International Journal of Pure and Applied Mathematics
Volume 63 No. 2 2010, 129-136

THREE POSITIVE SOLUTIONS OF NONLINEAR FIRST-ORDER BOUNDARY VALUE PROBLEMS ON TIME SCALES

Wen Shu[1 §], Deng Chunhua[2]
[1,2]Faculty of Mathematics and Physics
Huaiyin Institute of Technology
Huaiyin, Jiangsu, 223003, P.R. CHINA
[1]e-mail: wenshu119@163.com

Abstract: In this paper, existence criteria of three positive solutions to the following nonlinear first-order boundary value problem on time scales

$$\begin{cases} x^{\triangle}(t) = f(x(\sigma(t))), \quad t \in [0,T]_{\mathbf{T}}, \\ x(0) = \eta x(\sigma(T)), \end{cases}$$

are established by using the fixed-point theorem due to Ren et al [18]. An example is also given to illustrate the main result.

AMS Subject Classification: 26A33
Key Words: time scale, boundary value problem, positive solution, fixed point

1. Introduction

Let $\mathbf{T}$ be a time scale, i.e., $\mathbf{T}$ is a nonempty closed subset of R. Let 0, T be points in $\mathbf{T}$, an interval $(0,T)_{\mathbf{T}}$ denoting time scales interval, that is, $(0,T)_{\mathbf{T}} := (0,T) \cap \mathbf{T}$. Other types of intervals are defined similarly. The theory of dynamic equations on time scales has been a new important mathematical branch (see, for example, [1, 2, 9, 10, 16]) since it was initiated by Hilger [15]. At the same time, boundary value problems (BVPs) for dynamic equation on time scales

Received: May 20, 2010

§Correspondence author

have received considerable attention [3-7, 11-14, 17, 19-20].

In this paper, we are concerned with the existence of positive solutions for the following nonlinear first-order boundary value problem on time scale

$$\begin{cases} x^{\triangle}(t) = f(x(\sigma(t))), \ t \in [0,T]_{\mathbf{T}}, \\ x(0) = \eta x(\sigma(T)), \end{cases} \tag{1.1}$$

where $T > 0$ is fixed, $0, T \in \mathbf{T}$ and $f : [0,\infty) \to [0,\infty)$ is continuous.

We note that in [20], Sun considered the problem (1.1), by using a fixed-point theorem due to Avery et al [8], and obtained the existence of twin positive solutions.

In paper [18], Ren, Ge and Ren studied the problem $[\Phi_p(u'(t))]' + a(t)f(u) = 0$, $t \in (0,1)$, with boundary conditions $u'(0) = 0$, $u(1) + B_1(u'(\xi)) = 0$ or $u(0) - B_1(u'(\xi)) = 0$, $u'(1) = 0$, where $\xi \in (0,1)$. They obtained sufficient conditions for the existence of at least three positive solutions by applying the new fixed-point theorem due to theirs.

Motivated by [18] and [20], we shall show that the problem (1.1) has at least three positive solutions by means of the fixed-point theorem due to Ren et al[18].

Some definitions concerning time scales can be found in [2, 7, 9, 10].

In the remainder of this section, we state some fundamental definition and the fixed-point theorem due to Ren et al.

Definition 1.1. Let E be a real Banach space. A nonempty, closed, convex set $P \subset E$ is said to be a cone provided the following conditions are satisfied:

(i) if $x \in P$ and $\lambda \geq 0$, then $\lambda x \in P$;

(ii) if $x \in P$ and $-x \in P$, then $x = 0$.

Every cone $P \subset E$ induces an ordering in E given by

$$x \leq y \text{ if and only if } y - x \in P.$$

Definition 1.2. Given a nonnegative continuous functional γ on a cone P of a real Banach space E, for each $d > 0$, we define the set

$$P(\gamma, d) = \{x \in P \mid \gamma(x) < d\}.$$

To prove our main results, we need the following fixed-point theorem due to Ren et al in [18]. The origin in E is denoted by θ.

Theorem 1.1. *Let P be a cone in a real Banach space E. Let α, β and γ be increasing, nonnegative, continuous functionals on P, such that for some* *positive numbers c and M,*

$$\gamma(x) \leq \beta(x) \leq \alpha(x) \text{ and } \|x\| \leq M\gamma(x),$$

for all $x \in \overline{P(\gamma, c)}$. Suppose that there exist positive numbers a and b with $a < b < c$ and

$$F : \overline{P(\gamma, c)} \to P$$

is completely continuous operator such that:

(i) $\gamma(Fx) < c$, for $x \in \partial P(\gamma, c)$;

(ii) $\beta(Fx) > b$, for $x \in \partial P(\beta, b)$;

(iii) $P(\alpha, a) \neq \phi$ and $\alpha(Fx) < a$, for $x \in \partial(\alpha, a)$.

Then F has at least three fixed points $x_1, x_2, x_3 \in \overline{P(\gamma, c)}$ such that

$$0 \leq \alpha(x_1) < a < \alpha(x_2), \text{ with } \beta(x_2) < b < \beta(x_3),\ \gamma(x_3) < c.$$

Remark 1.1. If the restriction $F\theta \neq \theta$ is imposed in Theorem 1.1, then there is the slightly stronger conclusion as following:

F has at least three fixed points $x_1, x_2, x_3 \in \overline{P(\gamma, c)}$ such that

$$0 < \alpha(x_1) < a < \alpha(x_2), \text{ with } \beta(x_2) < b < \beta(x_3),\ \gamma(x_3) < c.$$

2. Existence of Three Positive Solutions

Throughout the rest of this paper, we always assume that $0 < \eta < 1$ and $f : [0, \infty) \to [0, \infty)$ is continuous.

We note that $x(t)$ is a solution of the BVP (1.1) if and only if

$$x(t) = \frac{1}{1-\eta}\left[\int_0^t f(x(\sigma(s)))\triangle s + \eta \int_t^{\sigma(T)} f(x(\sigma(s)))\triangle s\right].$$

Let $E = \{x \mid x : [0, \sigma(T)]_{\mathbf{T}} \to R \text{ is continuous}\}$ be endowed with $\|x\| = \max_{t\in[0,\sigma(T)]_{\mathbf{T}}} |x(t)|$, so E is a Banach space. Define cone $P \subset E$ by

$$P = \{x \in E \mid x \text{ is nondecreasing and } x(t) \geq \eta \|x\| \text{ on } [0, \sigma(T)]_{\mathbf{T}}\}.$$

Define an operator F: $P \to E$ by

$$(Fx)(t) = \frac{1}{1-\eta}\left[\int_0^t f(x(\sigma(s)))\triangle s + \eta \int_t^{\sigma(T)} f(x(\sigma(s)))\triangle s\right],\ t \in [0, \sigma(T)]_{\mathbf{T}}.$$

Lemma 2.1. (see [20]) *F: $P \to P$ is completely continuous.*

Let τ, $l \in \mathbf{T}$ be such that $0 < l < \tau < \sigma(T)$.

Define the increasing, nonnegative, continuous functionals γ, β, and α on P respectively as

$$\gamma(x) = \max_{t\in[0,l]_{\mathbf{T}}} x(t) = x(l), \quad \beta(x) = \min_{t\in[l,\tau]_{\mathbf{T}}} x(t) = x(l),$$

$$\alpha(x) = \max_{t\in[0,\tau]_{\mathbf{T}}} x(t) = x(\tau).$$

For each $x \in P$, it is easy to see that

$$\gamma(x) = \beta(x) \leq \alpha(x).$$

In addition, for each $x \in P$, $\gamma(x) = x(l) \geq \eta \|x\|$. Thus, $\|x\| \leq \frac{1}{\eta}\gamma(x)$, for $x \in P$.

We now state growth conditions on f so that the BVP (1.1) has at least three positive solutions.

Theorem 2.1. *Let $0 < a < \eta b < b < \frac{\eta^2(\sigma(T)-l)}{\sigma(T)}c$, and suppose that f satisfies the following conditions:*

(H_1) $f(x) < \frac{(1-\eta)c}{\sigma(T)}$, if $0 \leq x \leq \frac{c}{\eta}$,

(H_2) $f(x) > \frac{(1-\eta)b}{\eta(\sigma(T)-l)}$, if $b \leq x \leq \frac{b}{\eta}$,

(H_3) $f(x) < \frac{(1-\eta)a}{\sigma(T)}$, if $0 \leq x \leq \frac{a}{\eta}$.

Then the BVP (1.1) has at least three positive solutions x_1, x_2, x_3 satisfying

$$0 \leq \alpha(x_1) < a < \alpha(x_2), \text{ with } \beta(x_2) < b < \beta(x_3), \ \gamma(x_3) < c.$$

Proof. We first assert that $\gamma(Fx) < c$, for $x \in \partial P(\gamma, c)$.

Let $x \in \partial P(\gamma, c)$, then $\gamma(x) = \max_{t\in[0,l]_{\mathbf{T}}} x(t) = x(l) = c$. Since $\|x\| \leq \frac{1}{\eta}\gamma(x)$, we get

$$0 \leq x(t) \leq \frac{c}{\eta}, \quad t \in [0, \sigma(T)]_{\mathbf{T}}.$$

From (H_1), we have

$$\begin{aligned} \gamma(Fx) &= (Fx)(l) \\ &= \frac{1}{1-\eta}\left[\int_0^l f(x(\sigma(s)))\triangle s + \eta\int_l^{\sigma(T)} f(x(\sigma(s)))\triangle s\right] \\ &\leq \frac{1}{1-\eta}\int_0^{\sigma(T)} f(x(\sigma(s)))\triangle s \end{aligned}$$

$$\begin{aligned}
&< \frac{1}{1-\eta}\int_0^{\sigma(T)} \frac{(1-\eta)\,c}{\sigma(T)}\triangle s \\
&= c,
\end{aligned}$$

Therefore $\gamma(Fx) < c$, for $x \in \partial P(\gamma, c)$.

Secondly, we assert that $\beta(Fx) > b$, for $x \in \partial P(\beta, b)$.

Let $x \in \partial P(\beta, b)$, then $\beta(x) = \min_{t\in[l,\tau]_{\mathbf{T}}} x(t) = x(l) = b$. This implies $x(t) \geq b$, $t \in [l, \sigma(T)]_{\mathbf{T}}$, and since $\|x\| \leq \frac{1}{\eta}\gamma(x) = \frac{1}{\eta}\beta(x) = \frac{b}{\eta}$, we get

$$b \leq x(t) \leq \frac{b}{\eta}, \quad t \in [l, \sigma(T)]_{\mathbf{T}}.$$

From (H_2), we have

$$\begin{aligned}
\beta(Fx) &= (Fx)(l) \\
&= \frac{1}{1-\eta}\left[\int_0^{l} f(x(\sigma(s)))\triangle s + \eta\int_l^{\sigma(T)} f(x(\sigma(s)))\triangle s\right] \\
&\geq \frac{\eta}{1-\eta}\int_l^{\sigma(T)} f(x(\sigma(s)))\triangle s \\
&> \frac{\eta}{1-\eta}\int_l^{\sigma(T)} \frac{(1-\eta)\,b}{\eta(\sigma(T)-l)}\triangle s \\
&= b,
\end{aligned}$$

This implies $\beta(Fx) > b$, for $x \in \partial P(\beta, b)$.

Finally, we assert that $P(\alpha, a) \neq \phi$, and $\alpha(Fx) < a$, for $x \in \partial(\alpha, a)$.

Let $x(t) \equiv \frac{a}{2}$, $t \in [0, \sigma(T)]_{\mathbf{T}}$. Obviously, $\frac{a}{2} \in P(\alpha, a)$, and so $P(\alpha, a) \neq \phi$.

Let $x \in \partial(\alpha, a)$, then $\alpha(x) = \max_{t\in[0,\tau]_{\mathbf{T}}} x(t) = x(\tau) = a$. This means that $0 \leq x(t) \leq a$, $t \in [0, \tau]_{\mathbf{T}}$. Note that, $\|x\| \leq \frac{1}{\eta}\gamma(x) \leq \frac{1}{\eta}\alpha(x) = \frac{a}{\eta}$, we get

$$0 \leq x(t) \leq \frac{a}{\eta}, \quad t \in [0, \sigma(T)]_{\mathbf{T}}.$$

From (H_3), we have

$$\begin{aligned}
\alpha(Fx) &= (Fx)(\tau) \\
&= \frac{1}{1-\eta}\left[\int_0^{\tau} f(x(\sigma(s)))\triangle s + \eta\int_\tau^{\sigma(T)} f(x(\sigma(s)))\triangle s\right] \\
&\leq \frac{1}{1-\eta}\int_0^{\sigma(T)} f(x(\sigma(s)))\triangle s \\
&< \frac{1}{1-\eta}\int_0^{\sigma(T)} \frac{(1-\eta)\,a}{\sigma(T)}\triangle s
\end{aligned}$$

$$= \quad a,$$

as required.

Thus, all the conditions of Theorem 1.1 are satisfied. Hence, F has at least three fixed points x_1, x_2, x_3 satisfying

$$0 \leq \alpha(x_1) < a < \alpha(x_2), \text{ with } \beta(x_2) < b < \beta(x_3), \ \gamma(x_3) < c. \qquad \square$$

Remark 2.1. If adding in Theorem 2.1 the restriction $t_0 \in [0, T]_{\mathbf{T}}$ such that $f(x(\sigma(t_0))) \neq 0$, then $F\theta \neq \theta$. So by Remark 1.1, the BVP (1.1) has at least three positive solutions $x_1, x_2, x_3 \in \overline{P(\gamma, c)}$ such that

$$0 < \alpha(x_1) < a < \alpha(x_2), \text{ with } \beta(x_2) < b < \beta(x_3), \ \gamma(x_3) < c.$$

3. Examples

Let $\mathbf{T} = [0, \frac{1}{4}] \cup [\frac{1}{2}, 1]$. Take $T = 1$, $\eta = \frac{1}{2}$, $l = \frac{1}{4}$, $\tau = \frac{1}{2}$, and choose

$$f(x) = \begin{cases} \dfrac{1}{3}, & 0 \leq x \leq 2, \\ \dfrac{17}{6}x - \dfrac{16}{3}, & 2 \leq x \leq 4, \\ 6, & 4 \leq x \leq 8, \\ 9x - 66, & 8 \leq x \leq 9, \\ 15 & x \geq 9. \end{cases}$$

Choose $a = 1$, $b = 4$, $c = 32$, then we have $0 < a < \eta b < b < \frac{\eta^2(\sigma(T)-l)}{\sigma(T)}c$, and $f(x)$ satisfies

$$f(x) = \frac{1}{3} < \frac{1}{2} = \frac{(1-\eta)\,a}{\sigma(T)}, \quad 0 \leq x \leq 2;$$

$$f(x) = 6 > \frac{16}{3} = \frac{(1-\eta)b}{\eta(\sigma(T)-l)}, \quad 4 \leq x \leq 8;$$

$$f(x) \leq 15 < 16 = \frac{(1-\eta)c}{\sigma(T)}, \quad 0 \leq x \leq 64.$$

Thus by Theorem 2.1 and Remark 2.1, the BVP (1.1) has at least three positive solutions x_1, x_2, x_3 satisfying

$$0 < \alpha(x_1) < 1 < \alpha(x_2), \text{ with } \beta(x_2) < 4 < \beta(x_3), \ \gamma(x_3) < 32.$$

Acknowledgments

This paper is supported by the the Scientific Research Foundation of Huaiyin Institute of Technology (HGB0913, HGC0928).

References

[1] R.P. Agarwal, M. Bohner, Quadratic functionals for second order matrix equations on time scales, *Nonlinear Anal.*, **33** (1998), 675-692.

[2] R.P. Agarwal, M. Bohner, Basic calculus on time scales and some of its applications, *Results Math.*, **35** (1999), 3-22.

[3] R.P. Agarwal, M. Bohner, P.J.Y. Wong, Sturm-Liouville eigenvalue problems on time scales, *Appl. Math. Comput.*, **99** (1999), 153-166.

[4] D.R. Anderson, Solutions to second-order three-point problems on time scales, *J. Difference. Equations Appl.*, **8** (2002), 673-688.

[5] D.R. Anderson, R. Avery, J. Henderson, Existence of solutions for a one dimensional p-Laplacian on time-scales, *J. Difference. Equations Appl.*, **10** (2004), 889-896.

[6] E. Akin, Boundary value problems for a differential equation on a measure chain, *PanAmer. Math. J.*, **10** (2000), 17-30.

[7] F.M. Atici, G. SH. Guseinov, On Green's functions and positive solutions for boundary value problems on time scales, *J. Comput. Appl. Math.*, **141** (2002), 75-99.

[8] R.I. Avery, C.J. Chyan, J. Henderson, Twin solutions of boundary value problems for ordinary differential equations and finite difference equations, *Comput. Math. Appl.*, **42** (2001), 695-704.

[9] M. Bohner, A. Peterson, *Dynamic Equations on Time Scales: An Introduction with Applications*, Birkhäuser, Boston (2001).

[10] M. Bohner, A. Peterson, *Advances in Dynamic Equations on Time Scales*, Birkhäuser, Boston (2003).

[11] C.J. Chyan, J. Henderson, Eigenvalue problems for nonlinear differential equations on a measure chain, *J. Math. Anal. Appl*, **245** (2000), 547-559.

[12] L. Erbe, A. Peterson, Positive solutions for a nonlinear differential equation on a measure chain, *Math. Comput. Modelling.*, **32**, No-s: 5-6 (2000), 571-585.

[13] J. Henderson, Multiple solutions for $2m$-th-order Sturm-Liouville boundary value problems on a measure chain, *J. Difference Equations. Appl.*, **6** (2000), 417-429.

[14] J. Henderson, C.C. Tisdell, Topological transversality and boundary value problems on time scales, *J. Math. Anal. Appl.*, **289** (2004), 110-125.

[15] S. Hilger, Analysis on measure chains-a unified approach to continuous and discrete calculus, *Results Math.*, **18** (1990), 18-56.

[16] B. Kaymakcalan, V. Lakshmikantham, S. Sivasundaram, *Dynamical Systems on Measure Chains*, Kluwer Academic Publishers, Boston (1996).

[17] A.C. Peterson, Y.N. Raffoul, C.C. Tisdell, Three point boundary value problems on time scales, *J. Difference. Equations Appl.*, **10** (2004), 843-849.

[18] J.L. Ren, W.G. Ge, B.X. Ren, Existence of three positive solutions for quasi-linear boundary value problems, *Acta. Math. Appl. Sinica*, English Ser., **21** (2005), 353-358.

[19] H.R. Sun, W.T. Li, Positive solutions for nonlinear m-point boundary value problems on time scales, *Acta. Math. Sinica.*, **49** (2006), 369-380, In Chinese.

[20] J.P. Sun, Twin positive solutions of nonlinear first-order boundary value problems on time scales, *Nonlinear Anal.*, **68** (2008), 1754-1758.

[21] D.B. Wang, Three positive solutions of three-point boundary value problems for p-Laplacian dynamic equations on time scales, *Nonlinear Anal.*, **68** (2008), 2172-2180.