [5, 6]. Another excellent sources on dynamical systems on measure chains is the book [14].

A time scale $\mathbb{T}$ is a nonempty closed subset of $\mathbb{R}$. For $t \in \mathbb{T}$, the forward and back jump operators $\sigma, \rho : \mathbb{T} \to \mathbb{T}$ are well defined, respectively, by

$$\sigma(t) = \inf\{s \in \mathbb{T} : s > t\} \text{ and } \rho(t) = \sup\{s \in \mathbb{T} : s < t\}.$$

In this definition one put $\inf \emptyset := \sup \mathbb{T}$ and $\sup \emptyset := \inf \mathbb{T}$, where $\emptyset$ denotes the empty set. A point $t \in \mathbb{T}$ is called left-dense if $\rho(t) = t$, left-scattered if $\rho(t) < t$, right-dense if $\sigma(t) = t$, right-scattered if $\sigma(t) > t$. If $\mathbb{T}$ has a right-scattered minimum m, define $\mathbb{T}_{\kappa} = \mathbb{T} - \{m\}$; otherwise, set $\mathbb{T}_{\kappa} = \mathbb{T}$. If $\mathbb{T}$ has a left-scattered maximum M, define $\mathbb{T}^{\kappa} = \mathbb{T} - \{M\}$; otherwise, set $\mathbb{T}^{\kappa} = \mathbb{T}$. The forward graininess is $\mu(t) := \sigma(t) - t$. Similarly, the backward graininess is $\nu(t) := t - \rho(t)$.

If $f : \mathbb{T} \to \mathbb{R}$ is a function and $t \in \mathbb{T}^{\kappa}$, then the delta derivative of f at the point t is defined to the number $f^{\Delta}(t)$ (provided it exists) with the property that for any $\epsilon > 0$, there is a neighborhood $U \subset \mathbb{T}$ of t such that

$$\left| f(\sigma(t)) - f(s) - f^{\Delta}(t)(\sigma(t) - s) \right| \le \epsilon \left| \sigma(t) - s \right| \text{ for all } s \in U.$$

If $f : \mathbb{T} \to \mathbb{R}$ and $t \in \mathbb{T}_{\kappa}$, then the nabla derivative of f at the point t is defined by the number $f^{\nabla}(t)$ (provided it exists) with the property that for any $\epsilon > 0$, there is a neighborhood $U \subset \mathbb{T}$ of t such that

$$\left| f(\rho(t)) - f(s) - f^{\nabla}(t)(\rho(t) - s) \right| \le \epsilon \left| \rho(t) - s \right| \text{ for all } s \in U.$$

A function $f : \mathbb{T} \to \mathbb{R}$ is ld-continuous provided it is continuous at left dense points in $\mathbb{T}$ and its right sided limit exists (finite) at right dense points in $\mathbb{T}$.

Throughout this paper, it is assumed that:

(S1) $f : [0, T]_{\mathbb{T}} \times \mathbb{R} \to \mathbb{R}^+$ is continuous, and does not vanish identically on any closed subinterval of $[0, T]_{\mathbb{T}}$, where $\mathbb{R}^+$ denotes the nonnegative real numbers;

(S2) $h : \mathbb{T} \to \mathbb{R}^+$ is left dense continuous (i.e., $h \in C_{ld}(\mathbb{T}, \mathbb{R}^+)$), and does not vanish identically on any closed subinterval of $[0, T]_{\mathbb{T}}$, where $C_{ld}(\mathbb{T}, \mathbb{R}^+)$ denotes the set of all left dense continuous functionals from $\mathbb{T}$ to $\mathbb{R}^+$;

(S3) While discussing problem (1.5) and (1.6), we assume that if $\xi_{m-2} > 0$, then let $0 < \eta = \xi_{m-2}$, if $\xi_{m-2} = 0$, then let $\eta = \min\{t \in \mathbb{T} : t \geq \frac{T}{2}\}$, and there exists $r \in \mathbb{T}$ such that $\eta < r < T$ holds. While discussing problem (1.5) and (1.7), we assume that if $\xi_1' < T$, then let $\xi = \xi_1'$, if $\xi_1' = T$, then let $\xi = \max\left\{t \in \mathbb{T} : 0 < t \leq \frac{T}{2}\right\}$, and there exists $l \in \mathbb{T}$ such that $0 < l < \xi < T$ holds.

2. Some Lemmas

Now, we provide some background material from the theory of cones in Banach spaces (see [7]), and we then state fixed point theorems which we needed later.

Definition 2.1. Let E be a real Banach space. A nonempty, closed, convex set $P \subset E$ is said to be a cone provided the following conditions are satisfied: (i) If $x \in P$ and $\lambda \geq 0$, then $\lambda x \in P$; (ii) If $x \in P$ and $-x \in P$, then $x = 0$.

Every cone $P \subset E$ induces an ordering in E given by $x \leq y$ if and only if $y - x \in P$.

Definition 2.2. Given a cone P in a real Banach space E, a functional $\psi : P \to \mathbb{R}$ is said to be increasing on P, provided $\psi(x) \leq \psi(y)$ for all $x, y \in P$ with $x \leq y$.

Definition 2.3. A map α is said to be a nonnegative continuous concave functional on a cone P of a real Banach space E if $\alpha : P \to [0, \infty)$ is continuous and

$$\alpha(tx + (1-t)y) \geq t\alpha(x) + (1-t)\alpha(y)$$

for all $x, y \in P$ and $t \in [0, 1]$.

Similarly we say the map β is a nonnegative continuous convex functional on a cone P of a real Banach space E if $\beta : P \to [0, \infty)$ is continuous and

$$\beta(tx + (1-t)y) \leq t\beta(x) + (1-t)\beta(y)$$

for all $x, y \in P$ and $t \in [0, 1]$.

In order to prove the main results, we need the following lemmas.

Lemma 2.4. (see [7]) *Let P be a cone in a real Banach space E and U be a bounded, relatively open subset of P. Suppose that $A : \overline{U} \to P$ is a completely continuous operator and there exists a u_0 such that $x - Ax \neq \lambda u_0$ for all $x \in \partial U$ and $\lambda \geq 0$. Then $i(A, U, P) = 0$.*

Given a nonnegative continuous functional ψ on a cone P of a real Banach space E, we define, for each $d > 0$, the set

$$P(\gamma, d) = \{x \in P : \gamma(x) < d\}.$$

Lemma 2.5. *(see* [3]) Let P be a cone in a real Banach space E. Let α and γ be increasing, nonnegative continuous functional on P, and let θ be a nonnegative continuous functional on P with $\theta(0) = 0$ such that, for some $c > 0$ and $H > 0$,

$$\gamma(x) \leq \theta(x) \leq \alpha(x) \text{ and } \|x\| \leq H\gamma(x) \text{ for all } x \in \overline{P(\gamma, c)}.$$

Suppose there exist a completely continuous operator $A : \overline{P(\gamma, c)} \to P$ and $0 < a < b < c$ such that

$$\theta(\lambda x) \leq \lambda\theta(x) \text{ for } 0 \leq \lambda \leq 1 \text{ and } x \in \partial P(\theta, b),$$

and:

(i) $\gamma(Ax) > c$ for all $x \in \partial P(\gamma, c)$;

(ii) $\theta(Ax) < b$ for all $x \in \partial P(\theta, b)$;

(iii) $P(\alpha, a) \neq \emptyset$ and $\alpha(Ax) > a$ for $x \in \partial P(\alpha, a)$.

Then A has at least two fixed points x_1 and x_2 belonging to $\overline{P(\gamma, c)}$ satisfying

$$a < \alpha(x_1) \text{ with } \theta(x_1) < b \text{ and } b < \theta(x_2) \text{ with } \gamma(x_2) < c.$$

The following lemma can be found in [16].

Lemma 2.6. (see [16]) *Let P be a cone in a real Banach space E. Let α and γ be increasing, nonnegative continuous functional on P, and let θ be a nonnegative continuous functional on P with $\theta(0) = 0$ such that, for some $c > 0$ and $H > 0$,*

$$\gamma(x) \leq \theta(x) \leq \alpha(x) \text{ and } \|x\| \leq H\gamma(x) \text{ for all } x \in \overline{P(\gamma, c)}.$$

Suppose there exist a completely continuous operator $A : \overline{P(\gamma, c)} \to P$ and

$0 < a < b < c$ such that

$$\theta(\lambda x) \le \lambda\theta(x) \text{ for } 0 \le \lambda \le 1 \text{ and } x \in \partial P(\theta, b),$$

and:

(i) $\gamma(Ax) < c$ *for all* $x \in \partial P(\gamma, c)$;

(ii) $\theta(Ax) > b$ *for all* $x \in \partial P(\theta, b)$;

(iii) $P(\alpha, a) \ne \emptyset$ *and* $\alpha(Ax) < a$ *for* $x \in \partial P(\alpha, a)$.

Then A has at least two fixed points x_1 and x_2 belonging to $\overline{P(\gamma, c)}$ satisfying

$$a < \alpha(x_1) \text{ with } \theta(x_1) < b \text{ and } b < \theta(x_2) \text{ with } \gamma(x_2) < c.$$

3. Existence Results of Problems (1.5) and (1.6)

In this section, by using the fixed point theorems due to Avery and Henderson [3], we will discuss the existence of at least twin or arbitrary even positive solutions to problem (1.5) and (1.6) under some conditions.

Let E be a Banach space

$$E = C_{ld}\left([0, \sigma(T)]_{\mathbb{T}} \to \mathbb{R}\right)$$

with the norm

$$\|u\| = \sup_{t \in [0, \sigma(T)]_{\mathbb{T}}} |u(t)|,$$

and define the cone $P \subset E$ by

$$P = \left\{ \begin{array}{c} u \in E : u(t) \ge 0 \text{ for } t \in [0, \sigma(T)]_{\mathbb{T}} \text{ and} \\ u^{\Delta\Delta}(t) \le 0, u^{\Delta}(t) \ge 0 \text{ for } t \in [0, T]_{\mathbb{T}},\ u^{\Delta}(T) = 0 \end{array} \right\}.$$

In order to obtain the main results, we need to list the following lemma.

Lemma 3.1. *If $u \in P$, then*

(i) $u(t) \ge \frac{t}{\sigma(T)} \|u\| = \frac{t}{\sigma(T)} u(\sigma(T))$ *for* $t \in [0, \sigma(T)]_{\mathbb{T}}$;

(ii) $su(t) \le tu(s)$ *for* $s, t \in [0, \sigma(T)]_{\mathbb{T}}$ *and* $s \le t$.

Proof. (i) Since $u^{\Delta\Delta}(t) \le 0$, it follows that $u^{\Delta}(t)$ is nonincreasing. Hence, for $0 < t < \sigma(T)$,

$$u(t) - u(0) = \int_0^t u^{\Delta}(s)\Delta s \ge tu^{\Delta}(t)$$

and

$$u(\sigma(T)) - u(t) = \int_t^{\sigma(T)} u^{\Delta}(s)\Delta s \le (\sigma(T) - t)u^{\Delta}(t),$$

from which we have

$$u(t) \geq \frac{tu(\sigma(T)) + (\sigma(T) - t)u(0)}{\sigma(T)} \geq \frac{t}{\sigma(T)} u(\sigma(T)) = \frac{t}{\sigma(T)} \|u\| .$$

The proof is complete.

(ii) By using the similar way as in Lemma 3.1 in [20], we can prove it. □

Clearly, $\|u\| = u(\sigma(T))$ for $u \in P$. In view of (1.5) and (1.6), it is easy to have

$$u(t) = B_0 \left(\sum_{i=1}^{m-2} a_i \varphi_q \left(\int_{\xi_i}^{T} h(s) f(s, u(s)) \nabla s \right) \right) + \int_0^t \varphi_q \left(\int_\tau^T h(s) f(s, u(s)) \nabla s \right) \Delta\tau.$$

Define the operator $A : P \to E$ by

$$Au(t) = B_0 \left(\sum_{i=1}^{m-2} a_i \varphi_q \left(\int_{\xi_i}^{T} h(s) f(s, u(s)) \nabla s \right) \right) + \int_0^t \varphi_q \left(\int_\tau^T h(s) f(s, u(s)) \nabla s \right) \Delta\tau.$$

It is obvious that $A : P \to P$ is completely continuous.

For $u \in P$, we define the nonnegative, increasing, continuous functionals γ, θ and α by

$$\gamma(u) = \min_{t \in [\eta, r]_{\mathbb{T}}} u(t) = u(\eta), \quad \theta(u) = \max_{t \in [0, \eta]_{\mathbb{T}}} u(t) = u(\eta),$$

and

$$\alpha(u) = \max_{t \in [0, r]_{\mathbb{T}}} u(t) = u(r).$$

It is obvious that $\gamma(u) = \theta(u) \leq \alpha(u)$ for each $u \in P$. By Lemma 3.1, one obtains

$$\|u\| \leq \frac{\sigma(T)}{\eta} u(\eta) = \frac{\sigma(T)}{\eta} \gamma(u) \text{ for all } u \in P.$$

In addition, for positive constant b', we have

$$\theta(\lambda u) = \lambda \theta(u), \ 0 \leq \lambda \leq 1 \text{ and } u \in \partial P(\theta, b').$$

For the notational convenience, we denote

$$M = \left(A' \sum_{i=1}^{m-2} a_i + \eta \right) \varphi_q \left(\int_\eta^T h(s) \nabla s \right),$$

$$N=\left(B\sum_{i=1}^{m-2}a_i+\eta\right)\varphi_q\left(\int_0^T h(s)\nabla s\right),$$

and

$$L=\left(A'\sum_{i=1}^{m-2}a_i+r\right)\varphi_q\left(\int_r^T h(s)\nabla s\right).$$

We now present the main results in this section.

Theorem 3.2. *Suppose that there are positive numbers a', b', c' such that $0<a'<\frac{L}{N}b'<\frac{\eta L}{\sigma(T)N}c'$. In addition, $f(t,u)$ satisfies the following conditions:*

(i) $f(t,u)>\varphi_p\left(\frac{c'}{M}\right)$ *for* $t\in[\eta,\sigma(T)]_{\mathbb{T}}$ *and* $c'\le u\le\frac{\sigma(T)}{\eta}c'$;

(ii) $f(t,u)<\varphi_p\left(\frac{b'}{N}\right)$ *for* $t\in[0,\sigma(T)]_{\mathbb{T}}$ *and* $0\le u\le\frac{\sigma(T)}{\eta}b'$;

(iii) $f(t,u)>\varphi_p\left(\frac{a'}{L}\right)$ *for* $t\in[r,\sigma(T)]_{\mathbb{T}}$ *and* $a'\le u\le\frac{\sigma(T)}{r}a'$.

Then problem (1.5) *and* (1.6) *has at least two positive solutions* u_1 *and* u_2 *such that*

$$a'<\max_{t\in[0,r]_{\mathbb{T}}}u_1(t)\ \textit{with}\ \max_{t\in[0,\eta]_{\mathbb{T}}}u_1(t)<b'$$

and

$$b'<\max_{t\in[0,\eta]_{\mathbb{T}}}u_2(t)\ \textit{with}\ \max_{t\in[\eta,r]_{\mathbb{T}}}u_2(t)<c'.$$

Proof. By the definition of completely continuous operator A and its properties, it suffices to show that all the conditions of Lemma 2.5 hold with respect to A. From above analysis, it remains to show that (i)-(iii) of Lemma 2.5 hold.

In the first step, we verify that if $u\in\partial P(\gamma,c')$, then $\gamma(Au)>c'$.

If $u\in\partial P(\gamma,c')$, then

$$\gamma(u)=\min_{t\in[\eta,r]_{\mathbb{T}}}u(t)=u(\eta)=c'.$$

Lemma 3.1 implies that

$$\|u\|\le\frac{\sigma(T)}{\eta}u(\eta)=\frac{\sigma(T)}{\eta}c',$$

we have

$$c'\le u(t)\le\frac{\sigma(T)}{\eta}c',\ t\in[\eta,\sigma(T)]_{\mathbb{T}}.$$

Thus, by the condition (i) of Theorem 3.2, one has

$$\gamma(Au)=Au(\eta)=$$

$$B_0\left(\sum_{i=1}^{m-2} a_i\varphi_q\left(\int_{\xi_i}^{T} h(s)f\left(s,u(s)\right)\nabla s\right)\right)+\int_0^{\eta}\varphi_q\left(\int_{\tau}^{T} h(s)f\left(s,u(s)\right)\nabla s\right)\Delta\tau$$

$$\geq\left(A'\sum_{i=1}^{m-2} a_i+\eta\right)\varphi_q\left(\int_{\eta}^{T} h(s)f\left(s,u(s)\right)\nabla s\right)$$

$$>\left(A'\sum_{i=1}^{m-2} a_i+\eta\right)\varphi_q\left(\int_{\eta}^{T} h(s)\varphi_p\left(\frac{c'}{M}\right)\nabla s\right)$$

$$=\frac{c'}{M}\left(A'\sum_{i=1}^{m-2} a_i+\eta\right)\varphi_q\left(\int_{\eta}^{T} h(s)\nabla s\right)=c'.$$

In the second step, we show that $\theta(Au)<b'$ for $u\in\partial P(\theta,b')$.

If we choose $u\in\partial P(\theta,b')$, then

$$\theta(u)=\max_{t\in[0,\eta]_{\mathbb{T}}} u(t)=u(\eta)=b',$$

which implies

$$0\leq u(t)\leq b' \text{ for } t\in[0,\eta]_{\mathbb{T}}.$$

In view of Lemma 3.1, we have

$$\|u\|\leq\frac{\sigma(T)}{\eta}u(\eta)=\frac{\sigma(T)}{\eta}b'.$$

So

$$0\leq u(t)\leq\tfrac{\sigma(T)}{\eta}b',\ t\in[0,\sigma(T)]_{\mathbb{T}}.$$

Using (ii) in Theorem 3.2, we obtain

$$\theta(Au)=(Au)(\eta)=$$

$$B_0\left(\sum_{i=1}^{m-2} a_i\varphi_q\left(\int_{\xi_i}^{T} h(s)f\left(s,u(s)\right)\nabla s\right)\right)+\int_0^{\eta}\varphi_q\left(\int_{\tau}^{T} h(s)f\left(s,u(s)\right)\nabla s\right)\Delta\tau$$

$$<\left(B\sum_{i=1}^{m-2} a_i+\eta\right)\varphi_q\left(\int_{0}^{T} h(s)\varphi_p\left(\frac{b'}{N}\right)\nabla s\right)$$

$$=\frac{b'}{N}\left(B\sum_{i=1}^{m-2} a_i+\eta\right)\varphi_q\left(\int_{0}^{T} h(s)\nabla s\right)=b'.$$

Finally, we prove that

$$P(\alpha,a')\neq\emptyset$$

and

$$\alpha(Au) > a' \text{ for all } u \in \partial P(\alpha, a').$$

In fact, the constant function $\frac{a'}{2} \in P(\alpha, a')$. Moreover, for $u \in \partial P(\alpha, a')$, we have

$$\alpha(u) = \max_{t\in[0,r]_{\mathbb{T}}} u(t) = u(r) = a',$$

which implies

$$0 \le u(t) \le a' \text{ for } t \in [0, r]_{\mathbb{T}}.$$

In view of Lemma 3.1, we have

$$u(t) \le \|u\| \le \frac{\sigma(T)}{r} u(r) = \frac{\sigma(T)}{r} r'.$$

Hence

$$a' \le u(t) \le \frac{\sigma(T)}{r} a', \ t \in [r, \sigma(T)]_{\mathbb{T}}.$$

Using assumption (iii) in Theorem 3.2, one has

$$\begin{aligned} &\alpha(Au) = (Au)(r) = \\ &B_0 \left(\sum_{i=1}^{m-2} a_i \varphi_q \left(\int_{\xi_i}^{T} h(s) f(s, u(s)) \nabla s \right) \right) + \int_0^r \varphi_q \left(\int_\tau^T h(s) f(s, u(s)) \nabla s \right) \Delta\tau \\ &\qquad > \left(A' \sum_{i=1}^{m-2} a_i + r \right) \varphi_q \left(\int_r^T h(s) \varphi_p \left(\frac{a'}{L} \right) \nabla s \right) \\ &\qquad = \frac{a'}{L} \left(A' \sum_{i=1}^{m-2} a_i + r \right) \varphi_q \left(\int_r^T h(s) \nabla s \right) = a'. \end{aligned}$$

Thus, all the conditions in Lemma 2.5 are satisfied. Consequently, problem (1.5) and (1.6) has at least twin positive solutions u_1 and u_2, belonging to $\overline{P(\gamma, c')}$, and satisfing

$$a' < \max_{t\in[0,r]_{\mathbb{T}}} u_1(t) \text{ with } \max_{t\in[0,\eta]_{\mathbb{T}}} u_1(t) < b'$$

and

$$b' < \max_{t\in[0,\eta]_{\mathbb{T}}} u_2(t) \text{ with } \max_{t\in[\eta,r]_{\mathbb{T}}} u_2(t) < c'.$$

The proof is complete. □

From Theorem 3.2, it is difficulty for us to discuss examples under the condition. The following corollary replaces it with an easy verified condition.

Let

$$f_0 = \lim_{u\to 0^+} \frac{f(t,u)}{\varphi_p(u)} \text{ for } t \in [0,\sigma(T)]_{\mathbb{T}},$$

and

$$f_\infty = \lim_{u\to \infty} \frac{f(t,u)}{\varphi_p(u)} \text{ for } t \in [0,\sigma(T)]_{\mathbb{T}}.$$

Corollary 3.3. *Assume that f satisfies conditions:*

(i) $f_0 = \infty$, $f_\infty = \infty$;

(ii) *there exists $u_0 > 0$ such that*

$$f(t,u) < \varphi_p\left(\tfrac{1}{N}\right)\varphi_p\left(\tfrac{\eta}{\sigma(T)}u_0\right) \textit{ for } t \in [0,\sigma(T)]_{\mathbb{T}} \textit{ and } 0 \le u \le u_0.$$

Then problem (1.5) *and* (1.6) *has at least twin positive solutions.*

Proof. First, by the condition (ii), let $b' = \frac{\eta}{\sigma(T)}u_0$, one gets

$$f(t,u) < \varphi_p\left(\frac{1}{N}\right)\varphi_p(b') = \varphi_p\left(\frac{b'}{N}\right) \text{ for } t \in [0,\sigma(T)]_{\mathbb{T}} \text{ and } 0 \le u \le \frac{\sigma(T)}{\eta}b',$$

which implies that (ii) of Theorem 3.2 holds.

Second, choose K_3 sufficiently large satisfied

$$K_3L = K_3\left(A'\sum_{i=1}^{m-2} a_i + r\right)\varphi_q\left(\int_r^T h(s)\nabla s\right) > 1. \tag{3.1}$$

Since $f_0 = \infty$, there exists $r_1 > 0$ sufficiently small such that,

$$f(t,u) \ge \varphi_p(K_3)\varphi_p(u) = \varphi_p(K_3u) \text{ for } t \in [0,\sigma(T)]_{\mathbb{T}} \text{ and } 0 \le u \le r_1. \tag{3.2}$$

Without loss of generality, suppose $r_1 \le \frac{L\sigma(T)}{Nr}b'$. Choose $a' > 0$ such that $a' < \frac{r}{\sigma(T)}r_1$. For $a' \le u \le \frac{\sigma(T)}{r}a'$, we have

$$u \le r_1 \text{ and } a' < \frac{L}{N}b'.$$

Thus, by (3.1) and (3.2), we have

$$f(t,u) \ge \varphi_p(K_3u) \ge \varphi_p(K_3a') > \varphi_p\left(\frac{a'}{L}\right) \text{ for } t \in [0,\sigma(T)]_{\mathbb{T}}$$

$$\text{and } a' \le u \le \frac{\sigma(T)}{r}a',$$

this implies that (iii) of Theorem 3.2 is true.

Third, choose K_2 sufficiently large such that

$$K_2M = K_2\left(A'\sum_{i=1}^{m-2} a_i + \eta\right)\varphi_q\left(\int_\eta^T h(s)\nabla s\right) > 1.$$

Since $f_\infty = \infty$, there exists $r_2 > 0$ sufficiently large such that

$$f(t,u) \geq \varphi_p(K_2)\varphi_p(u) = \varphi_p(K_2 u) \text{ for } t \in [0,\sigma(T)]_{\mathbb{T}} \text{ and } u \geq r_2.$$

Without loss of generality, suppose $r_2 > \frac{\sigma(T)}{\eta} b'$. Choose $c' = r_2$, then

$$f(t,u) \geq \varphi_p(K_2 u) \geq \varphi_p(K_2 c') > \varphi_p(\tfrac{c'}{M}) \text{ for } t \in [0,\sigma(T)]_{\mathbb{T}} \text{ and } c' \leq u \leq \tfrac{\sigma(T)}{\eta} c',$$

which means that (i) of Theorem 3.2 holds.

From above analysis, we get

$$0 < a' < \tfrac{L}{N} b' < \tfrac{\eta L}{\sigma(T) N} c',$$

then, all conditions in Theorem 3.2 are satisfied. Hence, problem (1.5) and (1.6) has at least twin positive solutions. □

The following result gives the existence result of arbitrary even positive solutions of problem (1.5) and (1.6).

Theorem 3.4. *Let $i = 1, 2, \ldots, n, n \in \mathbb{N}$. Suppose that there are positive numbers a_i', b_i', c_i' such that*

$$0 < a_1' < \tfrac{L}{N} b_1' < \tfrac{\eta L}{\sigma(T)N} c_1' < a_2' < \tfrac{L}{N} b_2' < \tfrac{\eta L}{\sigma(T)N} c_2' < \ldots < a_n' < \tfrac{L}{N} b_n' < \tfrac{\eta L}{\sigma(T)N} c_n'.$$

In addition, $f(t,u)$ satisfies the following conditions:

(i) $f(t,u) > \varphi_p\left(\frac{c_i'}{M}\right)$ *for* $t \in [\eta, \sigma(T)]_{\mathbb{T}}$ *and* $c_i' \leq u \leq \frac{\sigma(T)}{\eta} c_i'$;

(ii) $f(t,u) < \varphi_p\left(\frac{b_i'}{N}\right)$ *for* $t \in [0, \sigma(T)]_{\mathbb{T}}$ *and* $0 \leq u \leq \frac{\sigma(T)}{\eta} b_i'$;

(iii) $f(t,u) > \varphi_p\left(\frac{a_i'}{L}\right)$ *for* $t \in [r, \sigma(T)]_{\mathbb{T}}$ *and* $a_i' \leq u \leq \frac{\sigma(T)}{r} a_i'$.

Then problem (1.5) *and* (1.6) *has at least* $2n$ *positive solutions.*

Proof. When $i = 1$, it is clear that Theorem 3.2 holds. Then we can obtain problem (1.5) and (1.6) has at least twin positive solutions u_1 and u_2, belonging to $\overline{P(\gamma, c')}$, and satisfing

$$a' < \max_{t \in [0,r]_{\mathbb{T}}} u_1(t) \text{ with } \max_{t \in [0,\eta]_{\mathbb{T}}} u_1(t) < b'$$

and

$$b' < \max_{t \in [0,\eta]_{\mathbb{T}}} u_2(t) \text{ with } \max_{t \in [\eta,r]_{\mathbb{T}}} u_2(t) < c'.$$

Following this way, we finish the proof by induction. □

Now, by using Lemma 2.6, we shall consider problem (1.5) and (1.6).

For convenience, we denote M', N' and L' by

$$M' = \left(B \sum_{i=1}^{m-2} a_i + \eta\right) \varphi_q\left(\int_0^T h(s)\nabla s\right),$$

$$N' = \left(A' \sum_{i=1}^{m-2} a_i + \eta \right) \varphi_q \left(\int_\eta^T h(s) \nabla s \right),$$

and

$$L' = \left(B \sum_{i=1}^{m-2} a_i + r \right) \varphi_q \left(\int_0^T h(s) \nabla s \right).$$

In view of Lemma 2.6, the following can be obtained by using the similar proof to that of Theorem 3.2 and we omit it here.

Theorem 3.5. *Assume that there are positive numbers a', b', c' such that $0 < a' < \frac{r}{\sigma(T)} b' < \frac{rN'}{\sigma(T)M'} c'$. Suppose $f(t,u)$ satisfies the following conditions:*

(i) $f(t,u) < \varphi_p \left(\frac{c'}{M'} \right)$ *for* $t \in [0, \sigma(T)]_{\mathbb{T}}$ *and* $0 \le u \le \frac{\sigma(T)}{\eta} c'$;

(ii) $f(t,u) > \varphi_p \left(\frac{b'}{N'} \right)$ *for* $t \in [\eta, \sigma(T)]_{\mathbb{T}}$ *and* $b' \le u \le \frac{\sigma(T)}{\eta} b'$;

(iii) $f(t,u) < \varphi_p \left(\frac{a'}{L'} \right)$ *for* $t \in [0, \sigma(T)]_{\mathbb{T}}$ *and* $0 \le u \le \frac{\sigma(T)}{r} a'$.

Then problem (1.5) *and* (1.6) *has at least twin positive solutions u_1 and u_2 such that*

$$a' < \max_{t \in [0,r]_{\mathbb{T}}} u_1(t) \text{ with } \max_{t \in [0,\eta]_{\mathbb{T}}} u_1(t) < b'$$

and

$$b' < \max_{t \in [0,\eta]_{\mathbb{T}}} u_2(t) \text{ with } \max_{t \in [\eta,r]_{\mathbb{T}}} u_2(t) < c'.$$

Corollary 3.6. *Assume that f satisfies conditions:*

(i) $f_0 = 0,\ f_\infty = 0$;

(ii) *there exists $u_0 > 0$ such that*

$f(t,u) > \varphi_p \left(\frac{1}{N'} \right) \varphi_p \left(\frac{\eta}{\sigma(T)} u_0 \right)$ *for* $t \in [\eta, \sigma(T)]_{\mathbb{T}}$ *and* $\frac{\eta}{\sigma(T)} u_0 \le u \le u_0$.

Then problem (1.5) *and* (1.6) *has at least twin positive solutions.*

Proof. In terms of Theorem 3.5, essentially the same reasoning as Corollary 3.3, the proof is easy and we omitted it. □

The following theorem can be obtained by using the similar way as in Theorem 3.4.

Theorem 3.7. *Let $i = 1, 2, \ldots, n, n \in \mathbb{N}$. Assume that there are positive numbers a'_i, b'_i, c'_i such that*

$$0 < a'_1 < \frac{r}{\sigma(T)} b'_1 < \frac{rN'}{\sigma(T)M'} c'_1 < a'_2 < \frac{r}{\sigma(T)} b'_2 < \frac{rN'}{\sigma(T)M'} c'_2 < \ldots$$

$$< a'_n < \frac{r}{\sigma(T)} b'_n < \frac{rN'}{\sigma(T)M'} c'_n.$$

Suppose $f(t,u)$ satisfies the following conditions:

(i) $f(t,u) < \varphi_p\left(\frac{c'_1}{M'}\right)$ *for* $t \in [0, \sigma(T)]_{\mathbb{T}}$ *and* $0 \le u \le \frac{\sigma(T)}{\eta} c'_i$;

(ii) $f(t,u) > \varphi_p\left(\frac{b'_i}{N'}\right)$ *for* $t \in [\eta, \sigma(T)]_{\mathbb{T}}$ *and* $b'_i \le u \le \frac{\sigma(T)}{\eta} b'_i$;

(iii) $f(t,u) < \varphi_p\left(\frac{a'_i}{L'}\right)$ *for* $t \in [0, \sigma(T)]_{\mathbb{T}}$ *and* $0 \le u \le \frac{\sigma(T)}{r} a'_i$.

Then problem (1.5) *and* (1.6) *has at least* $2n$ *positive solutions.*

4. Existence Results of Problems (1.5) and (1.7)

In this section, we consider positive solutions to problem (1.5) and (1.7).

Define the cone $P_1 \subset E$ by

$$P = \left\{ \begin{array}{c} u \in E : u(t) \ge 0 \text{ for } t \in [0, \sigma(T)]_{\mathbb{T}} \text{ and} \\ u^{\Delta\Delta}(t) \le 0, u^{\Delta}(t) \ge 0 \text{ for } t \in [0,T]_{\mathbb{T}},\ u^{\Delta}(0) = 0 \end{array} \right\}.$$

The following lemma is needed in our proof.

Lemma 4.1. *If $u \in P_1$, then:*

(i) $u(t) \ge \frac{\sigma(T)-t}{\sigma(T)} \|u\| = \frac{\sigma(T)-t}{\sigma(T)} u(0)$ *for* $t \in [0, \sigma(T)]_{\mathbb{T}}$;

(ii) $(\sigma(T)-t)u(s) \le (\sigma(T)-s)u(t)$ *for* $s,t \in [0,\sigma(T)]_{\mathbb{T}}$ *and* $s \le t$.

Clearly, $\|u\| = u(0)$ for $u \in P_1$. Define the operator $A_1 : P_1 \to E$ by

$$A_1 u(t) = B_1\left(\sum_{i=1}^{m-2} b_i \varphi_q\left(\int_0^{\xi'_i} h(s) f(s, u(s)) \nabla s\right)\right) + \int_t^{\sigma(T)} \varphi_q\left(\int_0^{\tau} h(s) f(s,u(s)) \nabla s\right) \Delta\tau.$$

By the similar way, we can obtain that $A_1 : P_1 \to P_1$ is completely continuous and every fixed point of A is a solution of the problem (1.5) and (1.7).

For $u \in P_1$, we define the nonnegative, increasing, continuous functionals γ_1, θ_1 and α_1 by

$$\gamma_1(u) = \min_{t \in [0,\xi]_{\mathbb{T}}} u(t) = u(\xi),$$

$$\theta_1(u) = \max_{t \in [\xi, \sigma(T)]_{\mathbb{T}}} u(t) = u(\xi),$$

and

$$\alpha_1(u) = \max_{t\in[l,\sigma(T)]_{\mathbb{T}}} u(t) = u(l).$$

It is obvious that

$$\gamma_1(u) = \theta_1(u) \le \alpha_1(u) \text{ for each } u \in P_1.$$

In terms of Lemma 4.1, one has

$$\|u\| \le \frac{\sigma(T)u(\xi)}{\sigma(T)-\xi} = \frac{\sigma(T)}{\sigma(T)-\xi}\gamma_1(u) \text{ for all } u \in P_1.$$

In addition, we also note that

$$\theta_1(\lambda u) = \lambda\theta_1(u),\ 0 \le \lambda \le 1 \text{ and } u \in \partial P(\theta, b').$$

For the notational convenience, we denote

$$M_1 = \left(A'\sum_{i=1}^{m-2} b_i + \sigma(T) - \xi\right)\varphi_q\left(\int_0^{\xi} h(s)\nabla s\right),$$

$$N_1 = \left(B\sum_{i=1}^{m-2} b_i + \sigma(T) - \xi\right)\varphi_q\left(\int_0^{T} h(s)\nabla s\right),$$

and

$$L_1 = \left(A'\sum_{i=1}^{m-2} b_i + \sigma(T) - l\right)\varphi_q\left(\int_0^{l} h(s)\nabla s\right).$$

In terms of operator A_1, essentially the same reasoning as above theorems and lemmas of problem (1.5) and (1.6), we have the following results.

Theorem 4.2. *Suppose that there are positive numbers a_1', b_1', c_1' such that*

$$0 < a_1' < \frac{L_1}{N_1}b_1' < \frac{(\sigma(T)-\xi)L_1}{\sigma(T)N_1}c_1'.$$

In addition, $f(t,u)$ satisfies the following conditions:

(i) $f(t,u) > \varphi_p\left(\frac{c_1'}{M_1}\right)$ *for $t \in [0,\xi]_{\mathbb{T}}$ and $c_1' \le u \le \frac{\sigma(T)}{\sigma(T)-\xi}c_1'$,*

(ii) $f(t,u) < \varphi_p\left(\frac{b_1'}{N_1}\right)$ *for $t \in [0,\sigma(T)]_{\mathbb{T}}$ and $0 \le u \le \frac{\sigma(T)}{\sigma(T)-\xi}b_1'$,*

(iii) $f(t,u) > \varphi_p\left(\frac{a_1'}{L_1}\right)$ *for $t \in [l,\sigma(T)]_{\mathbb{T}}$ and $a_1' \le u \le \frac{\sigma(T)}{\sigma(T)-l}a_1'$.*

Then problem (1.5) *and* (1.7) *has at least twin positive solutions u_1 and u_2 such that*

$$a_1' < \max_{t\in[l,\sigma(T)]_{\mathbb{T}}} u_1(t) \textit{ with } \max_{t\in[\xi,\sigma(T)]_{\mathbb{T}}} u_1(t) < b_1'$$

and

$$b_1' < \max_{t\in[\xi,\sigma(T)]_{\mathbb{T}}} u_2(t) \text{ with } \max_{t\in[0,\xi]_{\mathbb{T}}} u_2(t) < c_1'.$$

Corollary 4.3. *Assume that f satisfies conditions:*

(i) $f_0 = \infty,\ f_\infty = \infty$;

(ii) *there exists $u_0 > 0$ such that*

$$f(t,u) < \varphi_p\left(\frac{1}{N_1}\right)\varphi_p\left(\frac{\sigma(T)-\xi}{\sigma(T)}u_0\right) \text{ for } t \in [0,\sigma(T)]_{\mathbb{T}} \text{ and } 0 \le u \le u_0.$$

Then problem (1.5) *and* (1.7) *has at least twin positive solutions.*

Theorem 4.4. *Let $i = 1,2,\dots,n, n \in \mathbb{N}$. Suppose that there are positive numbers a_i', b_i', c_i' such that*

$$0 < a_1' < \frac{L_1}{N_1}b_1' < \frac{(\sigma(T)-\xi)L_1}{\sigma(T)N_1}c_1' < a_2' < \frac{L_1}{N_1}b_2' < \frac{(\sigma(T)-\xi)L_1}{\sigma(T)N_1}c_2' < \dots < a_n' < \frac{L_1}{N_1}b_n' < \frac{(\sigma(T)-\xi)L_1}{\sigma(T)N_1}c_n'.$$

In addition, $f(t,u)$ satisfies the following conditions:

(i) $f(t,u) > \varphi_p\left(\frac{c_i'}{M_1}\right)$ *for* $t \in [0,\xi]_{\mathbb{T}}$ *and* $c_i' \le u \le \frac{\sigma(T)}{\sigma(T)-\xi}c_i'$,

(ii) $f(t,u) < \varphi_p\left(\frac{b_i'}{N_1}\right)$ *for* $t \in [0,\sigma(T)]_{\mathbb{T}}$ *and* $0 \le u \le \frac{\sigma(T)}{\sigma(T)-\xi}b_i'$,

(iii) $f(t,u) > \varphi_p\left(\frac{a_i'}{L_1}\right)$ *for* $t \in [l,\sigma(T)]_{\mathbb{T}}$ *and* $a_i' \le u \le \frac{\sigma(T)}{\sigma(T)-l}a_i'$.

Then problem (1.5) *and* (1.7) *has at least $2n$ positive solutions.*

Now, we consider the problem (1.5) and (1.7) in view of Lemma 2.6. For convenience, we denote M', N' and L' by

$$M_1' = \left(B\sum_{i=1}^{m-2} b_i + \sigma(T) - \xi\right)\varphi_q\left(\int_0^T h(s)\nabla s\right),$$

$$N_1' = \left(A'\sum_{i=1}^{m-2} b_i + \sigma(T) - \xi\right)\varphi_q\left(\int_\xi^T h(s)\nabla s\right),$$

and

$$L_1' = \left(B\sum_{i=1}^{m-2} b_i + \sigma(T) - l\right)\varphi_q\left(\int_0^T h(s)\nabla s\right).$$

Theorem 4.5. *Assume that there are positive numbers* a_1', b_1', c_1' *such that*

$$0 < a_1' < \frac{\sigma(T) - L_1'}{\sigma(T)} b_1' < \frac{(\sigma(T) - L_1')N_1'}{\sigma(T)M_1'} c_1'.$$

Suppose $f(t,u)$ *satisfies the following conditions:*

(i) $f(t,u) < \varphi_p\left(\frac{c_1'}{M_1'}\right)$ *for* $t \in [0, \sigma(T)]_{\mathbb{T}}$ *and* $0 \le u \le \frac{\sigma(T)}{\sigma(T)-\xi} c_1'$,

(ii) $f(t,u) > \varphi_p\left(\frac{b_1}{N_1'}\right)$ *for* $t \in [\xi, \sigma(T)]_{\mathbb{T}}$ *and* $b_1' \le u \le \frac{\sigma(T)}{\sigma(T)-\xi} b_1'$,

(iii) $f(t,u) < \varphi_p\left(\frac{a_1'}{L_1'}\right)$ *for* $t \in [0, \sigma(T)]_{\mathbb{T}}$ *and* $0 \le u \le \frac{\sigma(T)}{\sigma(T)-l} a_1'$.

Then problem (1.5) *and* (1.7) *has at least twin positive solutions* u_1 *and* u_2 *such that*

$$a_1' < \max_{t \in [l, \sigma(T)]_{\mathbb{T}}} u_1(t) \text{ with } \max_{t \in [\xi, \sigma(T)]_{\mathbb{T}}} u_1(t) < b_1'$$

and

$$b_1' < \max_{t \in [\xi, \sigma(T)]_{\mathbb{T}}} u_2(t) \text{ with } \max_{t \in [0, \xi]_{\mathbb{T}}} u_2(t) < c_1'.$$

Corollary 4.6. *Assume that* f *satisfies conditions:*

(i) $f_0 = 0,\ f_\infty = 0$;

(ii) *there exists* $u_0 > 0$ *such that*

$$f(t,u) > \varphi_p\left(\frac{1}{N_1'}\right)\varphi_p\left(\frac{\sigma(T)-\xi}{\sigma(T)} u_0\right) \text{ for } t \in [\xi, \sigma(T)]_{\mathbb{T}} \text{ and } \frac{\sigma(T)-\xi}{\sigma(T)} u_0 \le u \le u_0.$$

Then problem (1.5) *and* (1.7) *has at least twin positive solutions.*

Theorem 4.7. *Let* $i = 1, 2, \ldots, n, n \in \mathbb{N}$. *Assume that there are positive numbers* a_i', b_i', c_i' *such that*

$$a_1' < \frac{\sigma(T) - L_1'}{\sigma(T)} b_1' < \frac{(\sigma(T) - L_1')N_1' c_1'}{\sigma(T)M_1'} < a_2' < \frac{\sigma(T) - L_1'}{\sigma(T)} b_2'$$
$$< \frac{(\sigma(T) - L_1')N_1' c_2'}{\sigma(T)M_1'} < \ldots < a_n' < \frac{\sigma(T) - L_1'}{\sigma(T)} b_n' < \frac{(\sigma(T) - L_1')N_1' c_n'}{\sigma(T)M_1'}.$$

Suppose $f(t,u)$ *satisfies the following conditions:*

(i) $f(t,u) < \varphi_p\left(\frac{c_i'}{M_1'}\right)$ *for* $t \in [0, \sigma(T)]_{\mathbb{T}}$ *and* $0 \le u \le \frac{\sigma(T)}{\sigma(T)-\xi} c_i'$,

(ii) $f(t,u) > \varphi_p\left(\frac{b_i}{N_1'}\right)$ *for* $t \in [\xi, \sigma(T)]_{\mathbb{T}}$ *and* $b_i' \le u \le \frac{\sigma(T)}{\sigma(T)-\xi} b_i'$,

(iii) $f(t,u) < \varphi_p\left(\frac{a_i'}{L_1'}\right)$ *for* $t \in [0, \sigma(T)]_{\mathbb{T}}$ *and* $0 \le u \le \frac{\sigma(T)}{\sigma(T)-l} a_i'$.

Then problem (1.5) *and* (1.7) *has at least* $2n$ *positive solutions.*

5. Examples

In this section, we present two simple examples to explain our results.

Example 5.1. Let

$$\mathbb{T} = \left\{2 - \left(\tfrac{1}{3}\right)^{\mathbb{N}_0}\right\} \cup \left\{0, \tfrac{1}{8}, \tfrac{1}{4}, \tfrac{1}{6}, \tfrac{1}{2}, 1, \tfrac{5}{4}, \tfrac{3}{2}, \tfrac{7}{4}, 2\right\} \cup \left[\tfrac{1}{20}, \tfrac{1}{10}\right]$$

and $T = 2$.

Consider the following boundary value problem with $p = 7$ and $k \in \mathbb{N}_0$.

$$\begin{gathered}\left(\varphi_p\left(u^{\Delta}(t)\right)\right)^{\nabla} + \left\{\sum_{k=0}^{7} t^k(\rho(t))^{7-k}\right\} t^{\nabla} f(t, u(t)) = 0,\ t \in [0,2]_{\mathbb{T}}, \\ u(0) - 4\left(u^{\Delta}\left(\tfrac{1}{4}\right) + u^{\Delta}(1)\right) = 0,\ u^{\Delta}(2) = 0,\end{gathered} \tag{5.1}$$

where

$$f(t,u) = \begin{cases} t + 2, & t \in [0,2]_{\mathbb{T}} \text{ and } 0 \le u \le 80, \\ t + p(u), & t \in [0,2]_{\mathbb{T}} \text{ and } 80 \le u \le 200, \\ t + 5 \times 10^5, & t \in [0,2]_{\mathbb{T}} \text{ and } 200 \le u \le 400, \\ t + s(u), & t \in [0,2]_{\mathbb{T}} \text{ and } u \ge 800, \end{cases}$$

where $p(u)$ and $s(u)$ satisfy

$$p(80) = 2,\ p(200) = 5 \times 10^5,\ s(400) = 5 \times 10^5,$$

and

$$p(u), s(u) : \mathbb{R} \to \mathbb{R}^{+} \text{is continuous.}$$

If

$$a_1(t) = \left\{\sum_{k=0}^{7} t^k(\rho(t))^{7-k}\right\} t^{\nabla},$$

then by Exercise 1.23 in [6], we have

$$(t^8)^{\nabla} = \left\{\sum_{k=0}^{7} t^k(\rho(t))^{7-k}\right\} t^{\nabla}.$$

It is obvious that $\xi_2 = \eta = 1$, $A = B = 4$ and $a_1 = a_2 = 1$. Choose $r = \frac{3}{2}$, a direct calculation shows that

$$M = 9\left(\int_1^2 \left\{\sum_{k=0}^{7} t^k(\rho(t))^{7-k}\right\} t^{\nabla}\nabla t\right)^{\frac{1}{6}} \approx 22.66,$$

$$N = 9\left(\int_0^2 \left\{\sum_{k=0}^{7} t^k(\rho(t))^{7-k}\right\} t^{\nabla}\nabla t\right)^{\frac{1}{6}} \approx 22.68,$$

and

$$L = 9.5\left(\int_{\frac{3}{2}}^{2}\left\{\sum_{k=0}^{7} t^k(\rho(t))^{7-k}\right\} t^{\nabla}\nabla t\right)^{\frac{1}{6}} \approx 23.52.$$

If we take $a' = 2$, $b' = 40$, $c' = 200$, then

$$0 < a' < \frac{L}{N}b' < \frac{\eta L}{\sigma(T)N}c',$$

$$f(t,u) = t + 2 < 30.10 \approx \varphi_p\left(\frac{b'}{N}\right) \text{ for } t \in [0,2]_{\mathbb{T}} \text{ and } 0 \le u \le \frac{\sigma(T)b'}{\eta} = 80,$$

$$f(t,u) = t + 5 \times 10^5 > 4.73 \times 10^5 \approx \varphi_p\left(\frac{c'}{M}\right)$$

$$\text{for } t \in [1,2]_{\mathbb{T}} \text{ and } 200 \le u \le \frac{\sigma(T)c'}{\eta} = 400,$$

$$f(t,u) = t + 2 > 3.78 \times 10^{-7} \approx \varphi_p\left(\frac{a'}{L}\right)$$

$$\text{for } t \in [1,2]_{\mathbb{T}} \text{ and } 2 \le u \le \frac{\sigma(T)a'}{r} = 2.667.$$

Therefore, all the conditions of Theorem 3.2 are satisfied. By Theorem 3.2, we see that the boundary value problem (5.1) has at least twin positive solutions u_1 and u_2 such that

$$2 < \max_{t\in[0,\frac{3}{2}]_{\mathbb{T}}} u_1(t) \text{ with } \max_{t\in[0,1]_{\mathbb{T}}} u_1(t) < 40$$

and

$$40 < \max_{t\in[0,1]_{\mathbb{T}}} u_2(t) \text{ with } \max_{t\in[1,\frac{3}{2}]_{\mathbb{T}}} u_2(t) < 200.$$

Example 5.2. Let

$$\mathbb{T} = \left\{2 - \left(\tfrac{1}{3}\right)^{\mathbb{N}_0}\right\} \cup \left\{0, \tfrac{1}{8}, \tfrac{1}{4}, \tfrac{1}{6}, \tfrac{1}{2}, 1, \tfrac{5}{4}, \tfrac{3}{2}, \tfrac{7}{4}, 2\right\}$$

and $T = 2$.

Consider the following boundary value problem

$$\begin{aligned} &(\varphi_p(u^{\triangle}(t)))^{\nabla} + h(t)f(t,u(t)) = 0, \ t \in [0,2]_{\mathbb{T}}, \\ &u(0) - 4\left(u^{\Delta}\left(\tfrac{1}{4}\right) + u^{\Delta}(1)\right) = 0, \ u^{\Delta}(1) = 0, \end{aligned} \tag{5.2}$$

where

$$h(t) = t + \rho(t) \text{ for } t \in [0,2]_{\mathbb{T}},$$

and

$$f(t,u(t)) = (t+\rho(t))|u(t)|^{p-2}u^2 + |u|^{p-2} \text{ for } t \in [0,2]_{\mathbb{T}}\,.$$

It is obvious that $\xi_2 = \eta = 1$, $A = B = 4$ and $a_1 = a_2 = 1$. Choose $r = \frac{3}{2}$, a direct calculation shows that

$$N = \left(B\sum_{i=1}^{m-2} a_i + \eta\right)\varphi_q\left(\int_0^T h(s)\nabla s\right) = 9\varphi_q\left(\int_0^2 t+\rho(t)\nabla s\right) = 9\varphi_q(4),$$

there exists

$$u_0 = 20\varphi_q(4)\varphi_q\left((t+\rho(t))|u(t)|^{p-2}u^2 + |u|^{p-2}\right) > 0$$

such that

$$f(t,u) < \varphi_p\left(\frac{1}{9\varphi_q(4)}\right)\varphi_p\left(\frac{1}{2}u_0\right) \text{ for } t \in [0,2]_{\mathbb{T}} \text{ and } 0 \le u \le u_0.$$

Hence, the condition (ii) of Corollary 3.3 holds.

Note that

$$f_0 = \lim_{u\to 0^+} \frac{(t+\rho(t))|u(t)|^{p-2}u^2 + |u|^{p-2}}{|u|^{p-2}u} = \infty \text{ for } t \in [0,2]_{\mathbb{T}}\,,$$

and

$$f_\infty = \lim_{u\to\infty} \frac{(t+\rho(t))|u(t)|^{p-2}u^2 + |u|^{p-2}u}{|u|^{p-2}u} = \infty \text{ for } t \in [0,2]_{\mathbb{T}}\,.$$

Hence, Corollary 3.3 implies that the boundary value problem (5.2) has at least twin solutions.

Acknowledgments

This work is supported by Fund of Education Ministry of Gansu Province (0809B-04).

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